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Blends and Alloys

R. Exel

Presented by David E. Handelman, FRSC

Abstract. Given two algebras A and B, sometimes assumed to be C*-algebras, we consider the question of putting algebra or C*-algebra structures on the tensor product $A \otimes B$. In the C*-case, assuming B to be two-dimensonal, we characterize all possible such C*-algebra structures in terms of an action of the cyclic group $\mathbb{Z}_2$. An example related to commuting squares is also discussed.

Résumé. Si A et B sont deux algèbres (resp. deux C*-algèbres), nous étudions dans cette note les structures possibles d'algèbre (resp. de C*-algèbre) qui peuvent être définies sur le produit tensoriel $A \otimes B$. Si A est une C*-algèbre, nous caractérisons toutes les structures de C*-algèbre sur le produit tensoriel $A \otimes \mathbb{C}^2$ par une action du groupe cyclique $\mathbb{Z}_2$. Nous présentons aussi un exemple associé aux carrés commutatifs.

1. **Introduction** When G is a group and $\alpha : G \to \mathrm{Aut}(A)$ is an action of G on a unital K-algebra A, one may form the the *crossed product algebra* (also known among algebraists as the *skew group algebra*) $A \rtimes_\alpha G$. As a vector space $A \rtimes_\alpha G$ is just the tensor product $A \otimes K(G)$, where $K(G)$ denotes the group algebra of G with coefficients in the base field K. The multiplication operation on $A \rtimes_\alpha G$ is given by

$$(a \otimes g)(b \otimes h) = a\alpha_g(b) \otimes gh, \qquad \forall\, a, b \in A \quad \forall\, g, h \in G.$$

Researchers working with crossed products are used to thinking that the above multiplication operation on $A \otimes K(G)$ has been twisted, or skewed by the group action in relation to the usual tensor product multiplication. Viewing things from this point of view, one can't help but to ask in how many other ways can the usual mutiplication on a tensor product algebra be similarly modified.

Returning to the example of crossed products above, it turns out that the maps

$$i : a \in A \mapsto a \otimes 1 \in A \rtimes_\alpha G$$

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and

$$j : y \in K(G) \mapsto 1 \otimes y \in A \rtimes_\alpha G$$

are algebra homomorphisms and, in addition, $A \rtimes_\alpha G$ is equal to the linear span of both $i(A)j\big(K(G)\big)$ and $j\big(K(G)\big)i(A)$.

Motivated by these properties we define a *blend* of algebras as being a quintuple

$$\mathcal{X} = (A, B, i, j, X),$$

where A, B and X are unital algebras (see below for the definition in the non-unital case), and

$$i : A \to X \quad \text{and} \quad j : B \to X$$

are unital homomorphisms such that the maps

$$i \circledast j \ : \ a \otimes b \in A \otimes B \quad \longmapsto \quad i(a)j(b) \in X,$$

and

$$j \circledast i \ : \ b \otimes a \in B \otimes A \quad \longmapsto \quad j(b)i(a) \in X$$

are surjective. If moreover $i \circledast j$ and $j \circledast i$ are one-to-one, we say that $\mathcal{X}$ is an *alloy*.

Working in the category of C*-algebras we introduce similar notions, but the above requirement that $i \circledast j$ and $j \circledast i$ are surjective is replaced by the weaker requirement that they have dense range.

Given an algebraic alloy, we may identify $A \otimes B$ with X under $i \circledast j$, and hence make $A \otimes B$ an algebra. Its multiplication operation will therefore satisfy

$$(a' \otimes 1)(a \otimes b) = a'a \otimes b \quad \text{and} \quad (a \otimes b)(1 \otimes b') = a \otimes bb'. \tag{1.1}$$

This is evidently not enough to characterize the whole multiplication operation in $A \otimes B$, but if one is also given the map

$$\tau : B \otimes A \to A \otimes B,$$

defined by $\tau = (i \circledast j)^{-1}(j \circledast i)$, then the product between $1 \otimes b$ and $a \otimes 1$ may be written in terms of τ as

$$(1 \otimes b)(a \otimes 1) = \tau(b \otimes a).$$

In fact the multiplication operation on $A \otimes B$ may be completely recovered, given τ, as follows:

$$\begin{aligned} (a_1 \otimes b_1)(a_2 \otimes b_2) &= (a_1 \otimes 1)(1 \otimes b_1)(a_2 \otimes 1)(1 \otimes b_2) \\ &= (a_1 \otimes 1)\tau(b_1 \otimes a_2)(1 \otimes b_2) \end{aligned} \tag{1.2}$$

while the final answer may be reached upon an application of (1.1).

Conversely, given a map $\tau : B \otimes A \to A \otimes B$, one may take the above answer as the *definition* of a multiplication operation on $A \otimes B$, which will evidently not always be associative but, at least in theory, one may spell out a condition on τ for the associativity to hold.

At this point I must confess that I could not find any nice looking associativity condition. This is perhaps an indication that this whole circle of ideas is thornier than one would initially believe. For starters, as already seen, this construction would encompass all group actions on algebras!

Of course similar thorny questions may be also asked in case A and B are C*-algebras, but then one wouldn't expect to get a C*-algebra right away, unless $A \otimes B$ is completed under some suitable norm.

In the purely algebraic situation this question has already been extensively treated, but almost always B is supposed to have some extra structure, like that of a Hopf algebra [2], [4] or some similar structure [7], [1].

Perhaps the most general approach to this problem is due to Brzeziński [3], where no algebra structure whatsoever is assumed on B, which is only assumed to be a vector space!

Among other things the beauty of Mathematics rests on the fact that, no matter how hard is the problem facing us, there is always an easier special case which may be effectively studied and which, one hopes, may lead the way to new ground.

Far from attempting a complete theory of blends or alloys in either the purely algebraic or the C*-algebraic context, the aim of this paper is to exploit a few concrete situations in which we have found some surprising amount of mathematical structure, but which apparently have not yet been discussed in the literature.

In our first example, a truly humbling experience, we study all possible C*-algebra structures on $A \otimes B$, when A is an arbitrary unital C*-algebra and $B = \mathbb{C}^2$. Curiously this turns out to be quite involving and the answer is that crossed products of A by actions of $\mathbb{Z}_2$ provide all possible examples!

This evidently begs for a generalization to $B = \mathbb{C}^n$, for $n > 2$, but unfortunately our methods do not seem to extend beyond the case $n = 2$.

Our second main example is based on Jones' basic construction, as generalized by Watatani [15] to C*-algebras. As initial data we consider a commuting square [11] of C*-algebras, meaning C*-algebras A, B, C and D, such that

$$\begin{array}{ccc} A & \supseteq & B \\ \cup\mathrm{I} & & \cup\mathrm{I} \\ C & \supseteq & D, \end{array}$$

and $D = B \cap C$. We also suppose we are given conditional expectations

$$E : A \to B \quad \text{and} \quad F : A \to C,$$

satisfying $EF = FE$. Evidently one then has that $G := EF$ is a conditional expectation onto D. Given such a commuting square we let M be the Hilbert

D-module obtained by completing A under the D-valued inner-product

$$\langle a, b\rangle = G(a^*b), \quad \forall a, b \in A.$$

We moreover let e, f and g be the projections on M obtained by extending E, F and G, respectively, to M. Viewing the left action of A on M as a homomorphism $\lambda : A \to \mathscr{L}(M)$, where $\mathscr{L}(M)$ denotes the algebra of all adjointable operators on M, we introduce our main players, namely the C*-algebras

$$K_g = \overline{\text{span}}\ \lambda(A)g\lambda(A),\ K_e = \overline{\text{span}}\ \lambda(A)e\lambda(A) \ \text{ and } \ K_f = \overline{\text{span}}\ \lambda(A)f\lambda(A).$$

We then show that there are natural maps

$$i_e : K_e \to M(K_g) \quad \text{and} \quad i_f : K_f \to M(K_g),$$

where $M(K_g)$ stands for the multiplier algebra of K_g. The main oustanding question left unresolved by this work, incidentally also the question that motivated me to consider the notion of a C*-blend, is whether or not

$$\big(K_e, K_f, i_e, i_f, K_g\big)$$

is a C*-blend (please see below for the definition of a C*-blend in the non-unital case).

Having little to say about this problem in its full generality we consider two special cases where we are able to give positive answers. The first one is the case in which G is of index finite type [15], while in the second case we assume that A is a unital commutative algebra and $D = \mathbb{C}\,1$.

After having discussed a preliminary notion of blends and alloys with George Elliott and Zhuang Niu, they have found an interesting example of a C*-alloy related to the irrational rotation C*-algebra, which we briefly describe below with their kind permission.

Last but not least I would like to thank V. Jones and G. Elliott for stimulating conversations while this work was being prepared.

2. General concepts

In this section we shall describe the main concepts to be treated in this work and which will be further developped in several different directions in the forthcoming sections.

We will assume througout that R is a commutative unital ring, quite often specialized to be the field of complex numbers. By an algebra we will always understand an associative R-algebra.

If A is an R-algebra we will denote by $M(X)$ the algebra of multipliers of X (see e.g. [5, 2.1] for a definition). In badly behaved cases X is not necessarily isomorphic to a subalgebra of $M(X)$, so we will assume that the natural map from X to $M(X)$ is injective and hence we will think of X as a subalgebra of $M(X)$. Please see the discussion following [5, 2.2].

Definition 2.1. Consider a quintuple $\mathcal{X} = (A, B, i, j, X)$, where A, B and X are R-algebras, and

$$i : A \to M(X) \quad \text{and} \quad j : B \to M(X)$$

are homomorphisms. Also consider the R-linear maps

$$i \circledast j \ : \ a \otimes b \in A \otimes_R B \quad \longmapsto \quad i(a)j(b) \in M(X),$$

and

$$j \circledast i \ : \ b \otimes a \in B \otimes_R A \quad \longmapsto \quad j(b)i(a) \in M(X).$$

We will say that $\mathcal{X}$ is:

(a) a *blend* when the ranges of both $i \circledast j$ and $j \circledast i$ coincide with X (or rather its cannonical copy within $M(X)$),
(b) an *alloy* when, in addition to (a), one has that $i \circledast j$ and $j \circledast i$ are one-to-one.

In either case, when A, B, and X are unital algebras, and both i and j preserve the unit, we will say that $\mathcal{X}$ is a *unital* blend or alloy.

As already mentioned in the introduction, if G is a group acting on a unital algebra A, then

$$\big(A, R(G), i, j, A \rtimes G\big)$$

is an example of a blend, which is easily seen to be an alloy.

Definition 2.2. Let $\mathcal{X}_1 = (A, B, i_1, j_1, X_1)$ and $\mathcal{X}_2 = (A, B, i_2, j_2, X_2)$ be blends. A *morphism* from $\mathcal{X}_1$ to $\mathcal{X}_2$ is a homomorphism

$$\phi : M(X_1) \to M(X_2),$$

such that $\phi i_1 = i_2$, and $\phi j_1 = j_2$. The morphism will be termed an *isomorphism* when ϕ is bijective. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are unital blends, we will say that ϕ is a *unital morphism* provided $\phi(1) = 1$.

Given a morphism ϕ, as above, notice that

$$\phi(i_1 \circledast j_1) = i_2 \circledast j_2. \tag{2.1}$$

Therefore it is evident that $\phi(X_1) = X_2$.

