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KMS States for the Generalized Gauge Action on Graph Algebras

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Abstract. Given a positive function on the set of edges of an arbitrary directed graph $E = (E^0, E^1)$, we define a one-parameter group of automorphisms on the C*-algebra of the graph $C^*(E)$, and study the problem of finding KMS states for this action. We prove that there are bijective correspondences between KMS states on $C^*(E)$, a certain class of states on its core, and a certain class of tracial states on $C_0(E^0)$. We also find the ground states for this action and give some examples.

Résumé. Étant donné une fonction positive sur l'ensemble des arcs d'un graphe orienté arbitraire $E = (E^0, E^1)$, nous définissons un groupe à un paramètre d'automorphismes de la C^*-algèbre du graphe $C^*(E)$, et nous étudions le problème de trouver les états KMS pour cette action. Nous prouvons qu'il existe des bijections entre les états KMS sur $C^*(E)$, une certaine classe d'états sur le core, et une certaine classe détats traciaux sur $C_0(E^0)$. Nous trouvons également les états fondamentaux pour cette action et nous donnons quelques exemples.

1. Introduction Given a directed graph $E = (E^0, E^1)$, we can associate to it a C*-algebra $C^*(E)$, and an interesting problem that arises is to find relations between the algebraic properties of the algebra and the combinatorial properties of the graph [14]. One such problem is to determine the set of KMS states for a certain action on the algebra.

Graph algebras are a generalization of Cuntz algebras and Cuntz-Krieger algebras. For the Cuntz algebra, there is a very natural action of the circle, the gauge action, which can be extended to an action of the real line. The KMS states for this action are studied in [13] and later generalized to a more general action of the line, that can be thought of as a generalized gauge action [5]. The same is done for the Cuntz-Krieger algebras [4], [6] and for Exel-Laca algebras [7].

Recently there were similar results proven for the C*-algebra associated to a finite graph. This is done in [10] for an arbitrary finite graph, in [9] for a

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certain class of finite graphs via groupoid C*-algebras and in [1] for the Toeplitz C*-algebra of the graph.

Our goal is to generalize these results to the case of an arbitrary graph. First we analyze which conditions the restrictions of a KMS state to the core of $C^*(E)$ and to $C_0(E^0)$ must satisfy. By using a description of the core as an inductive limit, we can build a KMS state on $C^*(E)$ from a tracial state on $C_0(E^0)$ satisfying the conditions found. We point out that in this paper it is not assumed that the graph has no sources and/or sinks, and therefore the graph C*-algebras considered may not be isomorphic to Exel-Laca algebras (as in [8]) - in particular, the results in [7] for Exel-Laca algebras may not apply.

In Section 2 we review some of the basic definitions and results about graph algebras as well as the description of the core as an inductive limit. In Section 3 we establish the results concerning KMS states, followed by a discussion on ground states in Section 4. In Section 5, some examples are given.

In our examples we show that, depending on the choice of the action on the algebra, there may be no KMS states or there may exist infinitely many KMS states.

2. Graph algebras

Definition 2.1. A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of nonempty sets E^0, E^1 and functions $r, s : E^1 \to E^0$; an element of E^0 is called a *vertex* of the graph, and an element of E^1 is called an *edge*. For an edge e, we say that $r(e)$ is the *range* of e and $s(e)$ is the *source* of e.

Remark 2.2. Whenever needed we assume that E^0 and E^1 have the discrete topology.

Definition 2.3. A vertex v in a graph E is called a *source* if $r^{-1}(v) = \varnothing$, and is said to be *singular* if it is either a source, or $r^{-1}(v)$ is infinite.

Definition 2.4. A *path of length* n in a graph E is a sequence $\mu = \mu_1\mu_2 \dots \mu_n$ such that $r(\mu_i + 1) = s(\mu_i)$ for all $i = 1, \dots, n-1$. We write $|\mu| = n$ for the length of μ and regard vertices as paths of length 0. We denote by E^n the set of all paths of length n and $E^* = \cup_{n\geq 0} E^n$. We extend the range and source maps to E^* by defining $s(\mu) = s(\mu_n)$ and $r(\mu) = r(\mu_1)$ if $n \geq 2$ and $s(v) = v = r(v)$ for $n = 0$.

Definition 2.5. Given a graph E, we define the *C^*-algebra of E* as the universal C*-algebra $C^*(E)$ generated by mutually orthogonal projections $\{p_v\}_{v\in E^0}$ and partial isometries $\{s_e\}_{e\in E^1}$ with mutually orthogonal ranges such that:

(1) $s_e^* s_e = p_{s(e)}$;
(2) $s_e s_e^* \leq p_r(e)$ for every $e \in E^1$;
(3) $p_v = \sum_{e\in r^{-1}(v)} s_e s_e^*$ for every $v \in E^0$ such that $0 < |r^{-1}(v)| < \infty$.

For a path $\mu = \mu_1 \dots \mu_n$, we denote the composition $s_{\mu_1} \dots s_{\mu_n}$ by s_μ, and for $v \in E^0$ we define s_v to be the projection p_v.

Propositions 2.6, 2.7, 2.9 and 2.10 below are found in [14] (as Corollary 1.15, Proposition 2.1, Proposition 3.2 and Corollary 3.3, respectively) in the context of row-finite graphs, but their proofs hold just the same for general graphs as above.

PROPOSITION 2.6. *For $\alpha, \beta, \mu, \nu \in E^*$ we have*

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_{\mu\alpha'} s_\beta^* & \text{if } \alpha = \nu\alpha' \\ s_\mu s_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise} \end{cases}$$

and $C^*(E) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}$.

PROPOSITION 2.7. *Let E be a graph. Then there is an action γ of $\mathbb{T}$ on $C^*(E)$, called a* gauge action*, such that $\gamma_z(s_e) = zs_e$ for every $e \in E^1$ and $\gamma_z(p_v) = p_v$ for every $v \in E^0$.*

DEFINITION 2.8. The *core* of the algebra $C^*(E)$ is the fixed-point subalgebra for the gauge action, denoted by $C^*(E)^\gamma$.

PROPOSITION 2.9. $C^*(E)^\gamma = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu),\ |\mu| = |\nu|\}$.

