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Nonnegatively Curved Geodesic Spheres in a Complex Hyperbolic Space

Sadahiro Maeda

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Abstract. We characterize geodesic spheres with sufficiently small radii in a complex hyperbolic space by using their geometric properties. These geodesic spheres are the only examples of hypersurfaces of type (A) having nonnegative sectional curvature in this ambient space.

Résumé. Nous caractérisons les sphères géodésiques de rayon suffisamment petit dans un espace hyperbolique complexe en utilisant leurs propriétés géométriques. Ces sphères géodésiques sont les seuls exemples d'hypersurfaces de type (A) qui ont courbure non-negative dans cet espace ambiant.

1. **Introduction** We denote by $\widetilde{M}_n(c)$ a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $c(\neq 0)$, namely it is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c according as c is positive or negative, which is called an n-dimensional *nonflat complex space form* of constant holomorphic sectional curvature c.

In the theory of real hypersurfaces M^{2n-1} isometrically immersed into $\widetilde{M}_n(c)$, hypersurfaces of type (A) are the most important examples (for details, see Preliminaries). When $c > 0$, every hypersurface of type (A) has nonnegative sectional curvature (see Proposition A). On the contrary, the horosphere, which is a typical example of hypersurfaces of type (A) in $\mathbb{C}H^n(c)$, has the sectional curvature K with $3c/4 \leqq K \leqq -c/4$. Motivated by this fact, we are interested in nonnegatively curved hypersurfaces of type (A) in $\mathbb{C}H^n(c)$.

We shall classify nonnegatively curved hypersurfaces M^{2n-1} of type (A) in $\mathbb{C}H^n(c)$ and characterize them in terms of the exterior differentiation $d\eta$ of the contact form η on M and the extrinsic shape of some geodesics of M. Our motivation is based on the following two facts on real hypersurfaces M of $\widetilde{M}_n(c)$:

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(1) There exist no real hypersurfaces M with closed contact form η, namely $d\eta \neq 0$ on each real hypersurface M;
(2) There exist no real hypersurfaces M all of whose geodesics are mapped to circles in the ambient space $\widetilde{M}_n(c)$.

The purpose of this paper is to prove the following:

THEOREM. *For a connected orientable real hypersurface M^{2n-1} in $\mathbb{C}H^n(c), n \geqq 2$, the following three conditions are mutually equivalent*:

(1) *M is locally congruent to a geodesic sphere $G(r)$ of radius r all of whose sectional curvatures are nonnegative. In this case, the radius r satisfies $0 < r \leqq \log 3/\sqrt{|c|}$.*
(2) *There exist a positive constant k such that M satisfies $d\eta(X,Y) = kg(X,\phi Y)$ for all $X, Y \in TM$ or $d\eta(X,Y) = -kg(X,\phi Y)$ for all $X, Y \in TM$, and a point x of M satisfying that every sectional curvature of M at x is nonnegative.*
(3) *There exists a positive constant k with $k \geqq \sqrt{|c|}$ such that at every point p of M all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p, \dot\gamma_i(0) = v_i \, (1 \leqq i \leqq 2n-2)$ are mapped to circles of the same positive curvature k in the ambient space $\mathbb{C}H^n(c)$ for some orthonormal vectors $v_1, \ldots, v_{2n-2} (\in T_pM)$ orthogonal to the characteristic vector ξ_p.*

In Conditions (2), (3), *k is expressed as*: $k = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)$, *so that the radius r of $G(r)$ is given by* $r = (1/\sqrt{|c|})\{\log(2k + \sqrt{|c|}) - \log(2k - \sqrt{|c|})\}$.

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2. Preliminaries

Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of $\widetilde{M}_n(c)$ through an isometric immersion. The ambient space $\widetilde{M}_n(c)$ is furnished with the standard Riemannian metric g and the canonical Kähler structure J. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2}$$

for arbitrary vector fields X and Y on M, where A is the shape operator of M in $\widetilde{M}_n(c)$. We denote by V_λ the eigenspace associated with the principal curvature λ, namely we set $V_\lambda = \{v \in TM | Av = \lambda v\}$. On M it is well-known that an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the structure (J, g) of the ambient space $\widetilde{M}_n(c)$, which is defined by $g(\phi X, Y) = g(JX, Y)$, $\xi = -J\mathcal{N}$ and $\eta(X) = g(\xi, X) = g(JX, \mathcal{N})$.

It follows from (1), (2) and $\widetilde{\nabla} J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{3}$$

$$\nabla_X \xi = \phi AX. \tag{4}$$

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned} g((R(X,Y)Z, W) = (c/4)\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\} \\ &+ g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W). \end{aligned} \tag{5}$$

Hence, the sectional curvature $K(X,Y)$ of the real plane spanned by a pair (X, Y) of orthonormal vectors is given by

$$K(X,Y) = (c/4)\big(1 + 3g(\phi X, Y)^2\big) + g(AX, X)g(AY, Y) - g(AX, Y)^2. \tag{6}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. Note that every tube of a sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface.