PROPOSITION 2.3. *Every morphism from a blend to an alloy is an isomorphism.*

PROOF. Let $\mathcal{X}_1 = (A, B, i_1, j_1, X_1)$ be a blend, let $\mathcal{X}_2 = (A, B, i_2, j_2, X_2)$ be an alloy, and let ϕ be a morphism from $\mathcal{X}_1$ to $\mathcal{X}_2$. Denote by ψ the restriction of ϕ to X_1, viewed as a map onto X_2, and observe that by (2.1) we have that

$$\psi(i_1 \circledast j_1) = i_2 \circledast j_2.$$

By hypothesis we have that $i_2 \circledast j_2$ is one-to-one, and hence so is $i_1 \circledast j_1$. Thus both $i_1 \circledast j_1$ and $i_2 \circledast j_2$ are bijective maps, respectively onto X_1 and X_2. Therefore

$$\psi = (i_2 \circledast j_2)(i_1 \circledast j_1)^{-1},$$

and hence ψ is bijective, and the reader may now prove that ϕ is necessarily bijective as well. □

3. C*-blends Let us now consider C*-algebraic versions of the above concepts, and hence we now assume that our base ring R is the field of complex numbers.

Definition 3.1. Consider a quintuple $\mathcal{X} = (A, B, i, j, X)$, where A, B and X are C*-algebras, and

$$i : A \to M(X) \quad \text{and} \quad j : B \to M(X)$$

are *-homomorphisms. Also consider the linear maps $i \circledast j$ and $j \circledast i$ described in Definition 2.1 (where $A \otimes_{\mathbb{C}} B$ and $B \otimes_{\mathbb{C}} A$ refer to the algebraic tensor product over the field of complex numbers). We will say that $\mathcal{X}$ is:

(a) a C*- blend when the ranges of both $i \circledast j$ and $j \circledast i$ are contained and dense in X,
(b) a C*-alloy when, in addition to (a), one has that $i \circledast j$ and $j \circledast i$ are one-to-one.

In either case, when A, B, and X are unital algebras, and both i and j preserve the unit, we will say that $\mathcal{X}$ is a *unital* C*-blend or C*-alloy.

Observe that, since the range of $j \circledast i$ is the adjoint of the range of $i \circledast j$, if the range of one of this maps is contained and dense in X, then so is the other.

The definition of a *C*-alloy* given above should be considered as tentative for the following reason: given C*-algebras A and B, think of the case of the minimal versus the maximal tensor products, here denoted $A \underset{min}{\otimes} B$ and $A \underset{max}{\otimes} B$, respectively. One evidently has two natural C*-alloys, namely

$$\mathcal{X}_{min} = (A, B, i, j, A \underset{min}{\otimes} B) \quad \text{and} \quad \mathcal{X}_{max} = (A, B, i, j, A \underset{max}{\otimes} B).$$

However, as an alloy, $\mathcal{X}_{min}$ does not satisfy any sensible generalization of 2.3, since the natural map

$$\phi : A \underset{max}{\otimes} B \to A \underset{min}{\otimes} B$$

provides a morphism of C*-blends which is not always an isomorphism. On the other hand it would be interesting to check if $\mathcal{X}_{max}$ satisfies anything like 2.3.

Note that a C*-blend is not necessarily an algebraic blend, since the requirement of having dense range is weaker than that of being surjective. However, occasionally we shall be interested in a concept which subsumes both.

Definition 3.2. A C*-blend (A, B, i, j, X) will be called *strict* if it also satisfies the conditions of Definition 2.1.a, that is, if $i \circledast j$ and $j \circledast i$ are onto X.

In all of our uses of the notion of strict C*-blends below, one of the algebras involved will be finite dimensional.

Remark 3.3. In the unital case, if (A, B, i, j, X) is either an algebraic or C*-alloy, we may identify A and B with the subalgebras of X given by $i(A)$ and $j(B)$, respectively. When such an identification is being made we will sometimes use the short hand notation (A, B, X) to indicate this.

An interesting source of examples of C*-blends is obtained from the theory of crossed products. Given a locally compact group G acting on a C*-algebra A, consider the full crossed product[1]

$$A \rtimes G.$$

Representing $A \rtimes G$ faithfully on a Hilbert space H, one deduces from [8, 7.6.4] that there exists a covariant representation (π, u), such that the given faithfull representation coincides with $\pi \times u$.

It is well known [8, 7.6.6] that the range of π lies in the multiplier algebra of $A \rtimes G$. Moreover, if

$$\rho : C^*(G) \to B(H)$$

denotes the integrated form of u, then the range of ρ is also contained in $M(A \rtimes G)$.

Proposition 3.4. *$\big(A, C^*(G), \pi, \rho, A \rtimes G\big)$ is a C*-blend.*

Proof. Let us first prove that the range of $\pi \circledast \rho$ is contained in $A \rtimes G$. For this let $a \in A$ and $f \in C^*(G)$, and write $f = \lim_n f_n$, with $f_n \in C_c(G)$.

Viewing $a \otimes f_n$ as an element of $C_c(G, A)$, and observing that

$$(\pi \times u)(a \otimes f_n) = \pi(a)\rho(f_n),$$

by [8, 7.6.4], one sees that $\pi(a)\rho(f_n)$ is in $A \rtimes G$. So

$$(\pi \circledast \rho)(a \otimes f) = \pi(a)\rho(f) = \lim_n \pi(a)\rho(f_n) \in A \rtimes G,$$

as claimed.

Next, let us check that the range of $\pi \circledast \rho$ is dense in $A \rtimes G$. For this, let us view $A \otimes C_c(G)$ as a subspace of $C_c(G, A)$ in the usual fashion. The former may be easily shown to be dense in the latter under the inductive limit topology.

On the other hand, by the very definition of crossed products, the image of $C_c(G, A)$ under $\pi \times u$ is a dense subalgebra of $A \rtimes G$. Therefore

$$(\pi \circledast \rho)\big(A \otimes C_c(G)\big) = \pi(A)\rho\big(C_c(G)\big) = (\pi \times u)\big(A \otimes C_c(G)\big)$$

is dense in $A \rtimes G$. Evidently we also have that $(\pi \circledast \rho)\big(A \otimes C^*(G)\big)$ is dense.

Upon taking adjoints, the same conclusions apply to $\rho \circledast \pi$, concluding the proof. □

If G is a finite group one may in fact prove the above to be a strict C*-alloy. In this case, one has that $C^*(G)$ is finite dimensional and this might lead one to expect C*-blends to be strict whenever one of the algebras involved is finite dimensional. However, in section 9 we shall present an example of a non strict C*-blend (A, B, i, j, X), in which B is a two-dimensional algebra.

[1] Similar conclusions may be obtained if we choose the reduced crossed product instead.

4. **The Elliott-Niu alloy** In this short section we describe, without proofs, an interesting example of a C*-algebra found by Elliott and Niu.

Given an irrational number θ, consider the unitary operators u and v on $L^2(S^1)$ defined by

$$u(\xi)|_z = z\xi(z) \quad \text{and} \quad v(\xi)|_z = \xi(e^{-2\pi i\theta}z),$$

for all $\xi \in L^2(S^1)$ and $z \in S^1$. It is well known that the C*-algebra generated by u and v is isomorphic to the irrational rotation C*-algebra A_θ. Since the latter is a crossed product of $C(S^1)$ by $\mathbb{Z}$, we may use 3.4 to obtain a blend (cf. Remark 3.3)

$$\big(C(S^1), C(S^1), A_\theta\big),$$

where the second occurence of $C(S^1)$ above is justified by the fact that it is isomorphic to $C^*(\mathbb{Z})$. Let χ be the characteristic function of the arc

$$J = \{e^{it} : 0 \leq t \leq 2\pi\theta\},$$

and let p be the operator on $L^2(S^1)$ given by

$$p(\xi)|_z = \chi(z)\xi(z),$$

so that p is just the spectral projection of u associated to the arc J.

Denote by B the closed *-algebra of operators on $L^2(S^1)$ generated by the set

$$\{u, v^k p v^{-k} : k \in \mathbb{Z}\}.$$

It is not hard to show that B is isomorphic to the algebra

$$C(S^1; \theta),$$

formed by all bounded, complex valued functions on the circle which are continuous at all points except possibly for points in the orbit of 1 under rotation by $2\pi\theta$, where the lateral limits exist.

Since B is clearly invariant under rotation by $2\pi\theta$ (conjugation by v), one may also form the crossed product $B \rtimes \mathbb{Z}$ and, as above, it is not surprising that

$$\big(B, C(S^1), B \rtimes \mathbb{Z}\big)$$

is also a blend. The interesting aspect of the Elliott-Niu example is that it extrapolates the realm of crossed products as follows.

Let q be the spectral projection of v (as opposed to u) associated to the arc J, and let C be the closed *-algebra of operators on $L^2(S^1)$ generated by the set

$$\{v, u^k q u^{-k} : k \in \mathbb{Z}\}.$$

Again it is possible to prove that C is isomorphic to $C(S^1; \theta)$.

THEOREM 4.1 (Elliott-Niu [6]). *The set*

$$X_\theta = \overline{\text{span}}\ \{bc : b \in B,\ c \in C\}$$

is a subalgebra of operators on $L^2(S^1)$*, and consequently* (B, C, X_θ) *is a C*-blend, which is in fact a C*-alloy.*

A proof of this result will hopefully appear soon.

In analogy with the highly influential isomorphism problem for irrational rotation C*-algebras ([13], [14]), one may ask:

Question. Given real numbers θ and θ', when are X_θ and $X_{\theta'}$ isomorphic?

5. **Algebraic alloys with** $B = R^2$ The examples based on crossed products above, together with the complexity and wide variety of group actions on C*-algebras, should be enough to convince the reader that the concept of a blend, in both the algebraic and the C* version, is a rather general and deep one. Understanding it in full is therefore a colossal task not likely to be accomplished in the near future.

In order to be able to say anything meaninful about blends and alloys we will make a drastic simplification by assuming that one of the algebras involved is as elementary as can possibly be, namely $B = \mathbb{C}^2$. In this case we will give a complete classification of unital strict C*-alloys.

In preparation for this we shall now study the algebraic counterpart, namely alloys in which B is the direct sum of two copies of the coefficient ring R.

As the above title suggests we shall fix, for the remainder of this section, a unital alloy (A, B, i, j, X) such that $B = R^2$, the latter viewed as an algebra with the usual coordinatewise operations. As already mentioned we will think of A and R^2 as subalgebras of X and we will refer to our alloy simply as (A, R^2, X). We will denote by p and q the standard idempotents of R^2, namely

$$p = (1, 0) \quad \text{and} \quad q = (0, 1).$$

PROPOSITION 5.1. *Every element* $c \in X$ *admits unique decompositions as*

$$c = ap + bq, \tag{5.1}$$

and

$$c = pa' + qb', \tag{5.2}$$

where $a, b, a', b' \in A$.