PROPOSITION 2.10. *There is a conditional expectation $\Phi : C^*(E) \to C^*(E)^\gamma$ such that $\Phi(s_\mu s_\nu^*) = [|\mu| = |\nu|] s_\mu s_\nu^*$.*

It is useful to describe the core as an inductive limit of subalgebras, as was done in an appendix in [2]. The idea is as follows. For $k \geq 0$ define the sets

$$F_k = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^k,\ s(\mu) = s(\nu)\},$$

$$\mathcal{E}_k = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^k \text{ and } s(\mu) = s(\nu) \text{ is singular}\},$$

$$C_k = F_0 + \cdots + F_k.$$

Also, for a given vertex v we define

$$F_k(v) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^k,\ s(\mu) = s(\nu) = v\}$$

so that

$$F_k = \bigoplus_{v \in E^0} F_k(v) \tag{2.1}$$

as a direct sum of C*-algebras.

LEMMA 2.11. *Let Λ be the set of all finite subsets of E^k and for $\lambda \in \Lambda$ define*

$$u_\lambda = \sum_{\mu \in \lambda} s_\mu s_\mu^*.$$

Then $\{u_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit of F_k consisting of projections.

PROOF. This is a direct consequence of Proposition 2.6. □

The following result is a combination of Proposition A.1 and Lemma A.2 in [2].

PROPOSITION 2.12. *With the notation as above for a graph E, the following hold for $k \geq 0$:*

(a) C_k is a C^*-subalgebra of $C^*(E)^\gamma$, F_{k+1} is an ideal in C_k, $C_k \subseteq C_{k+1}$ and

$$C^*(E)^\gamma = \varinjlim C_k.$$

(b) $F_k \cap F_{k+1} = \bigoplus \{F_k(v) : 0 < |r^{-1}(v)| < \infty\}$. ($C^*$-algebraic direct sum)
(c) $C_k = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k-1} \oplus F_k$. (vector space direct sum)

With the above, we can now prove the following.

PROPOSITION 2.13. *For each $k \geq 0$, $C_k \cap F_{k+1} = F_k \cap F_{k+1}$.*

PROOF. Obviously one has $F_k \cap F_{k+1} \subseteq C_k \cap F_{k+1}$. On the other hand, given $x \in C_k \cap F_{k+1}$, one can decompose x as sums in C_k and C_{k+1} with Proposition 2.12(c), use the fact that $F_k = \mathcal{E}_k \oplus (F_k \cap F_{k+1})$ and the uniqueness of the direct sum decompositions of x to conclude that $x \in F_k$. □

3. KMS states for the generalized gauge action In this section we define an action on $C^*(E)$ from a function $c : E^1 \to \mathbb{R}_+^*$ similar to what is done in [5] for the Cuntz algebras and in [6] for the Cuntz-Krieger algebras. We will always suppose that there is a constant $k > 0$ such that $c(e) > k$ for all $e \in E^1$ and that $\beta > 0$. Observe that in this case $c^{-\beta} \in C_b(E^1)$.

We extend a function as above to a function $c : E^* \to \mathbb{R}_+^*$ by defining $c(v) = 1$ if $v \in E^0$ and $c(\mu) = c(\mu_1) \ldots c(\mu_n)$ if $\mu = \mu_1 \cdots \mu_n \in E^n$.

PROPOSITION 3.1. *Given a function $c : E^1 \to \mathbb{R}_+^*$, there is a strongly continuous action $\sigma^c : \mathbb{R} \to \mathrm{Aut}(C^*(E))$ given by $\sigma_t^c(p_v) = p_v$ for all $v \in E^0$ and $\sigma_t^c(s_e) = c(e)^{it} s_e$ for all $e \in E^1$.*

PROOF. Let $T_e = c(e)^{it} s_e$ and note that T_e is a partial isometry with $T_e^* T_e = s_e^* s_e$ and $T_e T_e^* = s_e s_e^*$. It follows that the sets $\{p_v\}_{v \in E^0}$ and $\{T_e\}_{e \in E^1}$ satisfy the same relations as $\{p_v\}_{v \in E^0}$ and $\{s_e\}_{e \in E^1}$. By the universal property, there is a homomorphism $\sigma_t^c : C^*(E) \to C^*(E)$ such that $\sigma_t^c(p_v) = p_v$ for all $v \in E^0$ and $\sigma_t^c(s_e) = T_e = c(e)^{it} s_e$ for all $e \in E^1$.

It is easy to see that $\sigma^c_{t_1} \circ \sigma^c_{t_2} = \sigma^c_{t_1+t_2}$ and $\sigma^c_0 = Id$. Hence σ^c_t is an automorphism with inverse σ^c_{-t}.

To prove continuity, let $a \in C^*(E)$, $t \in \mathbb{R}$ and $\varepsilon > 0$. Take x to be a finite sum $x = \sum_{\mu,\nu\in E^*} \lambda_{\mu,\nu} s_\mu s_\nu^*$ such that $\|a - x\| < \varepsilon/3$. For each pair of paths μ, ν with $\lambda_{\mu,\nu} \neq 0$, there is $\delta_{\mu,\nu}$ such that

$$|c(\mu)^{it} c(\nu)^{-it} - c(\mu)^{iu} c(\nu)^{-iu}| < \frac{\varepsilon}{3\sum_{\mu,\nu\in E^*} \|\lambda_{\mu,\nu} s_\mu s_\nu^*\|}$$

for all $u \in \mathbb{R}$ with $|t - u| < \delta_{\mu,\nu}$. If we take δ to be the minimum of all such $\delta_{\mu,\nu}$, then for all $u \in \mathbb{R}$ with $|t - u| < \delta$ we have

$$\begin{aligned} \|\sigma^c_t(x) - \sigma^c_u(x)\| &= \left\| \sum_{\mu,\nu\in E^*} (c(\mu)^{it} c(\nu)^{-it} - c(\mu)^{iu} c(\nu)^{-iu}) \lambda_{\mu,\nu} s_\mu s_\nu^* \right\| \\ &< \frac{\varepsilon}{3\sum_{\mu,\nu\in E^*} \|\lambda_{\mu,\nu} s_\mu s_\nu^*\|} \sum_{\mu,\nu\in E^*} \|\lambda_{\mu,\nu} s_\mu s_\nu^*\| = \frac{\varepsilon}{3} \end{aligned}$$

and hence

$$\begin{aligned} &\|\sigma^c_t(a) - \sigma^c_u(a)\| = \|\sigma^c_t(a) - \sigma^c_t(x) + \sigma^c_t(x) - \sigma^c_u(x) + \sigma^c_u(x) - \sigma^c_u(a)\| \\ &\leq \|\sigma^c_t(a - x)\| + \|\sigma^c_t(x) - \sigma^c_u(x)\| + \|\sigma^c_u(x - a)\| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

□

From now on, we will write simply σ instead of σ^c. The next result shows that KMS states on $C^*(E)$ are determined by their values at the core of the algebra defined in 2.8.