LEMMA A ([3,4]). *Let M be a Hopf hypsurface of a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then the following hold.*

(1) *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + (c/2))\phi v$, where δ is the principal curvature associated with ξ. In particular, when $c > 0$, we have $A\phi v = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi v$.*
(2) *The principal curvature δ associated with ξ is constant locally.*

We recall the following real hypersurfaces which are the simplest examples of Hopf hypersurfaces.
When $c > 0$,

(A_1) a geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$,
(A_2) a tube of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic complex submanifold $\mathbb{C}P^\ell(c)$ with $1 \leqq \ell \leqq n-2$ in $\mathbb{C}P^n(c)$.

When $c < 0$,

(A_0) a horosphere HS in $\mathbb{C}H^n(c)$,
($A_{1,0}$) a geodesic sphere $G(r)$ of radius $r\,(0 < r < \infty)$ in $\mathbb{C}H^n(c)$,
($A_{1,1}$) a tube of of radius $r\,(0 < r < \infty)$ around a totally geodesic complex hypersurface $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$,

(A_2) a tube of radius $r\,(0 < r < \infty)$ around a totally geodesic complex submanifold $\mathbb{C}H^\ell(c)$ with $1 \leqq \ell \leqq n-2$.

Unifying these real hypersurfaces in $\widetilde{M}_n(c), n \geqq 2$, we call them *hypersurfaces of type* (A). The following is a well-known characterization of hypersurfaces of type (A) in $\widetilde{M}_n(c)$ (cf. [5]).

LEMMA B. *Let M be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then M is locally congruent to a hypersurface of type* (A) *if and only if $\phi A = A\phi$ holds on M, where A is the shape operator of M in this ambient space.*

It is well-known that every hypersurface of type (A) is a homogeneous real hypersurface in $\widetilde{M}_n(c)$, namely it is an orbit of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space $\widetilde{M}_n(c)$. For the later use we recall the classification theorem of homogeneous Hopf hypersurfaces in $\mathbb{C}H^n(c)$.

THEOREM A ([1]). *Let M^{2n-1} be a connected real hypersurface of $\mathbb{C}H^n(c), n \geqq 2$. Then M is a Hopf hypersurface all of whose principal curvatures are constant if and only if M is locally congruent to one of the following homogeneous real hypersurfaces*:

(A_0) *A horosphere in $\mathbb{C}H^n(c)$;*
($\mathrm{A}_{1,0}$) *A geodesic sphere of radius r $(0 < r < \infty)$;*
($\mathrm{A}_{1,1}$) *A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;*
(A_2) *A tube of radius r around a totally geodesic $\mathbb{C}H^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;*
(B) *A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.*

The following gives information on sectional curvatures of all hypersurfaces of type (A) in $\mathbb{C}P^n(c)$ (for details, see [2]).

PROPOSITION A. *The sectional curvature K of hypersurfaces of type (A) with radius $r\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ satisfies the following inequalities*:

(A_1) $(c/4)\cot^2(\sqrt{c}\,r/2) \leqq K \leqq c + (c/4)\cot^2(\sqrt{c}\,r/2)$;
(A_2) $0 \leqq K \leqq c + (c/4)\max\{\cot^2(\sqrt{c}\,r/2), \tan^2(\sqrt{c}\,r/2)\}$.

At the end of this section we explain the fundamental two facts (1) and (2) in Introduction. To do this, we first recall the definition $d\eta$, which is given by

$$(7)\quad d\eta(X,Y) = (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\} \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

It follows from (4) and (7) that $d\eta = 0$ if and only if $\phi A + A\phi = 0$. This, together with the fact that there exist no real hypersurfaces M with $\phi A + A\phi = 0$ on M (see Corollary 2.12 in [5]), implies Fact (1). We next review the definition of circles in Riemannian geometry. A smooth real curve $\gamma = \gamma(s)$ parametrized by its arclength s on a Riemannian manifold (M, g, ∇) is called a *circle* if it satisfies the following differential equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{8}$$

Here, $k^2 := g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)$ is automatically constant. A circle of null curvature, namely $k = 0$, is nothing but a geodesic. By easy computation, for a connected hypersurface M^n in a Riemannian manifold $\widetilde{M}^{n+1}$ through an isometric immersion, we can see that every geodesic γ on M^n is mapped to a circle in $\widetilde{M}^{n+1}$ if and only if M^n is totally umbilical in $\widetilde{M}^{n+1}$ and Trace A is constant locally on M^n, where A is the shape operator of M^n in $\widetilde{M}^{n+1}$. This, combined with the fact that there exist no totally umbilical real hypersurfaces M^{2n-1} in $\widetilde{M}_n(c)$, $c \neq 0$, yields Fact 2.