PROOF. This follows from the corresponding statements about decomposing an element in $A \otimes_R R^2$ as $a \otimes p + b \otimes q$, and similarly with respect to $R^2 \otimes_R A$. □

The structure of the multiplication operation on X is encoded by two crucial operators on A introduced in the following:

PROPOSITION 5.2. *There are unique linear maps* $\mathbf{E}, \mathbf{F} : A \to A$, *such that*

$$pap = \mathbf{E}(a)p \quad \text{and} \quad qaq = \mathbf{F}(a)q,$$

for all $a \in A$. *In addition, denoting by* $\mathbf{E}^{\perp} = id_A - \mathbf{E}$, *and* $\mathbf{F}^{\perp} = id_A - \mathbf{F}$, *we have that*

(i) $pa = \mathbf{E}(a)p + \mathbf{F}^{\perp}(a)q$,
(ii) $qa = \mathbf{E}^{\perp}(a)p + \mathbf{F}(a)q$,

for all $a \in A$.

PROOF. Given $a \in A$, we may use (5.1) to write

$$pa = x_a p + y_a q, \tag{5.3}$$

for a unique pair $(x_a, y_a) \in A^2$. The map

$$\mathbf{E} : a \in A \mapsto x_a \in A$$

is therefore well defined and may be easily proven to be linear. Right multiplying (5.3) by p then leads to

$$pap = x_a p = \mathbf{E}(a)p.$$

The uniqueness of $\mathbf{E}$ follows immediately from Proposition 5.1 and a similar reasoning applies to prove the existence and uniqueness of $\mathbf{F}$. As for the last part of the statement, given a in A, write

$$pa = x_a p + y_a q \quad \text{and} \quad qa = z_a p + w_a q.$$

Clearly $x_a = \mathbf{E}(a)$, and $w_a = \mathbf{F}(a)$, while

$$ap + aq = a(p+q) = a = (p+q)a = pa + qa = \big(\mathbf{E}(a) + z_a\big)p + \big(y_a + \mathbf{F}(a)\big)q.$$

Again by the uniqueness part of Proposition 5.1 we deduce that

$$z_a = a - \mathbf{E}(a) = \mathbf{E}^{\perp}(a) \quad \text{and} \quad y_a = a - \mathbf{F}(a) = \mathbf{F}^{\perp}(a).$$

This concludes the proof. □

The fact that $\mathbf{E}$ and $\mathbf{F}$ really do encode the multiplication operation on X is made clear by the following:

PROPOSITION 5.3. *Given* $c_1 = a_1 p + b_1 q$, *and* $c_2 = a_2 p + b_2 q$ *in* X, *one has that*

$$c_1 c_2 = \big(a_1 \mathbf{E}(a_2) + b_1 \mathbf{E}^{\perp}(a_2)\big)p + \big(a_1 \mathbf{F}^{\perp}(b_2) + b_1 \mathbf{F}(b_2)\big)q.$$

PROOF. Left for the reader. □

Because of the importance of these maps in describing the algebra structure of X they ought to be given a name.

Definition 5.4. We will refer to $(\mathbf{E}, \mathbf{F})$ as the *left intrinsic pair* for the alloy (A, R^2, X).

Some further important properties of $\mathbf{E}$ and $\mathbf{F}$ are studied next.

PROPOSITION 5.5. *Both* $\mathbf{E}$ *and* $\mathbf{F}$ *are idempotent operators and moreover, for every* a *and* b *in* A*, one has that*

(i) $\mathbf{E}(ab) = \mathbf{E}(a)\mathbf{E}(b) + \mathbf{F}^{\perp}(a)\mathbf{E}^{\perp}(b)$
(ii) $\mathbf{F}(ab) = \mathbf{E}^{\perp}(a)\mathbf{F}^{\perp}(b) + \mathbf{F}(a)\mathbf{F}(b)$
(iii) $\mathbf{E}^{\perp}(ab) = \mathbf{E}^{\perp}(a)\mathbf{E}(b) + \mathbf{F}(a)\mathbf{E}^{\perp}(b)$
(iv) $\mathbf{F}^{\perp}(ab) = \mathbf{E}(a)\mathbf{F}^{\perp}(b) + \mathbf{F}^{\perp}(a)\mathbf{F}(b)$.

PROOF. Given a in A, one has that

$$\mathbf{E}(a)p = pap = p(pap) = p\mathbf{E}(a)p = \mathbf{E}(\mathbf{E}(a))p,$$

whence $\mathbf{E}^2 = \mathbf{E}$, and one similarly proves that $\mathbf{F}^2 = \mathbf{F}$. In order to prove (i) notice that

$$\begin{aligned} \mathbf{E}(ab)p = (pa)bp &= \big(\mathbf{E}(a)p + \mathbf{F}^{\perp}(a)q\big)bp \\ &= \mathbf{E}(a)\mathbf{E}(b)p + \mathbf{F}^{\perp}(a)\mathbf{E}^{\perp}(b)p = \big(\mathbf{E}(a)\mathbf{E}(b) + \mathbf{F}^{\perp}(a)\mathbf{E}^{\perp}(b)\big)p, \end{aligned}$$

proving (i). The proof of (ii) follows similar lines. In order to prove (iii) we start with its right hand side:

$$\begin{aligned} \mathbf{E}^{\perp}(a)\mathbf{E}(b) + \mathbf{F}(a)\mathbf{E}^{\perp}(b) &= \big(a - \mathbf{E}(a)\big)\mathbf{E}(b) + \big(a - \mathbf{F}^{\perp}(a)\big)\mathbf{E}^{\perp}(b) \\ &= a\mathbf{E}(b) - \mathbf{E}(a)\mathbf{E}(b) + a\mathbf{E}^{\perp}(b) - \mathbf{F}^{\perp}(a)\mathbf{E}^{\perp}(b) \\ &= ab - \mathbf{E}(a)\mathbf{E}(b) - \mathbf{F}^{\perp}(a)\mathbf{E}^{\perp}(b) \overset{(i)}{=} ab - \mathbf{E}(ab) = \mathbf{E}^{\perp}(ab). \end{aligned}$$

One may similarly show that (ii) implies (iv). □

Another crucial piece of information to be extracted from (A, R^2, X) is as follows:

PROPOSITION 5.6. *The map*

$$\phi : a \in A \mapsto \mathbf{E}(a) + \mathbf{F}(a) - a \in A$$

is multiplicative.

PROOF. Given a and b in A we have

$$\begin{aligned}
\phi(ab) &= \mathbf{E}(ab) - (ab - \mathbf{F}(ab)) \\
&= \mathbf{E}(ab) - \mathbf{F}^{\perp}(ab) \overset{(5.5.i\&iv)}{=} \mathbf{E}(a)\mathbf{E}(b) + \mathbf{F}^{\perp}(a)\mathbf{E}^{\perp}(b) - \mathbf{E}(a)\mathbf{F}^{\perp}(b) - \mathbf{F}^{\perp}(a)\mathbf{F}(b) \\
&= \mathbf{E}(a)\big(\mathbf{E}(b) - \mathbf{F}^{\perp}(b)\big) + \mathbf{F}^{\perp}(a)\big(\mathbf{E}^{\perp}(b) - \mathbf{F}(b)\big) = \mathbf{E}(a)\phi(b) - \mathbf{F}^{\perp}(a)\phi(b) \\
&= \big(\mathbf{E}(a) - \mathbf{F}^{\perp}(a)\big)\phi(b) = \phi(a)\phi(b). \qquad \square
\end{aligned}$$

It is our next immediate goal to show that ϕ is indeed an automorphism of A. In order to do so we need to introduce the right-handed versions of the operators $\mathbf{E}$ and $\mathbf{F}$.

PROPOSITION 5.7. *There are unique linear maps $\mathbf{E}_*, \mathbf{F}_* : A \to A$, such that*

$$pap = p\mathbf{E}_*(a) \quad \text{and} \quad qaq = q\mathbf{F}_*(a) \quad \forall a \in A.$$

In addition,

(i) $ap = p\mathbf{E}_*(a) + q\mathbf{F}_*^{\perp}(a)$,
(ii) $aq = p\mathbf{E}_*^{\perp}(a) + q\mathbf{F}_*(a)$,

for all $a \in A$.

PROOF. By considering the opposite algebras we may think of the alloy (A^{op}, R^2, X^{op}) (since R^2 is commutative, it coincides with its oposite algebra). We may then apply 5.2 to obtain the corresponding versions of $\mathbf{E}$ and $\mathbf{F}$, which we denote by $\mathbf{E}_*$ and $\mathbf{F}_*$, respectively. The conditions in the statement are then obtained from the corresponding conditions in Proposition 5.2, once the order of the factors in all products are suitably reversed. $\square$

Definition 5.8. We shall refer to $(\mathbf{E}_*, \mathbf{F}_*)$ as the *right intrinsic pair* for the alloy (A, R^2, X).

The following result lists some relevant relations satisfied by $\mathbf{E}$ and $\mathbf{F}$ together with their right-handed versions.

PROPOSITION 5.9. *The maps $\mathbf{E}$, $\mathbf{F}$, $\mathbf{E}_*$ and $\mathbf{F}_*$ introduced above satisfy:*

(i) $\mathbf{E}\mathbf{E}_* = \mathbf{E}$,
(ii) $\mathbf{F}\mathbf{E}_* = \mathbf{E}_*$,
(iii) $\mathbf{F}\mathbf{F}_* = \mathbf{F}$,
(iv) $\mathbf{E}\mathbf{F}_* = \mathbf{F}_*$.

PROOF. Given a in A we have that

$$\mathbf{E}(a)p = pap = p\mathbf{E}_*(a) = \mathbf{E}\big(\mathbf{E}_*(a)\big)p + \mathbf{F}^{\perp}\big(\mathbf{E}_*(a)\big)q.$$

By Proposition 5.1 we then deduce that $\mathbf{E} = \mathbf{E}\mathbf{E}_*$, and

$$0 = \mathbf{F}^{\perp}\big(\mathbf{E}_*(a)\big) = \mathbf{E}_*(a) - \mathbf{F}\big(\mathbf{E}_*(a)\big),$$

whence $\mathbf{E}_* = \mathbf{F}\mathbf{E}_*$. The proofs of (iii) and (iv) are similar. $\square$

PROPOSITION 5.10. *The map ϕ introduced in Proposition 5.6 is an automorphism of A and its inverse is given by*

$$\phi_* := \mathbf{E}_* + \mathbf{F}_* - id_A.$$

PROOF. We have that

$$\begin{aligned}\phi\phi_* &= (\mathbf{E}+\mathbf{F}-id_A)(\mathbf{E}_*+\mathbf{F}_*-id_A)\\ &= \mathbf{E}\mathbf{E}_*+\mathbf{E}\mathbf{F}_*-\mathbf{E}+\mathbf{F}\mathbf{E}_*+\mathbf{F}\mathbf{F}_*-\mathbf{F}-\mathbf{E}_*-\mathbf{F}_*+id_A\\ &\overset{(5.9)}{=} \mathbf{E}+\mathbf{F}_*-\mathbf{E}+\mathbf{E}_*+\mathbf{F}-\mathbf{F}-\mathbf{E}_*-\mathbf{F}_*+id_A = id_A,\end{aligned}$$

and one may similarly prove that $\phi_*\phi = id_A$. □

Definition 5.11. The automorphism ϕ above shall be called the *intrinsic automorphism* of the alloy (A, R^2, X).