PROPOSITION 3.2. *Suppose $c : E^1 \to \mathbb{R}^*_+$ is such that $c(\mu) \neq 1$ for all $\mu \in E^* \setminus E^0$. If two (σ, β)-KMS states φ_1, φ_2 on $C^*(E)$ coincide at the core algebra $C^*(E)^\gamma$, then $\varphi_1 = \varphi_2$.*

PROOF. Taking an arbitrary $s_\mu s_\nu^*$ such that $s(\mu) = s(\nu)$, if $|\mu| = |\nu|$ then $s_\mu s_\nu^* \in C^*(E)^\gamma$ and thus $\varphi_1(s_\mu s_\nu^*) = \varphi_2(s_\mu s_\nu^*)$.

Suppose then that $|\mu| \neq |\nu|$, and denote the functional $\varphi_2 - \varphi_1$ by φ. Using the KMS condition, one obtains

$$\varphi(s_\mu s_\nu^*) = \varphi(s_\nu^* c(\mu)^{-\beta} s_\mu) = \begin{cases} c(\mu)^{-\beta} \varphi(s_{\nu'}^*) & \text{if } \nu = \mu\nu' \\ c(\mu)^{-\beta} \varphi(s_{\mu'}^*) & \text{if } \mu = \nu\mu' \\ 0 & \text{otherwise} \end{cases}.$$

It is therefore sufficient to show that $\varphi(s_\mu) = \varphi(s_\mu^*) = 0$ if $|\mu| \geq 1$. To see this, notice that if $C^*(E)$ has a unit, then

$$\varphi(s_\mu) = \varphi(s_\mu 1) = \varphi(1 c(\mu)^{-\beta} s_\mu) = c(\mu)^{-\beta} \varphi(s_\mu),$$

whence $\varphi(s_\mu) = 0$ since $c(\mu) \neq 1$ by hypothesis; the non-unital case is established analogously with the use of an approximate unit. □

THEOREM 3.3. *Suppose $c : E^1 \to \mathbb{R}_+^*$ is such that $c(\mu) \neq 1$ for all $\mu \in E^* \backslash E^0$. If φ is a (σ, β)-KMS state on $C^*(E)$ then its restriction $\omega = \varphi|_{C^*(E)^\gamma}$ to $C^*(E)^\gamma$ satisfies*

$$\omega(s_\mu s_\nu^*) = [\mu = \nu] c(\mu)^{-\beta} \omega(p_{s(\mu)}); \tag{3.1}$$

conversely, if ω is a state on $C^*(E)^\gamma$ satisfying (3.1) *then $\varphi = \omega \circ \Phi$ is a (σ, β)-KMS state on $C^*(E)$, where Φ is the conditional expectation from proposition* 2.10. *The correspondence thus obtained is bijective and preserves convex combinations.*

PROOF. Let φ be a (σ, β)-KMS state on $C^*(E)$ and ω its restriction to $C^*(E)^\gamma$. If μ, ν are paths such that $|\mu| = |\nu|$ and $s(\mu) = s(\nu)$ then

$$\omega(s_\mu s_\nu^*) = \varphi(s_\mu s_\nu^*) = \varphi(s_\nu^* \sigma_{i\beta}(s_\mu)) = \varphi(s_\nu^* c(\mu)^{-\beta} s_\mu)$$

$$= [\mu = \nu] c(\mu)^{-\beta} \varphi(p_{s(\mu)}) = [\mu = \nu] c(\mu)^{-\beta} \omega(p_{s(\mu)}).$$

Conversely, let ω be a state on $C^*(E)^\gamma$ satisfying (3.1) and $\varphi = \omega \circ \Phi$; we have to show that φ satisfies the KMS condition. By continuity and linearity, it is sufficient to verify this for elements $x = s_\mu s_\nu^*$ and $y = s_\zeta s_\eta^*$ where $\mu, \nu, \zeta, \eta \in E^*$ are paths such that $s(\mu) = s(\nu)$ and $s(\zeta) = s(\eta)$.

We need to check that $\varphi(xy) = \varphi(y \sigma_{i\beta}(x))$. First note that

$$xy = (s_\mu s_\nu^*)(s_\zeta s_\eta^*) = \begin{cases} s_{\mu\zeta'} s_\eta^* & \text{if } \zeta = \nu\zeta' \quad (1) \\ s_\mu s_{\eta\nu'}^* & \text{if } \nu = \zeta\nu' \quad (2) \\ 0 & \text{otherwise} \quad (3) \end{cases}$$

and

$$y \sigma_{i\beta}(x) = c(\mu)^{-\beta} c(\nu)^\beta (s_\zeta s_\eta^*)(s_\mu s_\nu^*) = c(\mu)^{-\beta} c(\nu)^\beta \begin{cases} s_{\zeta\mu'} s_\nu^* & \text{if } \mu = \eta\mu' \quad (a) \\ s_\zeta s_{\nu\eta'}^* & \text{if } \eta = \mu\eta' \quad (b) \\ 0 & \text{otherwise} \quad (c) \end{cases}.$$

There are nine cases to consider. In each case it must be checked whether the resulting paths have the same size, for they will be otherwise sent to 0 by Φ.