3. Proof of Theorem We first suppose Condition (2). It follows from $d\eta(X, Y) = \pm k g(X, \phi Y)$ for all $X, Y \in TM$, (7) and (4) that

$$0 = g(\phi AX, Y) - g(\phi AY, X) \mp 2kg(X, \phi Y) = g((\phi A + A\phi \pm 2k\phi)X, Y)$$

for each $X, Y \in TM$. So our real hypersurface M satisfies

$$\phi A + A\phi = \mp 2k\phi. \tag{9}$$

We then have $\phi A\xi = 0$, which shows that M is a Hopf hypersurface. We next take a principal curvature vector X orthogonal to ξ associated to a principal curvature λ. Hence, from Lemma A and Equation (9) we find that the principal curvature λ satisfies one of the following quadratic equations:

$$4\lambda^2 + 8k\lambda + c - 4k\delta = 0 \quad \text{or} \quad 4\lambda^2 - 8k\lambda + c + 4k\delta = 0. \tag{10}$$

Since k and δ are constant, this implies that λ is also constant on the connected real hypersurface M. Thus we can see that our real hypersurface M is locally congruent to a homogeneous Hopf hypersurface in $\mathbb{C}H^n(c)$ (see Theorem B).

We shall check (9) one by one for each homogeneous Hopf hypersurface M by standard computation (for examples see [5–7]). Let M be of type (A_0). Then we know that $A\phi + \phi A = \sqrt{|c|}\,\phi$. Let M be of type ($\mathrm{A}_{1,0}$). So, in this case we see that $A\phi + \phi A = \sqrt{|c|}\,\coth(\sqrt{|c|}\,r/2)\phi$. Let M be of type ($\mathrm{A}_{1,1}$). We find similarly that $A\phi + \phi A = \sqrt{|c|}\,\tanh(\sqrt{|c|}\,r/2)\phi$. Let M be of type (A_2). Note that $\phi V_{\lambda_1} = V_{\lambda_1}$ and $\phi V_{\lambda_2} = V_{\lambda_2}$ (see Lemma A). These imply that our real hypersurface M does not satisfy (9). Let M be of type (B). For each radius $r \in (0, \infty)$ we remark that $\phi V^0_{\lambda_1} = V^0_{\lambda_2}$ and $\phi V^0_{\lambda_2} = V^0_{\lambda_1}$, where $V^0_{\lambda_i} = \{v \in TM | Av = \lambda_i v,\ v \perp \xi\}$ for $i = 1, 2$ (see Lemma A). Hence our real hypersurface M satisfies (9). Therefore by virtue of the above discussion we can see that a real hypersurface M satisfies (9) if and only if M is of either type (A_0), type ($\mathrm{A}_{1,0}$), type ($\mathrm{A}_{1,1}$) or type (B).

We next investigate the sectional curvatures of these homogeneous real hypersurfaces. Let M be of type ($\mathrm{A}_{1,0}$). We take a pair of (X, Y) of orthonormal vectors that are orthogonal to ξ. In order to estimate the sectional curvature K of M, we calculate $K(\sin\theta\cdot X+\cos\theta\cdot\xi, Y)$. It follows from (6) that

$$K(\sin\theta\cdot X+\cos\theta\cdot\xi, Y)=(c/4)\{\sin^2\theta\bigl(1+3g(\phi X,Y)^2\bigr)-\coth^2(\sqrt{|c|}\,r/2)\}.$$

This gives the following inequalities:

$$c-(c/4)\coth^2(\sqrt{|c|}\,r/2)\leqq K\leqq(-c/4)\coth^2(\sqrt{|c|}\,r/2). \tag{11}$$

We remark that $K(X,\phi X)=c-(c/4)\coth^2(\sqrt{|c|}\,r/2)$ and $K(X,\xi)=(-c/4)\cdot\coth^2(\sqrt{|c|}\,r/2)$ for each unit vector X orthogonal to ξ. By easy computation we see that $c-(c/4)\coth^2(\sqrt{|c|}\,r/2)\geqq 0$ if and only if $0<r\leqq\log 3/\sqrt{|c|}$. Hence a geodesic sphere $G(r)$ has nonnegative sectional curvature at some point $x\in M$ if and only if the radius r satisfies $0<r\leqq\log 3/\sqrt{|c|}$. Let M be of type (A_0). Taking $r\to\infty$ in (11), we have $3c/4\leqq K\leqq -c/4$ at each point $x\in M$. Let M be of type ($\mathrm{A}_{1,1}$). For a unit vector with $AX=(\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)X$, from (6) we see

$$K(X,\phi X)=c+\frac{|c|}{4}\tanh^2\Bigl(\frac{\sqrt{|c|}\,r}{2}\Bigr)<c+\frac{|c|}{4}=\frac{3c}{4}<0. \tag{12}$$

Let M be of type (B). We take a unit vector X orthogonal to ξ with $AX=(\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)X$. Then it follows from (6) that $K(X,\phi X)=3c/4<0$. Therefore we can see that Conditions (1) and (2) are mutually equivalent.