It is interesting that ϕ may be extended to the whole of X, as shown in the next:

PROPOSITION 5.12. *There exists a unique automorphism Φ of X, extending ϕ, and such that $\Phi(p) = q$, and $\Phi(q) = p$.*

PROOF. Given c in X, let $c = ap + bq$ be its unique decomposition according to Proposition 5.1 and put

$$\Phi(c) = \phi(b)p + \phi(a)q.$$

In order to show that Φ is multiplicative we first notice that

$$\phi\mathbf{E} = \mathbf{F}\phi \quad \text{and} \quad \phi\mathbf{F} = \mathbf{E}\phi, \tag{5.4}$$

as the reader may easily verify by writing $\phi = \mathbf{F} - \mathbf{E}^{\perp} = \mathbf{E} - \mathbf{F}^{\perp}$. Given c_1 and c_2 in A, write $c_i = a_ip + b_iq$, for $i = 1, 2$, so that

$$\begin{aligned}\Phi(c_1c_2) &\overset{(5.3)}{=} \phi\big(b_1\mathbf{F}(b_2) + a_1\mathbf{F}^{\perp}(b_2)\big)p + \phi\big(a_1\mathbf{E}(a_2) + b_1\mathbf{E}^{\perp}(a_2)\big)q\\ &= \Big(\phi(b_1)\mathbf{E}\big(\phi(b_2)\big) + \phi(a_1)\mathbf{E}^{\perp}\big(\phi(b_2)\big)\Big)p\\ &\quad + \Big(\phi(a_1)\mathbf{F}\big(\phi(a_2)\big) + \phi(b_1)\mathbf{F}^{\perp}\big(\phi(a_2)\big)\Big)q\\ &\overset{(5.3)}{=} \big(\phi(b_1)p + \phi(a_1)q\big)\big(\phi(b_2)p + \phi(a_2)q\big) = \Phi(c_1)\Phi(c_2).\end{aligned}$$

This shows that Φ is an endomorphism of X.

Glancing at the definition of Φ one immediately checks that it is injective and surjective and hence that Φ is indeed an automorphism. Uniqueness of Φ is also evident. □

6. **Conditional expectations** We continue enforcing the standing hypothesis set out at the beginning of the previous section, namely that (A, R^2, X) is a fixed unital algebraic alloy.

From now on we shall be interested in a concept borrowed from probability theory:

Definition 6.1. By a *conditional expectation* from X to A we shall mean a linear map

$$G : X \to A,$$

which is an A-bimodule map and which coincides with the identity on A.

PROPOSITION 6.2. *There is a one-to-one correspondence between the set of all conditional expectations G from X to A and the set of all elements h in A such that*

$$ha = \phi(a)h + \mathbf{F}^{\perp}(a) \forall a \in A. \tag{6.1}$$

The correspondence is given by $G \mapsto h = G(p)$.

PROOF. We first claim that, for every $a \in A$, we have that

$$pa = \phi(a)p + \mathbf{F}^{\perp}(a). \tag{6.2}$$

This follows from

$$pa = \mathbf{E}(a)p + \mathbf{F}^{\perp}(a)(1-p) = \big(\mathbf{E}(a) - \mathbf{F}^{\perp}(a)\big)p + \mathbf{F}^{\perp}(a) = \phi(a)p + \mathbf{F}^{\perp}(a).$$

Given a conditional expectation G, let $h = G(p)$. Then

$$ha = G(p)a = G(pa) \overset{(6.2)}{=} G\big(\phi(a)p + \mathbf{F}^{\perp}(a)\big) = \phi(a)G(p) + \mathbf{F}^{\perp}(a) = \phi(a)h + \mathbf{F}^{\perp}(a).$$

On the other hand, given any h satisfying (6.1), define

$$G : ap + bq \in X \mapsto ah + b(1-h) \in A.$$

Then, evidently G is a left-A-linear map, which coincides with the identity on A. Moreover, for any $x \in A$, we have

$$\begin{aligned}
G\big((ap+bq)x\big) &= G\big((a-b)px + bx\big) \\
&= (a-b)G(px) + bx \overset{(6.2)}{=} (a-b)G\big(\phi(x)p + \mathbf{F}^{\perp}(x)\big) + bx \\
&= (a-b)\big(\phi(x)h + \mathbf{F}^{\perp}(x)\big) + bx \overset{(6.1)}{=} (a-b)hx + bx \\
&= G(ap+bq)x.
\end{aligned}$$

Therefore G is a conditional expectation. Finally, it is clear that the correspondence $G \mapsto h$ is injective. □

Condition (6.1) is quite interesting because it relates the intrinsic map $\mathbf{F}$ to the intrinsic automorphism ϕ. Exploring this condition a bit further we obtain:

PROPOSITION 6.3. *Given any conditional expectation $G : X \to A$, let $h = G(p)$ and $k = 1 - h$. Then,*

(i) $\mathbf{E}(a) = ha + \phi(a)k$
(ii) $\mathbf{F}(a) = ka + \phi(a)h$
(iii) $\mathbf{E}^{\perp}(a) = ka - \phi(a)k$
(iv) $\mathbf{F}^{\perp}(a) = ha - \phi(a)h$

for every $a \in A$.

PROOF. Point (iv) follows immediately from (6.1). As for (ii) we have

$$\mathbf{F}(a) = a - \mathbf{F}^{\perp}(a) \overset{(iv)}{=} a - \big(ha - \phi(a)h\big) = (1-h)a + \phi(a)h = ka + \phi(a)h,$$

The proofs of (i) and (iii) follow similar lines. □

The following gives an important covariance condition relating ϕ and the images of p and q under a conditional expectation.

PROPOSITION 6.4. *Given a conditional expectation G from X to A, let $h = G(p)$, and $k = G(q) = 1 - h$. Then*

$$hka = \phi^2(a)hk \quad \forall a \in A.$$

PROOF. We initially observe that

$$\phi\mathbf{E}^{\perp} = -\mathbf{F}^{\perp}\mathbf{E}^{\perp} = \mathbf{F}^{\perp}\phi,$$

which may be easily proven by writing $\phi = \mathbf{E} - \mathbf{F}^{\perp} = \mathbf{F} - \mathbf{E}^{\perp}$. Secondly, by Proposition 6.3.ii we have that $ka = \mathbf{E}^{\perp}(a) + \phi(a)k$, for every $a \in A$. Therefore

$$\begin{aligned} hka &= h\big(\mathbf{E}^{\perp}(a) + \phi(a)k\big) \overset{(6.1)}{=} \phi\big(\mathbf{E}^{\perp}(a)\big)h + \mathbf{F}^{\perp}\big(\mathbf{E}^{\perp}(a)\big) + \phi^2(a)hk + \mathbf{F}^{\perp}\big(\phi(a)\big)k \\ &= -\mathbf{F}^{\perp}\big(\mathbf{E}^{\perp}(a)\big)h + \mathbf{F}^{\perp}\big(\mathbf{E}^{\perp}(a)\big) + \phi^2(a)hk - \mathbf{F}^{\perp}\big(\mathbf{E}^{\perp}(a)\big)k \\ &= -\mathbf{F}^{\perp}(\mathbf{E}^{\perp}(a))\big(h + k\big) + \mathbf{F}^{\perp}\big(\mathbf{E}^{\perp}(a)\big) + \phi^2(a)hk = \phi^2(a)hk. \end{aligned}$$ □

In the presence of the automorphism Φ of Proposition 5.12, it is interesting to characterize the conditional expectations which commute with Φ.

PROPOSITION 6.5. *Let $G : X \to A$ be a conditional expectation and let $h = G(p)$. Then the following are equivalent*

(i) $G\Phi = \Phi G$,
(ii) $\phi(h) = 1 - h$.

PROOF. Assuming (i) we have

$$\phi(h) = \Phi(h) = \Phi(G(p)) = G(\Phi(p)) = G(q) = G(1-p) = 1-h.$$

Conversely, if (ii) is known to hold, given any $c \in X$, write $c = ap+bq$, according to Proposition 5.1 and notice that

$$\begin{aligned} G\big(\Phi(c)\big) &= G\big(\phi(b)p + \phi(a)q\big) = \phi(b)h + \phi(a)(1-h) \\ &= \phi(ah + b(1-h)) = \phi(G(c)) = \Phi(G(c)). \end{aligned}$$

□

Given the relevance of conditional expectations satisfying the above conditions we make the following:

Definition 6.6. A conditional expectation $G : X \to A$ satisfying the equivalent conditions of Proposition 6.5 will be called *covariant*.

An interesting question is whether or not there exists a covariant conditional expectation. In the following sections we will give an affirmative answer in the context of C*-algebras.

7. Strict C*-alloys with $B = \mathbb{C}^2$ From this point on we shall soup up the working hypothesis of the previous two sections by assuming that we are given a unital strict C*-alloy (A, B, i, j, X) in which $B = \mathbb{C}^2$.

Since the present situation is a special case of the situation treated above we may use all of the results so far obtained, including the existence of the intrinsic maps.

Among the consequences of our strengthened standing hypothesis notice that the projections p and q are now self-adjoint.

As before we will refer to our alloy using the simplified notation $(A, \mathbb{C}^2, X)$.

As seen in Proposition 5.3, the left intrinsic pair encodes the multiplication operation on X. In the present case we will now show that the star operation may also be recovered from the left intrinsic pair.

PROPOSITION 7.1. *Given $a, b \in A$, let $c = ap + bq$. Then*

$$c^* = \big(\mathbf{E}(a^*) + \mathbf{E}^{\perp}(b^*)\big)p + \big(\mathbf{F}^{\perp}(a^*) + \mathbf{F}(b^*)\big)q.$$

PROOF. Left for the reader. □

Let us now study the continuity of the intrinsic maps.

PROPOSITION 7.2. *$\mathbf{E}$, $\mathbf{F}$ and ϕ are bounded linear maps.*

PROOF. We first prove the continuity of $\mathbf{E}$ using the Closed Graph Theorem. For this, assume that $\{a_n\}_n$ is a sequence of elements in A, converging to some $a \in A$, and such that $\mathbf{E}(a_n) \to b$. Then

$$bp = \lim_{n\to\infty} \mathbf{E}(a_n)p = \lim_{n\to\infty} pa_np = pap = \mathbf{E}(a)p,$$

so, $b = \mathbf{E}(a)$, and the continuity of $\mathbf{E}$ is established. Similarly one proves that $\mathbf{F}$ is continuous and hence so if ϕ. □

Other important properties relating the intrinsic maps and the metric structure of the C*-algebras involved are discussed next.

PROPOSITION 7.3. *There exists a constant $K > 0$, such that, for all a in A, one has that*

(i) $\|ap\| \geq K\|a\|$,
(ii) $\|aq\| \geq K\|a\|$,
(iii) $\|\mathbf{E}(a^*a)\| \geq K^2\|a\|^2$,
(iv) $\|\mathbf{F}(a^*a)\| \geq K^2\|a\|^2$.

PROOF. Observing that $X = Ap + Aq$, one concludes that

$$Ap = Xp = \{c \in X : cq = 0\}.$$

This implies that Ap is closed in X, and hence that it is a Banach space. Since the map

$$\lambda : a \in A \mapsto ap \in Ap,$$

is a continuous bijection, the Open Mapping Theorem implies that it is bicontinuous. The constant $K = \|\lambda^{-1}\|^{-1}$ may then be shown to satisfy (i).

A similar constant can be chosen satisfying (ii) and so we may rename K to be the smallest of the two, and both (i) and (ii) will be satisfied with the same constant K.