Case 1-a. In this case $\zeta = \nu\zeta'$ and $\mu = \eta\mu'$ so that $|\zeta| = |\nu| + |\zeta'|$ and $|\mu| = |\eta| + |\mu'|$. We claim that $|\mu\zeta'| = |\mu| + |\zeta'| = |\eta|$ if and only if $|\zeta\mu'| = |\zeta| + |\mu'| = |\nu|$, and in this case $\mu = \eta$ and $\nu = \zeta$. In fact,

$$|\mu| + |\zeta'| = |\eta| \Leftrightarrow |\eta| + |\mu'| + |\zeta'| = |\eta| \Leftrightarrow |\mu'| + |\zeta'| = 0$$

$$\Leftrightarrow |\nu| + |\zeta'| + |\mu'| = |\nu| \Leftrightarrow |\zeta| + |\mu'| = |\nu|.$$

Observe that, in this case, we have $|\mu'| + |\zeta'| = 0$ so that $|\mu'| = |\zeta'| = 0$, and hence $\mu = \eta$, $\nu = \zeta$.

It follows that, if $|\mu\zeta'| \neq |\eta|$, then

$$\varphi(xy) = \omega \circ \Phi(xy) = \omega(0) = \omega \circ \Phi(y\sigma_{i\beta}(x)) = \varphi(y\sigma_{i\beta}(x))$$

and, if $|\mu\zeta'| = |\eta|$, we get

$$\varphi(xy) = \varphi(s_\mu s_\mu^*) = \omega(s_\mu s_\mu^*) = c(\mu)^{-\beta}\omega(p_{s(\mu)})$$

and on the other hand

$$\begin{aligned}\varphi(y\sigma_{i\beta}(x)) &= c(\mu)^{-\beta}c(\nu)^{\beta}\varphi(s_\nu s_\nu^*) = c(\mu)^{-\beta}c(\nu)^{\beta}\omega(s_\nu s_\nu^*) \\ &= c(\mu)^{-\beta}c(\nu)^{\beta}c(\nu)^{-\beta}\omega(p_{s(\mu)}) = c(\mu)^{-\beta}\omega(p_{s(\mu)}).\end{aligned}$$

Case 1-b. Now, we have that $\zeta = \nu\zeta'$ and $\eta = \mu\eta'$ so that $|\zeta| = |\nu\zeta'| = |\nu| + |\zeta'|$ and $|\eta| = |\mu\eta'| = |\mu| + |\eta'|$; as before, we can check that $|\mu| + |\zeta'| = |\eta|$ if and only if $|\zeta| = |\nu| + |\eta'|$. If that is not the case then $\varphi(xy) = 0 = \varphi(y\sigma_{i\beta}(x))$. If the equivalent conditions are true then

$$\varphi(xy) = \varphi(s_{\mu\zeta'}s_\eta^*) = \omega(s_{\mu\zeta'}s_\eta^*) = [\mu\zeta' = \eta]c(\eta)^{-\beta}\omega(p_{s(\eta)})$$

and

$$\varphi(y\sigma_{i\beta}(x)) = c(\mu)^{-\beta}c(\nu)^{\beta}\varphi(s_\zeta s_{\nu\eta'}^*) = c(\mu)^{-\beta}c(\nu)^{\beta}[\zeta = \nu\eta']c(\zeta)^{-\beta}\omega(p_{s(\zeta)}).$$

Since $\zeta = \nu\zeta'$ and $\eta = \mu\eta'$, we have that $\mu\zeta' = \eta$ if and only if $\zeta = \nu\eta'$ and if both are true, then $\zeta' = \eta'$ and

$$\begin{aligned}c(\mu)^{\beta}c(\nu)^{-\beta}c(\zeta)^{-\beta} &= c(\mu)^{-\beta}c(\nu)^{\beta}c(\nu)^{-\beta}c(\eta')^{-\beta} = c(\mu)^{-\beta}c(\eta')^{-\beta} \\ &= c(\mu)^{-\beta}c(\zeta')^{-\beta} = c(\eta)^{-\beta}.\end{aligned}$$

From our original hypothesis, we have that $s(\eta) = s(\zeta)$ so we conclude that $\varphi(xy) = \varphi(y\sigma_{i\beta}(x))$.

Case 1-c. In this case $\varphi(y\sigma_{i\beta}(x)) = 0$, so we need to check that $\varphi(xy) = 0$. As with the previous case, we have that $\varphi(xy) = [\mu\zeta' = \eta]c(\eta)^{-\beta}\omega(p_{s(\eta)})$; however, in case (c) $\mu\zeta' \neq \eta$ for all ζ' and therefore $\varphi(xy) = 0$.

The other cases are analogous to these three, except for case 3-c, where $\varphi(xy) = 0 = \varphi(y\sigma_{i\beta}(x))$ since $xy = 0 = y\sigma_{i\beta}(x)$.

That the correspondence obtained is bijective follows from Proposition 3.2 and that it preserves convex combinations is immediate. □

Next, we want to show that there is also a bijective correspondence between (σ, β)-KMS states on $C^*(E)$ and a certain class of tracial states on $C_0(E^0)$. We build this correspondence by first describing a correspondence between this class of tracial states on $C_0(E^0)$ and states ω on $C^*(E)^\gamma$ satisfying (3.1).

The conditions found for the states on $C_0(E^0)$ are similar to those in [12], although as discussed in [10], their results cannot be used directly for an arbitrary graph; nevertheless, the results of Theorem 1.1 of [12] still apply in the general setting, and we use them to build a certain kind of transfer operator on the dual of $C_0(E^0)$.

Let us first recall how to construct $C^*(E)$ as C*-algebra associated to a C*-correspondence [11]. If we let $A = C_0(E^0)$, then $C_c(E^1)$ has a pre-Hilbert A-module structure given by

$$\langle \xi, \eta \rangle (v) = \sum_{e \in s^{-1}(v)} \overline{\xi(e)}\eta(e) \quad \text{for} \quad v \in E^0,$$

$$(\xi a)(e) = \xi(e)a(s(e)) \quad \text{for} \quad e \in E^1,$$

where $\xi, \eta \in C_c(E^1)$ and $a \in A$; it follows that the completion X of $C_c(E^1)$ with respect to the norm given by $\|\xi\| = \| \langle \xi, \xi \rangle \|^{1/2}$ is a Hilbert A-module. A representation $i_X : A \to \mathcal{L}(X)$ is then defined by

$$i_X(a)(\xi)(e) = a(r(e))\xi(e) \quad \text{for} \quad v \in E^0,$$

where $\mathcal{L}(X)$ is the C*-algebra of adjointable operators on X.