We next verify Condition (1) implies Condition (3). Let $\gamma=\gamma(s)$ be an arbitrary geodesic on our real hypersurface M whose initial vector $\dot\gamma(0)$ is perpendicular to the characteristic vector ξ_p. It follows from (4), Lemma B, the symmetry of A and the skew-symmetry of ϕ that

$$\begin{aligned}\dot\gamma(g(\dot\gamma,\xi))&=g(\dot\gamma,\nabla_{\dot\gamma}\xi)=g(\dot\gamma,\phi A\dot\gamma)=g(\dot\gamma,A\phi\dot\gamma)\\&=g(A\dot\gamma,\phi\dot\gamma)=-g(\phi A\dot\gamma,\dot\gamma)=0,\end{aligned}$$

which, combined with $g(\dot\gamma(0),\xi_{\gamma(0)})=0$, shows that $\dot\gamma(s)$ is orthogonal to $\xi_{\gamma(s)}$ for every s. Thus we have $A\dot\gamma(s)=(\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)\dot\gamma(s)$ for $-\infty<s<\infty$. This, together with (1) and (2), implies that the geodesic γ is mapped to a circle of the same positive curvature $(\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)$ in $\mathbb{C}H^n(c)$. Here, by the assumption $0<r\leqq\log 3/\sqrt{|c|}$ we see $(\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)\geqq\sqrt{|c|}$. Thus we obtain Condition (3).

Conversely, we suppose Condition (3). Let $\gamma_i=\gamma_i(s)\,(1\leqq i\leqq 2n-2)$ be geodesics on M satisfying Condition (3). Then it follows from (8) that

$$\widetilde\nabla_{\dot\gamma_i}(\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i)=-k^2\dot\gamma_i. \tag{13}$$

On the other hand, from (1) and (2) we obtain

$$\widetilde{\nabla}_{\dot\gamma_i}(\widetilde{\nabla}_{\dot\gamma_i}\dot\gamma_i) = g((\nabla_{\dot\gamma_i}A)\dot\gamma_i, \dot\gamma_i)\mathcal{N} - g(A\dot\gamma_i, \dot\gamma_i)A\dot\gamma_i. \tag{14}$$

Comparing the tangential components of (13) and (14), we have

$$g(A\dot\gamma_i(s), \dot\gamma_i(s))A\dot\gamma_i(s) = k^2\dot\gamma_i(s) \quad \text{for } 1 \leqq i \leqq 2n-2,$$

which, combined with $k \neq 0$, yields that $Av_i = kv_i$ or $Av_i = -kv_i$ for $1 \leqq i \leqq 2n-2$ at the point $p = \gamma(0)$. Note that ξ is principal. Indeed, $g(A\xi, v_i) = g(\xi, Av_i) = 0$ for $1 \leqq i \leqq 2n-2$. Thus we know that our real hypersurface M is a Hopf hypersurface having at most three constant principal curvatures δ, k and $-k$. Therefore by the principal curvatures of homogeneous Hopf hypersurfaces in $\mathbb{C}H^n(c)$ we find that our Hopf hypersurface M has two distinct constant principal curvatures either δ, k or $\delta, -k$. Then without loss of generality M has two distinct constant principal curvatures δ and $k(\geqq \sqrt{|c|}\,)$. Therefore our discussion guarantees that our real hypersurface M is locally congruent to a geodesic sphere $G(r)$ with $0 < r \leqq \log 3/\sqrt{|c|}$. Thus we have Condition (1). □

As an immediate consequence of the discussion in the proof of our Theorem we get the following:

PROPOSITION 1. *In the class of all totally η-umbilical hypersurfaces M^{2n-1} in $\mathbb{C}H^n(c)$, $n \geqq 2$, the geodesic spheres $G(r)$ of radius r with $0 < r \leqq \log 3/\sqrt{|c|}$ are the only real hypersurfaces having nonnegative sectional curvature.*

For the definition of totally η-umbilical hypersurfaces, refer to [5].

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Department of Mathematics, Saga University, Saga, 840-8502, Japan
e-mail: smaeda@ms.saga-u.ac.jp