Next notice that

$$\|\mathbf{E}(a^*a)\| \geq \|\mathbf{E}(a^*a)p\| = \|pa^*ap\| = \|(ap)^*ap\| = \|ap\|^2 \geq K^2\|a\|^2,$$

proving (iii), while (iv) follows by a similar reasoning. □

When dealing with C*-algebras, conditional expectations are always required to be positive. When a conditional expectation G also satisfies

$$G(xx^*) = 0 \Rightarrow \ x = 0,$$

we say that G is *faithful*.

The following will be helpful later.

LEMMA 7.4. *Given a faithful conditional expectation $G : X \to A$, let*

$$h = G(p) \quad \text{and} \quad k = G(q) = 1 - h.$$

Then, for every $a \in A$, either $ah = 0$, or $ak = 0$, imply that $a = 0$.

PROOF. Assuming that $ah = 0$, notice that

$$G\big((ap)(ap)^*\big) = G(apa^*) = aG(p)a^* = aha^* = 0,$$

so $ap = 0$, and hence Proposition 5.1 applies to give $a = 0$. The same conclusion may similarly be obtained if we assume that $ak = 0$. □

Let us now prove some useful properties of covariant conditional expectations. Recall from Definition 6.6 that a conditional expectation is said to be covariant when $\phi\big(G(p)\big) = 1 - G(p)$.

COROLLARY 7.5. *For any covariant conditional expectation $G : X \to A$, one has that*

(i) *$G(p)$ is invertible,*
(ii) *$1 - G(p)$ is invertible,*
(iii) *G satisfies the Pimsner–Popa finite index condition [10],*
(iv) *G is faithful.*

PROOF. Let $h = G(p)$. Since G is positive we have that h is positive as well, and so the spectrum of h, here denoted $\sigma(h)$, is contained in the interval $[0, +\infty)$. Arguing by contradiction, suppose that $0 \in \sigma(h)$. Consider the real valued functions

$$f_n : [0, +\infty) \to \mathbb{R}$$

defined by

$$f_n(t) = \begin{cases} 1 - nt, & \text{if } t \leq 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

It is then easy to see that the element $a_n := f_n(h)$ is positive and satisfies

$$\|a_n\| = 1 \quad \text{and} \quad \|a_n h\| \leq \frac{1}{4n}.$$

We then have

$$\mathbf{E}(a_n) \overset{(6.3.i)}{=} ha_n + \phi(a_n)(1 - h) \overset{(6.5.ii)}{=} ha_n + \phi(a_n h),$$

whence

$$\|\mathbf{E}(a_n)\| \leq \|ha_n\| + \|\phi\|\|a_n h\| \leq \frac{1 + \|\phi\|}{4n}.$$

However, plugging $a = a_n^{1/2}$ in Proposition 7.3.ii leads to

$$\|\mathbf{E}(a_n)\| \geq K^2 \|a_n^{1/2}\|^2 = K^2 \|a_n\| = K^2,$$

bringing about a contradiction. It follows that $0 \notin \sigma(h)$, proving (i). Similarly one proves (ii).

In order to prove (iii), observe that $1 - h = G(q)$, and hence $1 - h$ is also positive. Being invertible, we have that

$$1 - h \geq \alpha > 0,$$

for some real number α. Since the same analysis applies to h, we may also assume that $h \geq \alpha$.

Given $c = ap + bq \in X$, we then have

$$\begin{aligned} G(cc^*) &= G\big((ap+bq)(pa^*+qb^*)\big) = G(apa^*+bqb^*) \\ &= aha^* + b(1-h)b^* \geq \alpha(aa^*+bb^*) \geq \alpha(apa^*+bqb^*) = \alpha cc^*. \end{aligned}$$

This proves (iii), and (iv) is then an obvious consequence. □

Let us now discuss the consequences of left-right symmety imposed by the existence of the star operation.

Definition 7.6. If f is any linear map between two *-algebras, we let f^* be given by

$$f^*(x) = f(x^*)^*.$$

Evidently f^* is also linear. In addition f^* coincides with f if and only if f is *-preserving.

PROPOSITION 7.7. *The components of the left and right intrinsic pairs and the intrinsic automorphisms satisfy:*

(i) $\mathbf{E}^* = \mathbf{E}_*$,
(ii) $\mathbf{F}^* = \mathbf{F}_*$,
(iii) $\phi^* = \phi^{-1}$,
(iv) $\Phi^* = \Phi^{-1}$.

PROOF. Given a in A, observe that

$$p\mathbf{E}_*(a) = pap = (pa^*p)^* = \big(\mathbf{E}(a^*)p\big)^* = p\mathbf{E}(a^*)^* = p\mathbf{E}^*(a),$$

so we deduce from Proposition 5.1 that $\mathbf{E}_*(a) = \mathbf{E}^*(a)$. The proof that $\mathbf{F}_*(a) = \mathbf{F}^*(a)$ is similar. With respect to (iii), recall from 5.10 that

$$\phi^{-1} = \phi_* = \mathbf{E}_* + \mathbf{F}_* - id_A = \mathbf{E}^* + \mathbf{F}^* - id_A = \phi^*.$$

Addressing the last point, observe that, given $a, b \in A$,

$$\Phi^*(ap+bq) = \big(\Phi(pa^*+qb^*)\big)^* = \big(q\phi(a^*) + p\phi(b^*)\big)^* = \phi^*(a)q + \phi^*(b)p,$$

from where it is evident that $\Phi\Phi^*$ is the identity map, hence concluding the proof. □

Recall from [9, 4.1] that if ρ is an automorphism of a C*-algebra, its *dual* ρ' is defined by

$$\rho' = (\rho^*)^{-1}.$$

Clearly ρ is a *-automorphism if and only if $\rho'\rho = id$.

According to [9, 6.1], ρ is said to be self-dual if $\rho' = \rho$. If, in addition, its spectrum[2] consists of non-negative real numbers, one says that ρ is positive.

For the convenience of the reader we state below a slight variant of the "polar decomposition for isomorphisms" from [9], which we shall use in the sequell.

[2]Here ρ is viewed simply as a bounded linear transformation and hence one may speak of its spectum in the usual way.

THEOREM 7.8. [9, 7.1] *Any automorphism ρ of a C*-algebra is written uniquely as $\rho = \pi\gamma$, where π is a *-automorphism and γ is a positive automorphism.*

An application of these ideas gives us some important information.

PROPOSITION 7.9. *The automorphism Φ introduced in Proposition 5.12 is self-dual. In addition, if*

$$\Phi = \Pi\,\Gamma$$

is the polar decomposition of Φ, then Π is an involution commuting with Γ.

PROOF. Since $\Phi^{-1} = \Phi^*$ by Proposition 7.7.iv, we deduce that $\Phi' = \Phi$. We then have that

$$\Phi = \Phi' = (\Pi\,\Gamma)' = \Gamma'\Pi' = \Gamma\,\Pi^{-1} = \Pi^{-1}(\Pi\,\Gamma\,\Pi^{-1}).$$

It is evident that Π^{-1} is a *-automorphism and that $\Pi\,\Gamma\,\Pi^{-1}$ is positive. By the uniqueness part of [9, 7.1] we conclude that $\Pi^{-1} = \Pi$, and $\Pi\,\Gamma\,\Pi^{-1} = \Gamma$, concluding the proof. □

By construction (see Proposition 5.12) we have that Φ preserves A and interchanges p and q. These properties are reflected in the components of the polar decomposition as follows:

PROPOSITION 7.10.

(i) $\Gamma(A) = A = \Pi(A)$,
(ii) $\Gamma(p) = p$, *and* $\Gamma(q) = q$,
(iii) $\Pi(p) = q$, *and* $\Pi(q) = p$.

PROOF. By the discussion following [9, 6.3], one has that $\Phi^2 = \Phi'\Phi$ is positive. Moreover Γ is the "square root" of Φ^2 in the sense of the analytical functional calculus, where by square root we mean the principal branch of the complex square root function defined on the open right half-plane.

Using Runge's Theorem we may find a sequence $\{f_n\}_n$ of polynomials converging uniformly to the above mentioned square root function over some open neighborhood of the spectrum of Φ^2. It follows that

$$\Gamma = \sqrt{\Phi^2} = \lim_{n\to\infty} f_n(\Phi^2). \tag{7.1}$$

Since $\Phi^2(A) = A$, we then conclude that $\Gamma(A) \subseteq A$.

Observing that $\Gamma^{-1} = \sqrt{\Phi^{-2}}$, the same reasoning above gives $\Gamma^{-1}(A) \subseteq A$, hence proving that $\Gamma(A) = A$. To conclude the proof of (i) it is now enough to observe that

$$\Pi(A) = \Phi\big(\Gamma^{-1}(A)\big) = \Phi(A) = A.$$

Since Φ interchanges p and q, as already observed, one has that Φ^2 fixes p and q. Using (7.1) we then have that

$$\Gamma(p) = \lim_{n\to\infty} f_n(\Phi^2)(p) = \lim_{n\to\infty} f_n(1)\; p = p,$$

and likewise $\Gamma(q) = q$. Consequently

$$\Pi(p) = \Phi\big(\Gamma^{-1}(p)\big) = \Phi(p) = q,$$

and similarly $\Pi(q) = p$. □

Since ϕ is the restriction of Φ to A, we may easily obtain the polar decomposition of the former, knowing that of the latter:

PROPOSITION 7.11. *Let π and γ be the restrictions of Π and Γ to A, respectively. Then π and γ are automorphisms of A and*

$$\phi = \pi\gamma$$

is the polar decomposition of ϕ. Moreover π is an involution commuting with γ.

PROOF. By Proposition 7.10.i we have that A is invariant under Π and Γ and hence π and γ are indeed automorphisms of A. By [9, 6.3] we have that γ is positive and it is evident that π is a *-automorphism. Since it is clear that $\phi = \pi\gamma$, the uniqueness of the polar decomposition [9, 7.1] warrants that to be the polar decomposition of ϕ. The last part of the statement follows from Proposition 7.9. □

We may now prove the existence of covariant conditional expectations.

THEOREM 7.12. *Suppose that there exists a faithful conditional expectation $\hat{G} : X \to A$. Then there exists another faithful conditional expectation G such that, setting $h = G(p)$, one has*

(i) $\pi(h) = 1 - h$,
(ii) $\phi(h) = 1 - h$,

and therefore G is covariant.

PROOF. Since Π is an involutive *-automorphism of X preserving A, it is clear that $\Pi\hat{G}\Pi$ is another conditional expectation onto A, and so is the map G defined by

$$G(c) = \frac{\hat{G}(c) + \Pi\hat{G}\Pi(c)}{2} \quad \forall c \in X.$$

It is easy to see that G is also faithful, since, for all $c \in X$, one has that

$$0 \leq \hat{G}(c^*c) \leq 2\,G(c^*c).$$

Letting

$$h := G(p) \overset{(7.10.iii)}{=} \frac{\hat{G}(p) + \pi\big(\hat{G}(q)\big)}{2},$$

we claim that $\pi(h) = 1 - h$. In order to prove it notice that

$$\begin{aligned}\pi(h) &= \frac{\pi\big(\hat{G}(p)\big) + \hat{G}(q)}{2} = \frac{\pi\big(\hat{G}(1-q)\big) + \hat{G}(1-p)}{2} \\ &= 1 - \frac{\pi\big(\hat{G}(q)\big) + \hat{G}(p)}{2} = 1 - h,\end{aligned}$$

proving (i). We next claim that $\phi^2(h) = h$. Indeed, if $k = 1-h$, then Proposition 6.4 applies and hence

$$hkh = \phi^2(h)hk,$$

which says that

$$\big(h - \phi^2(h)\big)hk = 0,$$

so the claim follows from (7.4).