Let $\mathcal{K}(X)$ be the C*-subalgebra of $\mathcal{L}(X)$ generated by the operators $\theta_{\xi,\eta}$ given by $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle$. For each $e \in E^1$, let $\chi_e \in C_c(E^1)$ be the characteristic function of $\{e\}$ and observe that

$$\left\{ t_\lambda = \sum_{e \in \lambda} \theta_{\chi_e, \chi_e} \right\}_{\lambda \in \Lambda},$$

where Λ is the set of all finite subsets of E^1, is an approximate unit of $\mathcal{K}(X)$. It is essentially the same approximate unit given by Lemma 2.11.

If τ is a tracial state on $C_0(E^0)$, the expression

$$\mathrm{Tr}_\tau(T) = \lim_{\lambda \to \infty} \sum_{e \in \lambda} \tau\left(\langle \chi_e, T\chi_e \rangle\right)$$

where $T \in \mathcal{L}(X)$, defines a trace Tr_τ on $\mathcal{L}(X)$ according to Theorem 1.1 of [12].

For a function $c : E^1 \to \mathbb{R}_+^*$ as in the beginning of the section and $\beta > 0$, we have that $c^{-\beta} \in C_b(E^1)$ and so it defines a positive operator on $\mathcal{L}(X)$ by pointwise multiplication which commutes with $i_X(a)$ for all $a \in A$. We can thus define a trace on $C_0(E^0)$ as follows.

DEFINITION 3.4. Given c and β as above and τ a tracial state on $C_0(E^0)$, we define a trace $\mathcal{F}_{c,\beta}(\tau)$ on $C_0(E^0)$ by

$$\mathcal{F}_{c,\beta}(\tau)(a) = \mathrm{Tr}_\tau(i_X(a)c^{-\beta}).$$

Now, observe that $C_0(E^0) \cong \overline{\operatorname{span}}\{p_v\}_{v\in E^0}$; regarding this as an equality, for a given tracial state τ on $C_0(E^0)$ we will write $\tau(p_v) = \tau_v$. For $v \in E^0$, it can be verified that

$$\mathcal{F}_{c,\beta}(\tau)(p_v) = \lim_{D\to r^{-1}(v)} \sum_{e\in D} c(e)^{-\beta}\tau_{s(e)},$$

where the limit is taken on finite subsets D of $r^{-1}(v)$, and $\mathcal{F}_{c,\beta}(\tau)(p_v) = 0$ if $r^{-1}(v) = \varnothing$.

Remark 3.5. By Theorem 1.1 of [12], if $\mathcal{F}_{c,\beta}(\tau)(a) < \infty$ for all $a \in C_0(E^0)$, then $\mathcal{F}_{c,\beta}(\tau)$ is actually a positive linear functional; also, if $V \subseteq E^0$ and $\mathcal{F}_{c,\beta}(\tau)(p_v) < \infty$ for all $v \in V$ then $\mathcal{F}_{c,\beta}(\tau)$ is a positive linear functional on $\overline{\operatorname{span}}\{p_v : v \in V\}$.

Definition 3.6. For a vertex $v \in E^0$ and a positive integer n, we define

$$r^{-n}(v) = \{\mu \in E^n : r(\mu) = v\}.$$

Lemma 3.7. *If $\mathcal{F}_{c,\beta}(\tau)(p_v) \le \tau_v$ for all $v \in E^0$ then*

$$\lim_{D\to r^{-n}(v)} \sum_{\mu\in D} c(\mu)^{-\beta}\tau_{s(\mu)} \le \tau_v$$

for all $v \in E^0$ and for all $n \in \mathbb{N}^*$.

Proof. This is proven by induction. The case $n = 1$ is the hypothesis. Now suppose it is true for n, then

$$\begin{aligned}
&\lim_{D\to r^{-(n+1)}(v)} \sum_{\mu\in D} c(\mu)^{-\beta}\tau_{s(\mu)}\\
&= \lim_{D\to r^{-n}(v)} \sum_{\nu\in r^{\le n}} c(\nu)^{-\beta} \sum_{e\in r^{-1}(s(\nu))} c(e)^{-\beta}\tau_{s(e)}[\nu e \in D]\\
&\le \lim_{D\to r^{-n}(v)} \sum_{\nu\in D} c(\nu)^{-\beta}\tau_{s(\nu)} \le \tau_v
\end{aligned}$$

where the first inequality is true due to the fact that since c is a positive function then the net $\sum_{e\in D} c(e)^{-\beta}\tau_{s(e)}$ for finite subsets D of $r^{-1}(s(\nu))$ is nondecreasing and less than or equal to $\tau_{s(\nu)}$ by hypothesis. The last inequality is the induction hypothesis. □

The next lemma is found in [7] for unital algebras, but their proof carries out the same in the non-unital case by using an approximate unit instead of a unit.

Lemma 3.8 (Exel-Laca). *Let B be a C*-algebra, A be a C*-subalgebra such that an approximate unit of A is also an approximate unit of B and I a closed bilateral ideal of B such that $B = A + I$. Let φ be a state on A and ψ a linear positive functional on I such that $\varphi(x) = \psi(x)$ $\forall x \in A \cap I$ and $\overline{\psi}(x) \le \varphi(x)$ $\forall x \in A^+$, where $\overline{\psi}(x) = \lim_\lambda \psi(bu_\lambda)$ for an approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$ of I. Then there is a unique state Φ on B such that $\Phi|_A = \varphi$ and $\Phi|_I = \psi$.*

We want to use this lemma for $A = C_n$, $I = F_{n+1}$ and $B = C_{n+1}$, defined in Section 2. For that, we first note that F_{n+1} is indeed an ideal of C_{n+1} by Proposition 2.12 and that the approximate unit for F_0 given by Lemma 2.11 is also an approximate unit of C_n for all n. We also need to know what the intersection $A \cap I$ is, and for that we need a preliminary result.

Lemma 3.9. *Suppose c, β and τ are such that $\mathcal{F}_{c,\beta}(\tau)(a) \leq \tau(a)$ for all $a \in C_0(E^0)^+$, then for each $k \geq 1$ there is a unique positive linear functional ψ_k on F_k defined by*

$$\psi_k(s_\mu s_\nu^*) = [\mu = \nu] c(\mu)^{-\beta} \tau_{s(\mu)}. \tag{3.2}$$

Proof. Since $\{s_\mu s_\nu^* : \mu, \nu \in E^k,\ s(\mu) = s(\nu)\}$ is linearly independent, Equation 3.2 defines a unique linear functional on $\operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in E^k,\ s(\mu) = s(\nu)\}$. To extend to the closure, it is sufficient to prove that ψ_k is continuous.

If $x \in \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in E^k,\ s(\mu) = s(\nu)\}$ then

$$x = \sum_{v \in V} \sum_{(\mu,\nu) \in G_v} a^v_{\mu,\nu} s_\mu s_\nu^*$$

where V is a finite subset of E^0 and G_v is a finite subset of $\{(\mu, \nu) \in E^n \times E^n : s(\mu) = s(\nu) = v\}$. Using the decomposition given by Equation 2.1 and observing that $\{s_\mu s_\nu^* : (\mu, \nu) \in G_v\}$ can be completed to generators of a matrix algebra, we have that

$$\|x\| = \max_{v \in V} \left\| \sum_{(\mu,\nu) \in G_v} a^v_{\mu,\nu} s_\mu s_\nu^* \right\| = \max_{v \in V} \|(a^v_{\mu,\nu})_{\mu,\nu}\|$$

where the last norm is the matrix norm.

If Tr is the usual matrix trace we have

$$\begin{aligned}
|\psi_k(x)| &= \left| \psi_k \left(\sum_{v \in V} \sum_{(\mu,\nu) \in G_v} a^v_{\mu,\nu} s_\mu s_\nu^* \right) \right| \\
&= \left| \sum_{v \in V} \sum_{(\mu,\nu) \in G_v} a^v_{\mu,\nu} [\mu = \nu] c(\mu)^{-\beta} \tau_{s(\mu)} \right| \\
&= \left| \sum_{v \in V} \operatorname{Tr}((a^v_{\mu,\nu})_{\mu,\nu} \operatorname{diag}(c(\mu)^{-\beta} \tau_{s(\mu)}) \right| \\
&\leq \sum_{v \in V} \left| \operatorname{Tr}((a^v_{\mu,\nu})_{\mu,\nu} \operatorname{diag}(c(\mu)^{-\beta} \tau_{s(\mu)}) \right| \\
&\leq \sum_{v \in V} \|(a^v_{\mu,\nu})_{\mu,\nu}\| \sum_{\mu : (\mu,\mu) \in G_v} c(\mu)^{-\beta} \tau_{s(\mu)} \leq^{\text{lemma 3.7}} \\
&\leq \sum_{v \in V} \|(a^v_{\mu,\nu})_{\mu,\nu}\| \tau_v \leq \max_{v \in V} (\|(a^v_{\mu,\nu})_{\mu,\nu}\|) \sum_{v \in V} \tau_v = \|x\| \sum_{v \in V} \tau_v \leq \|x\|
\end{aligned}$$

where the last inequality comes from the fact that τ comes from a probability measure on a discrete space. □

THEOREM 3.10. *If ω is a state on $C^*(E)^\gamma$ satisfying (3.1) then its restriction τ to $C_0(E^0)$ satisfies:*

(K1) $\mathcal{F}_{c,\beta}(\tau)(a) = \tau(a)$ for all $a \in \overline{\text{span}}\{p_v : 0 < |r^{-1}(v)| < \infty\}$;
(K2) $\mathcal{F}_{c,\beta}(\tau)(a) \leq \tau(a)$ for all $a \in C_0(E^0)^+$.

Conversely, if τ is a tracial state on $C_0(E^0)$ satisfying (K1) and (K2) then there is a unique state ω on $C^*(E)^\gamma$ satisfying (3.1). This correspondence preserves convex combinations.

PROOF. Let ω be a state on $C^*(E)^\gamma$ satisfying (3.1) and τ its restriction to $C_0(E^0)$. By Remark 3.5, to establish (K1) it is sufficient to consider $a = p_v$ where $v \in E^0$ is such that $0 < |r^{-1}(v)| < \infty$, and in this case

$$\begin{aligned} \tau(p_v) = \omega(p_v) &= \omega\left(\sum_{e\in r^{-1}(v)} s_e s_e^*\right) = \sum_{e\in r^{-1}(v)} c(e)^{-\beta}\omega(p_{s(e)}) \\ &= \sum_{e\in r^{-1}(v)} c(e)^{-\beta}\tau_{s(e)} = \mathcal{F}_{c,\beta}(\tau)(p_v). \end{aligned}$$

For (K2), let $a \in C_0(E^0)^+$ and write $a = \sum_{v\in E^0} a_v p_v$; again, by Remark 3.5 it is sufficient to show the result for $a = p_v$ where $v \in E^0$. If $0 < |r^{-1}(v)| < \infty$, then we have an equality as shown above. If $|r^{-1}(v)| = 0$, then $\mathcal{F}_{c,\beta}(\tau)(p_v) = 0 \leq \tau(p_v)$. If $|r^{-1}(v)| = \infty$, then

$$\begin{aligned} \mathcal{F}_{c,\beta}(\tau)(p_v) &= \lim_{D\to r^{-1}(v)} \sum_{e\in D} c(e)^{-\beta}\tau_{s(e)} = \lim_{D\to r^{-1}(v)} \sum_{e\in D} \omega(s_e s_e^*) \\ &= \lim_{D\to r^{-1}(v)} \sum_{e\in D} \omega(p_v s_e s_e^*) \leq \omega(p_v) = \tau(p_v). \end{aligned}$$

To see the inequality above, we observe that $s_e s_e^*$ are mutually orthogonal projections that commute with p_v so that

$$\begin{aligned} p_v - \sum_{e\in D} p_v s_e s_e^* &= p_v\left(1 - \sum_{e\in D} s_e s_e^*\right) \\ &= \left(1 - \sum_{e\in D} s_e s_e^*\right) p_v \left(1 - \sum_{e\in D} s_e s_e^*\right) \geq 0. \end{aligned}$$

Now, let τ be a tracial state on $C_0(E^0)$ satisfying (K1) and (K2). We will use Lemma 3.8 and the discussion after it. Observe that $F_0 = C_0(E^0)$ and let