Evidently we also have that $\Phi^2(h) = h$ so, if $\{f_n\}_n$ is the sequence of polynomials employed in the proof of (7.10), we obtain

$$\Gamma(h) = \lim_{n\to\infty} f_n(\Phi^2)(h) = \lim_{n\to\infty} f_n(1)\ h = h.$$

It follows that

$$\phi(h) = \Phi(h) = \Pi\big(\Gamma(h)\big) = \Pi(h) = \pi(h) = 1 - h.$$

The last statement follows from (6.5). □

The existence of covariant conditional expectations allows us to prove that the automorphism γ of Proposition 7.11 is inner, as follows:

PROPOSITION 7.13. *Let G be a covariant conditional expectation and let $h = G(p)$ and $k = 1 - h$. Then*

$$\gamma(a) = (hk)^{1/2}a(hk)^{-1/2} \quad \forall a \in A.$$

PROOF. Since both h and k are invertible by Corollary 7.5 we may use Proposition 6.4 to write

$$(hk)a(hk)^{-1} = \phi^2(a) = \phi'\phi(a) = \gamma^2(a),$$

where we have used that the intrinsic automorphism ϕ is self-dual. The conclusion then follows from [9, 6.6]. □

We now summarize our main results so far.

THEOREM 7.14. *Let $(A, \mathbb{C}^2, X)$ be a unital strict C^*-alloy such that there exists a faithful conditional expectation $\hat{G} : X \to A$. Then there exists a $*$-automorphism π of A, and a positive invertible element h in A, such that*

(i) $\pi^2 = id$,
(ii) $\pi(h) = 1 - h$.

Moreover, letting $k = 1 - h$, one has that the intrinsic automorphism and the components of the left intrinsic pair are entirely determined in terms of π and h by

$$\begin{aligned} \phi(a) &= \pi\big((hk)^{1/2} a (hk)^{-1/2}\big) = (hk)^{1/2}\ \pi(a)\ (hk)^{-1/2}, \\ \mathbf{E}(a) &= ha + \phi(a)k, \\ \mathbf{F}(a) &= ka + \phi(a)h, \end{aligned}$$

for all a in A. Finally, the map

$$G : ap + bq \in X \mapsto ah + bk \in A \tag{7.2}$$

is a faithful covariant conditional expectation satisfying the Pimsner–Popa finite index condition.

Proof. Let G be the faithful covariant conditional expectation given by Theorem 7.12, let $h = G(p)$, and let $k = 1 - h = G(q)$. Therefore

$$G(ap + bq) = ah + bk \quad \forall a, b \in A,$$

and the last statement is then a consequence of Corollary 7.5.

Since p is positive and G preserves positivity, it is clear that h is positive. By Corollary 7.5 we have that h and k are invertible.

Letting π be as in Proposition 7.11, one sees that (i) follows from the last statement in (7.11), while (ii) is the content of Proposition 7.12.i.

The first formula above for ϕ is a consequence of Propositions 7.11 and 7.13, while the second one follows from the fact that hk is a fixed point for π, as one may easily verify using (ii). The expressions for $\mathbf{E}$ and $\mathbf{F}$ above in turn follow from Proposition 6.3.i–ii. □

The relevance of the above result is that the whole algebraic and analytical structure of X may be recovered from π and h. This is because, once the intrinsic pair is known, we may use Propositions 5.3 and 7.1 to recover both the multiplication and star operations on X. The norm may also be recovered since the norm on any C*-algebra is encoded in its *-algebraic structure: the norm of an element x coincides with the square root of the spectral radius of x^*x. It is therefore meaningful to give the folowing:

Definition 7.15. We will say that the pair (π, h) is the *fundamental data* for the alloy $(A, \mathbb{C}^2, X)$.

8. **Strict C*-alloys and crossed products by $\mathbb{Z}_2$** As seen in the previous section, the whole structure of a unital strict C*-alloy may be recovered from its fundamental data. A natural question arising from this is whether or not one may construct a C*-alloy from a pair (π, h), where π is a *-automorphism of a unital C*-algebra A, and h is a positive invertible element in A satisfying Proposition 7.14.i-ii.

This may in fact be done by first *defining* operators $\mathbf{E}$ and $\mathbf{F}$ on A using the formulas provided by Proposition 7.14. On the vector space $X = A \oplus A$, where a pair (a, b) is formally denoted $ap + bq$, we may then introduce a *-algebra structure by employing formulas in Propositions 5.3 and 7.1. It may then be shown that $(A, \mathbb{C}^2, i, j, X)$ is a strict C*-alloy, where

$$i : a \in A \mapsto (a, a) \in X \quad \text{and} \quad j : (\lambda, \mu) \in \mathbb{C}^2 \mapsto (\lambda, \mu) \in X.$$

The situation is however even more interesting in the sense that π alone provides enough information to construct X, while h is necessary only in order to locate p and q within X. In order to explain this in detail let us first quickly analyze the C*-alloy arising from an action of $\mathbb{Z}_2$.

If A is a unital C*-algebra and π is an involutive *-automorphism of A we may form an action of $\mathbb{Z}_2$ on A by mapping the generator of $\mathbb{Z}_2$ to π. We may therefore consider the crossed product algebra

$$X = A \rtimes_\pi \mathbb{Z}_2.$$

The unitary element implementing the action of π, here denoted by ϖ, is clearly a self-adjoint unitary. Therefore, defining

$$p = \frac{1 + \varpi}{2} \quad \text{and} \quad q = \frac{1 - \varpi}{2},$$

we have that p and q are complementary projections which therefore generate a copy of $\mathbb{C}^2$ within X. Thinking of A as a subalgebra of X in the usual way, observe that, for every a in A, one has that

$$\begin{aligned} pap &= \frac{1}{4}(1 + \varpi)a(1 + \varpi) = \frac{1}{4}(a + \varpi a + a\varpi + \varpi a \varpi) \\ &= \frac{1}{4}\big(a + \pi(a)\varpi + a\varpi + \pi(a)\big) = \frac{a + \pi(a)}{2}\,\frac{1 + \varpi}{2} = \mathbf{E}(a)p, \end{aligned}$$

where $\mathbf{E}(a)$ is defined to be $(a + \pi(a))/2$. Routine calculations show, in fact, that $(A, \mathbb{C}^2, X)$ is a strict C*-alloy with left intrinsic pair $(\mathbf{E}, \mathbf{E})$, meaning that $\mathbf{F} = \mathbf{E}$. The intrinsic automorphism, namely $\phi = \mathbf{E} + \mathbf{F} - id$, therefore coincides with π.

Recall that the usual conditional expectation $G : X \to A$ is given by

$$G(a + b\varpi) = a \quad \forall a, b \in A.$$

The elements h and k, which played crucial roles above, are therefore given by

$$h = G(p) = \frac{1}{2} \quad \text{and} \quad k = 1 - h = \frac{1}{2}.$$

We have thus proven:

PROPOSITION 8.1. *If π is an involutive *-automorphism of a unital C*-algebra A, and $X = A \rtimes_\pi \mathbb{Z}_2$, then $(A, \mathbb{C}^2, X)$ is a strict C*-alloy. Moreover its intrinsic automorphism coincides with π, and $h = 1/2$.*

In our next result we shall prove that, under suitable conditions, the above example is essentially the only one.

THEOREM 8.2. *Let $(A, \mathbb{C}^2, X)$ be a unital strict C*-alloy. Suppose moreover that there exists a faithful conditional expectation $\hat{G} : X \to A$, and let (π, h) be provided by Proposition 7.14. Then there exists a *-isomorphism*

$$\rho : A \rtimes_\pi \mathbb{Z}_2 \to X,$$

coinciding with the identity on A, and such that

$$\rho(\varpi) = (hk)^{-1/2}(kp - hq),$$

and

$$\rho\big((hk)^{1/2}\varpi + h\big) = p,$$

where ϖ is the unitary element implementing the action of π, and $k = 1 - h$.

PROOF. By direct computation, using the expression for ϕ given in Proposition 7.14, one may show that

$$\phi(h) = k \quad \text{and} \quad \phi(k) = h. \tag{8.1}$$

From now on the proof will consist of a series of claims, starting with:

Claim 1. $\mathbf{E}(hk) = hk = \mathbf{F}(hk)$, and $\mathbf{E}^\perp(hk) = 0 = \mathbf{F}^\perp(hk)$.

In order to verify this we compute

$$\mathbf{E}(hk) \overset{(6.3.i)}{=} h(hk) + \phi(hk)k \overset{(8.1)}{=} hhk + khk = (h + k)hk = hk,$$

while

$$\mathbf{F}(hk) \overset{(6.3.ii)}{=} k(hk) + \phi(hk)h \overset{(8.1)}{=} khk + khh = kh(k + h) = hk.$$

Claim 2. $\mathbf{E}(h^{-1}k) = 1 = \mathbf{F}(k^{-1}h)$.

We have

$$\mathbf{E}(h^{-1}k) \overset{(6.3.i)}{=} h(h^{-1}k) + \phi(h^{-1}k)k \overset{(8.1)}{=} h(h^{-1}k) + (k^{-1}h)k = k + h = 1,$$

while

$$\mathbf{F}(k^{-1}h) \overset{(6.3.ii)}{=} k(k^{-1}h) + \phi(k^{-1}h)h \overset{(8.1)}{=} k(k^{-1}h) + (h^{-1}k)h = h + k = 1.$$

We will now introduce the element of X which will correspond to the implementing unitary in the crossed product. Let

$$u = (hk)^{-1/2}(kp - hq).$$

It is our next immediate goal to show that u is a self-adjoint unitary. We begin with the following:

Claim 3. u is an isometry.

We have

$$\begin{aligned} u^*u &= (kp - hq)^*(hk)^{-1}(kp - hq) = (pk - qh)(hk)^{-1}(kp - hq) \\ &= (ph^{-1} - qk^{-1})(kp - hq) = ph^{-1}kp + qk^{-1}hq \\ &= \mathbf{E}(h^{-1}k)p + \mathbf{F}(k^{-1}h)q = p + q = 1. \end{aligned}$$

Claim 4. hk commutes with p and q.

Observe that

$$phk \overset{(5.2.i)}{=} \mathbf{E}(hk)p + \mathbf{F}^{\perp}(hk)q = hkp.$$

Since $q = 1 - p$, it is clear that hk also commutes with q.

Our next claim refers to the second factor of u, namely the element defined by

$$v = kp - hq.$$

Claim 5. One has that $v = p - h$, and hence v is self-adjoint.

Notice that

$$v = kp - hq = (1 - h)p - hq = p - hp - hq = p - h(p + q) = p - h.$$

Since both p and h are self-adjoint, the claim follows.

Claim 6. u is self-adjoint.

Since hk commutes with p and q, it is clear that hk also commutes with v. Therefore $(hk)^{-1/2}$ commutes with v. The claim is then a consequence of the obvious fact that the product of two commuting self-adjoint elements is again self-adjoint.

Claim 7. u is unitary.

We have already seen that $u^*u = 1$, so the result follows from the fact that $u = u^*$.

Claim 8. For every a in A one has that $\pi(a) = uau^{-1}$.