$\psi_0 = \tau$. For $n \geq 1$, by Lemma 3.9 there exists a positive linear functional ψ_n on F_n defined by

$$\psi_n(s_\mu s_\nu^*) = [\mu = \nu] c(\mu)^{-\beta} \tau_{s(\mu)}.$$

Let us show by induction that there is a unique state φ_n on C_n such that the restriction to F_n is ψ_n. For $n = 1$, we use Lemma 3.8 with $A = C_0(E^0)$, $I = F_1$, $B = C_1$, $\varphi = \tau$ and $\psi = \psi_1$. By Proposition 2.13, in this case $A \cap I = \overline{\text{span}}\{p_v : v \in E^0,\ 0 < |r^{-1}(v)| < \infty\}$ and if $p_v \in A \cap I$ then

$$\psi(p_v) = \psi_1(p_v) = \psi_1(s_v s_v^*) = \tau_v = \tau(p_v).$$

Using the approximate unit given by Lemma 2.11, for any $v \in E^0$ we have

$$\begin{aligned}
\overline{\psi}(p_v) = \overline{\psi_1}(p_v) &= \lim_{\lambda \to \infty} \psi_1(p_v u_\lambda) = \lim_{D \to r^{-1}(v)} \sum_{e \in D} \psi_1(s_e s_e^*) \\
&= \lim_{D \to r^{-1}(v)} \sum_{e \in D} c(e)^{-\beta} \tau_{s(e)} = \mathcal{F}_{c,\beta}(\tau)(p_v) \leq \tau(p_v),
\end{aligned}$$

where the last inequality is exactly (K2).

Now suppose that there is a unique state φ_n on C_n such that the restriction to F_n is ψ_n and let us show that this is also true for $n+1$. We set $A = C_n$, $I = F_{n+1}$, $B = C_{n+1}$, $\varphi = \varphi_n$ and $\psi = \psi_{n+1}$ on Lemma 3.8. By Proposition 2.13, we have that $A \cap I = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^n, s(\mu) = s(\nu), |\mu| = |\nu|, 0 < |r^{-1}(s(\mu))| < \infty\}$. Let $s_\mu s_\nu^* \in A \cap I$. Since $0 < |r^{-1}(s(\mu))| < \infty$ we have that

$$\begin{aligned}
\psi(s_\mu s_\nu^*) = \psi_{n+1}(s_\mu s_\nu^*) &= \sum_{e \in r^{-1}(s(\mu))} \psi_{n+1}(s_{\mu e} s_{\nu e}^*) \\
&= \sum_{e \in r^{-1}(s(\mu))} [\mu e = \nu e] c(\mu e)^{-\beta} \tau_{s(\mu e)} \\
&= \sum_{e \in r^{-1}(s(\mu))} [\mu = \nu] c(\mu)^{-\beta} c(e)^{-\beta} \tau_{s(e)} \\
&= [\mu = \nu] c(\mu)^{-\beta} \sum_{e \in r^{-1}(s(\mu))} c(e)^{-\beta} \tau_{s(e)} \\
&= [\mu = \nu] c(\mu)^{-\beta} \mathcal{F}_{c,\beta}(\tau)(p_{s(\mu)}) \\
&= [\mu = \nu] c(\mu)^{-\beta} \tau(p_{s(\mu)}) = \psi_n(s_\mu s_\nu^*) = \varphi_n(s_\mu s_\nu^*).
\end{aligned}$$

Again, using the approximate unit given by Lemma 2.11, if $s_\mu s_\nu^* \in C_n$, then

$$\begin{aligned}
\overline{\psi}(s_\mu s_\nu^*) = \overline{\psi_{n+1}}(s_\mu s_\nu^*) &= \lim_{\lambda\to\infty} \psi_{n+1}(s_\mu s_\nu^* u_\lambda) \\
&= \lim_{D\to r^{\leq n+1-|\nu|}(s(\mu))} \sum_{\zeta\in D} \psi_{n+1}(s_{\mu\zeta}s_{\nu\zeta}^*) \\
&= \lim_{D\to r^{\leq n+1-|\nu|}(s(\nu))} \sum_{\zeta\in D} [\mu\zeta = \nu\zeta] c(\nu\zeta)^{-\beta}\tau_{s(\nu\zeta)} \\
&= \lim_{D\to r^{\leq n+1-|\nu|}(s(\nu))} \sum_{\zeta\in D} [\mu = \nu] c(\nu)^{-\beta} c(\zeta)^{-\beta}\tau_{s(\zeta)} \\
&= [\mu=\nu]c(\nu)^{-\beta} \lim_{D\to r^{\leq n+1-|\nu|}(s(\nu))} \sum_{\zeta\in D} c(\zeta)^{-\beta}\tau_{s(\zeta)} \\
&\leq [\mu=\nu]c(\nu)^{-\beta}\tau_{s(\nu)} = \varphi_n(s_\mu s_\nu^*),
\end{aligned}$$

where the inequality is given by Lemma 3.7, which is a consequence of (K2).

By the description of the core $C^*(E)^\gamma$ as an inductive limit of the C_n, we can define a state ω as the inductive limit of φ_n. By construction, ω satisfies (3.1) and, since each φ_n is uniquely defined by (3.1), so is ω.

Finally, it is easily seen that the correspondence built preserves convex combinations by construction. □

4. Ground states

In this section, we let a function $c : E^1 \to \mathbb{R}_+^*$ be given and define a one-parameter group of automorphisms σ as in the last section.

The following definition of a ground state will be used [3].

Definition 4.1. We say that ϕ is a *σ-ground state* if for all $a, b \in C^*(E)^a$, the entire analytic function $\zeta \mapsto \phi(a\sigma_\zeta(b))$ is uniformly bounded in the region $\{\zeta \in \mathbb{C} : \mathrm{Im}(\zeta) \geq 0\}$, where $C^*(E)^a$ is the set of analytic elements for σ.

Proposition 4.2. *If τ is a tracial state on $C_0(E^0)$ such that* $\mathrm{supp}(\tau) \subseteq \{v \in E^0 : v \text{ is singular}\}$ *then there is a unique state ϕ on $C^*(E)$ such that*

(i) *$\phi(p_v) = \tau(p_v)$ for all $v \in E^0$;*
(ii) *$\phi(s_\mu s_\nu^*) = 0$ if $|\mu| > 0$ or $|\nu| > 0$.*

Proof. First, observe that a state ϕ satisfying (ii) is uniquely determined by its values on $C^*(E)^\gamma$ because (ii) implies that $\phi = \phi|_{C^*(E)^\gamma} \circ \Phi$, where Φ is the conditional expectation given by Proposition 2.10.

Given τ as in the statement of the proposition, a state ω on $C^*(E)^\gamma$ can be built in the same way as in the proof of Theorem 3.10. For each n, use Lemma 3.8 with $A = C_n$, $B = C_{n+1}$, $I = F_{n+1}$, $\psi_n \equiv 0$ and φ_n is given by the previous step, where for the first step we have $\varphi_0 = \tau$. For $\omega = \varinjlim \varphi_n$, we have that $\phi = \omega \circ \Phi$ satisfies (i) and (ii) and is unique by construction. □

PROPOSITION 4.3. *If c is such that $c(e) > 1$ for all $e \in E^1$, then a state ϕ on $C^*(E)$ is a σ-ground state for σ if and only if $\phi(s_\mu s_\nu^*) = 0$ whenever $|\mu| > 0$ or $|\nu| > 0$.*

PROOF. If ϕ is a ground state then for each pair $\mu, \nu \in E^*$ the function $\zeta \mapsto |\phi(s_\mu \sigma_\zeta(s_\nu^*))|$ is bounded on the upper half of the complex plane. If $\zeta = x + iy$ then

$$|\phi(s_\mu \sigma_\zeta(s_\nu^*))| = |\phi(s_\mu c(\nu)^{-i\zeta} s_\nu^*)| = |c(\nu)^{y-ix}\phi(s_\mu s_\nu^*)| = c(\nu)^y |\phi(s_\mu s_\nu^*)|.$$

If $|\nu| > 0$, we have that $c(\nu) > 1$ and so the only possibility for the above function to be bounded is if $\phi(s_\mu s_\nu^*) = 0$. It is shown analogously that if $|\mu| > 0$ then $\phi(s_\mu s_\nu^*) = 0$.

For the converse, observe that if $|\mu| = |\nu| = 0$ then $|\phi(s_\mu \sigma_\zeta(s_\nu^*))| = |\phi(s_\mu s_\nu^*)| \leq 1$. It can be now readily verified that if $\phi(s_\mu s_\nu^*) = 0$ whenever $|\mu| > 0$ or $|\nu| > 0$ then ϕ is a ground state. □

THEOREM 4.4. *If c is such that $c(e) > 1$ for all $e \in E^1$ then there is a bijective correspondence, given by restriction, between σ-ground states ϕ and tracial states τ on $C_0(E^0)$ such that* $\operatorname{supp}(\tau) \subseteq \{v \in E^0 : v \text{ is singular}\}$.

PROOF. This is an immediate consequence of Propositions 4.2 and 4.3. Just note that if ϕ is a σ-ground state and $v \in E^0$ is not singular then

$$\phi(p_v) = \phi\left(\sum_{e \in r^{-1}(v)} s_e s_e^*\right) = 0.$$

□

5. Examples

In this section we give two examples with infinite graphs and study the KMS states on the C*-algebras associated to these graphs.

EXAMPLE 5.1 (The Cuntz algebra $\mathcal{O}_\infty$). Let $E^0 = \{v\}$ be any unitary set and $E^1 = \{e_n\}_{n\in\mathbb{N}}$ any countably infinite set with $r(e_n) = s(e_n) = v$ $\forall n \in \mathbb{N}$, then $C^*(E) \cong \mathcal{O}_\infty$.

If $c(e_n) = e$ (Euler's number) then we have the usual gauge action. In this case, $\mathcal{F}_{c,\beta}(\tau)(p_v) = \infty$ so that condition (K2) from Theorem 3.10 is not satisfied and we have no KMS states for finite β. Since we have only one state on $C_0(E^0)$ and v is a singular vertex, by Theorem 4.4 there exists a unique ground state.

Now if $c(e_n) = a_n$ where $a_n \in (1, \infty)$ is such that there is $\beta > 0$ for which $\sum_{n=0}^\infty a_n^{-\beta}$ converges, then there exists $\beta_0 > 0$ such that $\sum_{n=0}^\infty a_n^{-\beta_0} = 1$. Observing that $\mathcal{F}_{c,\beta}(\tau)(p_v) = \sum_{n=0}^\infty a_n^{-\beta}$ and using again the fact that there exists only one state on $C_0(E^0)$, we conclude from Theorems 3.3 and 3.10 that there is no KMS state for $\beta < \beta_0$, there exists a unique KMS state for each $\beta \geq \beta_0$ and, as with the gauge action, there is a unique ground state.

EXAMPLE 5.2 (A graph with infinitely many sources). Let $E^0 = \{v_n\}_{n\in\mathbb{N}}$ and $E^1 = \{e_n\}_{n\in\mathbb{N}\setminus\{0\}}$ be countably infinite sets and define $r(e_n) = v_0$ and $s(e_n) = v_n$ for all $n \in \mathbb{N} \setminus \{0\}$.

Again, let $a_n \in (1, \infty)$, $n \in \mathbb{N} \setminus \{0\}$, be such that $\sum_{n=1}^{\infty} a_n^{-\beta}$ converges for some $\beta > 0$. For $n \neq 0$ we have that $\mathcal{F}_{c,\beta}(\tau)(p_{v_n}) = 0$ and for $n = 0$ we have $\mathcal{F}_{c,\beta}(\tau)(p_{v_0}) = \sum_{n=1}^{\infty} a_n^{-\beta}\tau_{v_n}$. Condition (K1) of Theorem 3.10 is trivially satisfied, and for condition (K2) we need $\sum_{n=1}^{\infty} a_n^{-\beta}\tau_{v_n} \leq \tau_{v_0}$.

If $\tau_{v_0} > 0$, since $0 \leq \tau_{v_n} \leq 1$ for all n there exists $\beta_0 > 0$ such that $\sum_{n=1}^{\infty} a_n^{-\beta_0}\tau_{v_n} = \tau_{v_0}$ so that (K2) is verified for all $\beta \geq \beta_0$ and so there are infinitely many KMS states. And for $\beta < \beta_0$ (K2) is not satisfied so there are no KMS states.

For ground states, since all vertices are singular, we have no restriction on τ_{v_0}; every state τ on $C_0(E^0)$ gives a ground state on $C^*(E)$.

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