Recalling from Claim 5 that $v = p - h$, one has

$$va = pa - ha \overset{(6.1\ \&6.2)}{=} \phi(a)p + \mathbf{F}^{\perp}(a) - \phi(a)h - \mathbf{F}^{\perp}(a) = \phi(a)(p-h) = \phi(a)v.$$

Since $v = (hk)^{1/2}u$, one sees that v is invertible, and so

$$\phi(a) = vav^{-1}.$$

Consequently

$$\pi(a) = (hk)^{-1/2}\phi(a)(hk)^{1/2} = (hk)^{-1/2}vav^{-1}(hk)^{1/2} = uau^{-1}.$$

By the universal property of crossed products there is a *-homomorphism

$$\rho : A \rtimes_\pi \mathbb{Z}_2 \to X,$$

extending the identity map on A, and sending the implementing unitary ϖ to u.

We then have that

$$\rho\big((hk)^{1/2}\varpi + h\big) = (hk)^{1/2}\rho(\varpi) + h = (hk)^{1/2}u + h = v + h = p.$$

It now remains to show that ρ is bijective. By the computation above we see that p is in the range of ρ, and this in turn shows that ρ is onto.

Let G be the faithful conditional expectation provided by Proposition 7.14. Then $G(p) = h$, and hence

$$G(u) = G\big((hk)^{-1/2}(p-h)\big) = (hk)^{-1/2}\big(G(p) - h\big) = 0.$$

If H is the standard conditional expectation from $A \rtimes_\pi \mathbb{Z}_2$ to A, namely that which is given by

$$H(a + b\varpi) = a \quad \forall a, b \in A,$$

we claim that $G\rho = \rho H$. In fact, we have

$$G\big(\rho(a + b\varpi)\big) = G(a + bu) = a + bG(u) = a = \rho\big(H(a + b\varpi)\big).$$

In order to prove that ρ is one-to-one, assume that $x \in A \rtimes_\pi \mathbb{Z}_2$ is such that $\rho(x) = 0$. Then

$$0 = G\big(\rho(xx^*)\big) = \rho\big(H(xx^*)\big).$$

Since $H(xx^*)$ lies in A, and since ρ coincides with the identity on A, and hence is injective there, we deduce that $H(xx^*) = 0$. Finally, since H is faithful, we conclude that $x = 0$, thus proving that ρ is injective. □

9. **A non strict finite dimensional example** When dealing with tensor products, one is used to believing that most analytical problems disappear provided one of the factors is finite dimensional.

Contrary to such expectations, in this section we shall give an example of a unital C*-blend (A, B, X), with B being a finite dimensional algebra, which is not strict. In particular, upon identifying $A \otimes B$ with a subspace of X, we may view $A \otimes B$ as a normed space which will turn out not to be complete, regardless of the fact that B is finite dimensional.

In order to prepare for the construction of our counterexample we first consider the following elementary example: let X denote the C*-algebra formed by all 2×2 complex matrices. Given any real number r in the open interval $(0, 1)$, consider the element of X given by

$$p(r) = \begin{bmatrix} r & \sqrt{r - r^2} \\ \sqrt{r - r^2} & 1 - r \end{bmatrix},$$

which is easily seen to be a projection. We shall also consider its complementary projection

$$q(r) = 1 - p(r).$$

We shall now describe two subalgebras of X which will form a blend of C*-algebras. On the one hand we will let A be the subalgebra formed by all diagonal matrices and, on the other, B will be the subalgebra generated by $p(r)$ and $q(r)$. Clearly B is isomorphic to $\mathbb{C}^2$, so we will identify B and $\mathbb{C}^2$ from now on (however we will not give much attention to the fact that A is also isomorphic to $\mathbb{C}^2$).

We leave it for the reader to prove that the triple (A, B, X), henceforth also referred to as $(A, \mathbb{C}^2, X)$, is a C*-blend. It is also easy to see that, if $a = \begin{bmatrix} x & \\ & y \end{bmatrix} \in A$, then

$$\mathbf{E}(a) = \big(rx + (1 - r)y\big)I_2 \quad \text{and} \quad \mathbf{F}(a) = \big((1 - r)x + ry\big)I_2, \tag{9.1}$$

where I_2 is the identity 2×2 matrix. Consider the C*-algebra

$$X^\infty := \ell_\infty(X)$$

formed by all bounded sequences of elements in X, under pointwise operations. It is evident that

$$A^\infty := \ell_\infty(A)$$

sits as a subalgebra of X^∞.

PROPOSITION 9.1. *Given any sequence $\{r_n\}_n$ of real numbers in the open interval $(0, 1)$, let p and q be the elements of X^∞ given by*

$$P = \big(p(r_n)\big)_{n \in \mathbb{N}} \quad \text{and} \quad Q = 1 - P.$$

*Then $A^\infty P + A^\infty Q$ is a *-subalgebra of X^∞, containing A^∞.*

PROOF. Throughout this proof we will write p_n for $p(r_n)$. Moreover, when r is replaced by r_n, the maps described in 9.1 will be written $\mathbf{E}_n$ and $\mathbf{F}_n$, respectively.

We first claim that

$$PA^\infty \subseteq A^\infty P + A^\infty Q.$$

In order to see this, let $a \in A^\infty$. Then

$$Pa = (p_n a_n)_n = (\mathbf{E}_n(a_n)p_n + \mathbf{F}_n^\perp(a_n)q_n)_n = (\mathbf{E}_n(a_n))_n P + (\mathbf{F}_n^\perp(a_n))_n Q.$$

Noticing that the $\mathbf{E}_n$ and the $\mathbf{F}_n$ are uniformly bounded, the above calculation implies the claim.

It is clear that A^∞ is contained in $A^\infty P + A^\infty Q$, and hence also

$$QA^\infty = (1-P)A^\infty \subseteq A^\infty + PA^\infty \subseteq A^\infty P + A^\infty Q.$$

It follows that

$$PA^\infty + QA^\infty \subseteq A^\infty P + A^\infty Q.$$

Taking the adjoint on both sides above, we deduce the reverse inclusion and hence that

$$PA^\infty + QA^\infty = A^\infty P + A^\infty Q.$$

In particular it follows that $A^\infty P + A^\infty Q$ is a self-adjoint set and it is now easy to prove all remaining assertions. □

Based on the above result, the closure of $A^\infty P + A^\infty Q$, here denoted by

$$\widetilde{X} = \overline{A^\infty P + A^\infty Q},$$

is seen to be a C*-algebra and consequently

$$(A^\infty, \mathbb{C}^2, \widetilde{X}) \tag{9.2}$$

is a C*-blend, where we identify $\mathbb{C}^2$ with the subalgebra of $\widetilde{X}$ spanned by P and Q.

The main question we wish to address here is related to whether or not $A^\infty P + A^\infty Q$ is closed. In fact we wish to prove that this is not the case when the r_n tend to zero.

PROPOSITION 9.2. *Let $\{r_n\}_n$ be a sequence in $(0,1)$ such that*

$$\lim_{n\to\infty} r_n = 0.$$

Then $A^\infty P + A^\infty Q$ is not closed in $\widetilde{X}$ and hence $(A^\infty, \mathbb{C}^2, \widetilde{X})$ is a C-blend which is not strict.*

PROOF. We shall suppose, by way of contradiction, that $A^\infty P + A^\infty Q$ is closed, and hence that $\widetilde{X} = A^\infty P + A^\infty Q$.

From this it follows that

$$A^\infty P = \widetilde{X}P = \{x \in \widetilde{X} : xQ = 0\},$$

which is therefore closed in $\widetilde{X}$. The map

$$a \in A^\infty \to aP \in A^\infty P$$

is clearly continuous and bijective, and hence, by the Open Mapping Theorem, we deduce that it is bounded below, that is, there exists a constant $K > 0$, such that

$$\|aP\| \geq K\|a\| \quad \forall a \in A^\infty.$$

Given a positive integer m, let $a = (a_n)_n$ be the element of A^∞ given by

$$a_n = \delta_{n,m} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}.$$

Then aP has a single nonzero coordinate in the m^{th} position, and that coordinate is given by

$$a_m p_m = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \begin{bmatrix} r_m & \sqrt{r_m - r_m^2} \\ \sqrt{r_m - r_m^2} & 1 - r_m \end{bmatrix} = \begin{bmatrix} r_m & \sqrt{r_m - r_m^2} \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$0 < K = K\|a\| \leq \|aP\| = \|a_m p_m\| = \sqrt{r_m} \overset{m\to\infty}{\longrightarrow} 0,$$

a contradiction. □

10. **C*-blends and commuting squares** Let us now explore a different class of C*-blends, not necessarily coming from group actions. The initial data we shall use in the construction to be described in this section is a commuting square [11], by which we mean that we are given C*-algebras A, B, C and D, such that

$$\begin{array}{ccc} A & \supset & B \\ \cup & & \cup \\ C & \supset & D, \end{array}$$

and $D = B \cap C$. One is moreover given conditional expectations

$$E : A \to B \quad \text{and} \quad F : A \to C,$$

satisfying $EF = FE$. Evidently one then has that $G := EF$ is a conditional expectation onto D.

Let us now quickly describe Watatani's version [15] of the celebrated Jones' basic construction relative to G. One first introduces a D-valued inner product on A by the formula

$$\langle a, b\rangle = G(a^*b), \quad \forall a, b \in A. \tag{10.1}$$

The completion of A relative to the norm arising from this inner product is a right Hilbert D-module, which we shall denote by M. The left action of A on itself may be shown to extend to a *-homomorphism

$$\lambda : A \to \mathscr{L}(M),$$

where $\mathscr{L}(M)$ denotes the C*-algebra of all adjointable operators on M. The fact that

$$G(a)^*G(a) \leq G(a^*a) \quad \forall a \in A,$$

which may be easily proven by noting that $G(x^*x) \geq 0$, where $x = a - G(a)$, implies that G extends to a bounded linear operator on M, which we denote by g, and which may be shown to be a projection in $\mathscr{L}(M)$, often referred to as the *Jones projection*. It is easy to show that

$$g\lambda(a)g = \lambda\big(G(a)\big)g \quad \forall a \in A. \tag{10.2}$$

One may similarly show that E and F extend to bounded operators on M, respectively denoted e and f, providing projections e and f in $\mathscr{L}(M)$ satisfying

$$e\lambda(a)e = \lambda(E(a))e \quad \text{and} \quad f\lambda(a)f = \lambda(F(a))f \quad \forall a \in A. \tag{10.3}$$

The fact that $G = EF = FE$ immediately implies that

$$g = ef = fe. \tag{10.4}$$

Given $a_1, b_1, a_2, b_2 \in A$, observe that

$$\big(\lambda(a_1)g\lambda(b_1)\big)\big(\lambda(a_2)g\lambda(b_2)\big) = \lambda\big(a_1G(b_1a_2)\big)g\lambda(b_2),$$

which implies that the linear span of $\lambda(A)g\lambda(A)$ is an algebra, easily seen to be self adjoint. We thus obtain a closed *-subalgebra of $\mathscr{L}(M)$ by setting

$$K_g = \overline{\text{span}}\ \lambda(A)g\lambda(A).$$

In an entirely similar fashion we have the closed *-subalgebras

$$K_e = \overline{\text{span}}\ \lambda(A)e\lambda(A) \quad \text{and} \quad K_f = \overline{\text{span}}\ \lambda(A)f\lambda(A).$$

PROPOSITION 10.1. *One has that K_eK_g, K_gK_e, K_fK_g and K_gK_f are all contained in K_g, and hence both K_e and K_f may be naturally mapped into the multiplier algebra $M(K_g)$. We denote these maps by*

$$i_e : K_e \to M(K_g) \quad \text{and} \quad i_f : K_f \to M(K_g).$$

PROOF. Given $a, b, c, d \in A$, and using that $g = eg$, we have that

$$\big(\lambda(a)e\lambda(b)\big)\big(\lambda(c)g\lambda(d)\big) = \lambda(a)e\lambda(bc)eg\lambda(d) = \lambda\big(aE(bc)\big)g\lambda(d) \in K_g,$$

which proves that $K_eK_g \subseteq K_g$. The remaining inclusions may be proven similarly. □

Question 10.2. Is $\big(K_e, K_f, i_e, i_f, K_g\big)$ a C*-blend?

My interest in the whole idea of C*-blends actually arose from this question, to which I still do not have a definitive answer in its full generality. However we at least have:

PROPOSITION 10.3. *If either B or C contain an approximate identity for A, then the ranges of $i \circledast j$ and $j \circledast i$ are both dense in K_g.*

PROOF. By taking adjoints, it is enough to prove that

$$K_g \subseteq \overline{\text{span}}\ K_eK_f.$$

Let $\{u_i\}_i$ be an approximate identity for A contained, say, in B. Then, given x in the dense image of A within M, we have that

$$e\lambda(u_i)f(x) = E(u_iF(x)) = u_iE(F(x)) = u_iG(x) = \lambda(u_i)g(x),$$

which says that $e\lambda(u_i)f = \lambda(u_i)g$. Therefore, given $a, b \in A$, we have

$$\begin{aligned}\lambda(a)g\lambda(b) &= \lim_i \lambda(a)\lambda(u_i)g\lambda(b) = \lim_i \lambda(a)e\lambda(u_i)f\lambda(b) \\ &= \lim_i \big(\lambda(a)e\lambda(u_i^{1/2})\big)\big(\lambda(u_i^{1/2})f\lambda(b)\big) \in \overline{\text{span}}\ K_eK_f.\end{aligned}$$ □

This may now be used to state a sufficient condition for Question 10.2.

PROPOSITION 10.4. *Suppose that either B or C contain an approximate identity for A. If*

$$e\lambda(A)f \subseteq K_g,$$

then $\big(K_e, K_f, i_e, i_f, K_g\big)$ is a C-blend.*

PROOF. Once in possession Proposition 10.3 it is enough to prove that the range of $i_e \circledast i_f$, and consequently also of $i_f \circledast i_e$, is contained in K_g. In other words we must show that

$$K_eK_f \subseteq K_g.$$

Given $a, b, c, d \in A$, we have

$$\big(\lambda(a)e\lambda(b)\big)\big(\lambda(c)f\lambda(d)\big) = \lambda(a)e\lambda(bc)f\lambda(d) \subseteq \lambda(a)K_g\lambda(d) \subseteq K_g.$$ □

This is as much as we can say in the present generality, so let us now consider a somewhat restrictive special case. Recall from [15], that the conditional expectation G is said to be of *index finite type* provided there is a finite set

$$\{u_1, \dots, u_n\} \subseteq A,$$

called a *quasi-basis*, such that

$$a = \sum_{i=1}^{n} u_i G(u_i^* a) \quad \forall a \in A. \tag{10.5}$$

Let us assume for the time being that G is of index finite type. As a consequence G is necessarily faithful [15, 2.1.5] and, as one may easily show, λ is injective. We shall therefore identify A with its image under λ without further warnings.

Some other aspects of the present situation are also much simplified because of the following:

PROPOSITION 10.5. *If G is of index finite type, let $\{u_1, \dots, u_n\}$ be a quasi-basis. Then*

(i) $\sum_{i=1}^{n} u_i g u_i^* = 1$,
(ii) span $AgA = K_g = \mathscr{L}(M) = L_D(M)$, *where the latter refers to the set of all D-linear maps on M.*

PROOF. Initially notice that, by [15, 2.1.5], A is already complete with the norm arising from (10.1), so $M = A$. For every $a \in M$, one then has that

$$\sum_{i=1}^{n} u_i g u_i^*(a) = \sum_{i=1}^{n} u_i G(u_i^* a) = a,$$

proving (i). As for (ii), we first notice that the inclusions

$$\text{span } AgA \subseteq K_g \subseteq \mathscr{L}(M) \subseteq L_D(M)$$

are all evident. Given a D-linear map T on M, notice that, for every $a \in M$,

$$T(a) = T\Big(\sum_{i=1}^{n} u_i G(u_i^* a)\Big) = \sum_{i=1}^{n} T(u_i) G(u_i^* a) = \sum_{i=1}^{n} T(u_i) g u_i^* a,$$

so we see that

$$T = \sum_{i=1}^{n} T(u_i) g u_i^* \in \text{span } AgA. \qquad \square$$

Evidently both e and f are D-linear, as are all operators on M in sight, so we have that $eAf \subseteq K_g$. Employing Proposition 10.4 we then have the following affirmative answer to Question 10.2:

PROPOSITION 10.6. *If G is of index finite type then $\big(K_e, K_f, i_e, i_f, K_g\big)$ is a C*-blend.*

Let us now give another partial answer to Question 10.2, this time without assuming finiteness of the index. Still under the assumption that we are given a commuting square as above, we will now assume the following:

STANDING HYPOTHESIS 1. *A is a unital commutative algebra and $D = \mathbb{C}1$.*

As a consequence G is necessarily of the form

$$G(a) = \phi(a)1 \quad \forall a \in A, \tag{10.6}$$

where ϕ is a state of A.

Being a right Hilbert D-module, M is nothing but a Hilbert space while λ is just the GNS representation associated to ϕ. Using λ we will view M as a left A module, and hence we will adopt the notation

$$a\eta := \lambda(a)\eta \quad \forall a \in A,\ \eta \in M.$$

Denoting the image of 1 in M by ξ, we evidently have the well known GNS formula

$$\phi(a) = \langle a\xi, \xi\rangle \quad \forall a \in A.$$

Let us now prove a crucial inequality to be used later.

LEMMA 10.7. *Given $a \in A$, and $c_1, \ldots, c_n \in C$, one has that*

$$\sum_{i=1}^{n} \|eac_i\xi\|^2 \leq \|E(a^*a)\|\ \|\mu\|,$$

where μ is the $n \times n$ scalar matrix with $\mu_{ij} = \langle c_i\xi, c_j\xi\rangle$, for $i, j = 1, \ldots, n$.

PROOF. Since $E(1) = 1$, one has that $e(\xi) = \xi$, and hence for every $a \in A$ we have

$$ea(\xi) = eae(\xi) = E(a)e(\xi) = E(a)\xi,$$

and similarly

$$fa(\xi) = F(a)\xi \quad \forall a \in A. \tag{10.7}$$

We then have

$$\begin{aligned} \sum_{i=1}^{n} \|eac_i\xi\|^2 &= \sum_{i=1}^{n} \|E(ac_i)\xi\|^2 \\ &= \sum_{i=1}^{n} \phi\big(E(c_i^*a^*)E(ac_i)\big) = \phi(b), \end{aligned} \tag{10.8}$$

where $b := \sum_{i=1}^n E(c_i^* a^*)E(ac_i)$. Let m be the $n \times n$ matrix over A given by

$$m = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ c_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_n & 0 & \dots & 0 \end{pmatrix}.$$

Observing that

$$E(m^* a^*)E(am) = \begin{pmatrix} b & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

we see that

$$\|b\| = \|E(m^* a^*)E(am)\| = \|E(am)\|^2.$$

Recall from [12, 2.9] that

$$\|E(x^* y)\|^2 \leq \|E(x^* x)\| \, \|E(y^* y)\| \quad \forall x, y \in A,$$

so we have

$$\|b\| = \|E(am)\|^2 \overset{(!)}{=} \|E(ma)\|^2 \leq \|E(mm^*)\| \, \|E(a^* a)\|. \tag{10.9}$$

In case the reader is interested in attempting to eliminate the commutativity hypothesis assumed in Standing Hypothesis 1, we have marked the only two places in which we used the commutativity of A with an exclamation mark, the first one appearing in the calculation just above.

The entry (i, j) of the matrix $E(mm^*)$ is given by

$$E(c_i c_j^*) = EF(c_i c_j^*) = \phi(c_i c_j^*) \overset{(!)}{=} \phi(c_j^* c_i) = \langle c_j^* c_i \xi, \xi \rangle = \langle c_i \xi, c_j \xi \rangle,$$

which precisely says that $E(mm^*)$ coincides with the matrix μ in the statement. We therefore have

$$\sum_{i=1}^n \|eac_i \xi\|^2 \overset{(10.8)}{=} \phi(b) \leq \|b\| \overset{(10.9)}{\leq} \|\mu\| \|E(a^* a)\|. \qquad \square$$

Recall that ξ is the image of 1 in M, and hence that $g(\xi) = \xi$. Moreover, given $a, b, c \in A$, let $\eta = c\xi$, and notice that

$$\begin{aligned} (agb^*)(\eta) &= (agb^*)(c\xi) = agb^* cg\xi = aG(b^* c)g\xi = aG(b^* c)\xi \\ &= \phi(b^* c)a\xi = \langle c\xi, b\xi \rangle a\xi = \langle \eta, b\xi \rangle a\xi. \end{aligned}$$

Since the set of vectors of the form $\eta = c\xi$ is dense in A, this shows that

$$(agb^*)(\eta) = \langle \eta, b\xi\rangle a\xi, \tag{10.10}$$

for every η in M. Given any two vectors ζ_1 and ζ_2 in M, consider the rank-one operator Ω_{ζ_1,ζ_2}, defined by

$$\Omega_{\zeta_1,\zeta_2}(\eta) = \langle \eta, \zeta_2\rangle\zeta_1 \quad \forall \eta \in M.$$

By 10.10, we then have that

$$\lambda(a)g\lambda(b^*) = \Omega_{a\xi,b\xi}.$$

Since ξ is cyclic, we may approximate any given ζ_1 and ζ_2 by vectors of the form $a\xi$ and $b\xi$, respectively, and hence one sees that Ω_{ζ_1,ζ_2} belongs to K_g. This proves the following:

PROPOSITION 10.8. *K_g coincides with the algebra of all compact operators on M.*

We now plan to use Proposition 10.4 in order to obtain an affirmative answer to Question 10.2. Under Standing Hypothesis 1 we have that D contains the unit of A, so the first part of the hypothesis of Proposition 10.4 is granted. Therefore we must only verify that $e\lambda(A)f$ consists of compact operators in order to reach our conclusion.

THEOREM 10.9. *For every $a \in A$, one has that $e\lambda(a)f$ is a Hilbert-Schmidt operator on M. Moreover, denoting the Hilbert-Schmidt norm by $\|\cdot\|_2$, one has that*

$$\|e\lambda(a)f\|_2 \leq \|E(a^*a)\|^{1/2} \quad \forall a \in A.$$

As a consequence $e\lambda(A)f$ consists of compact operators and hence $\big(K_e, K_f, i_e, i_f, K_g\big)$ is a C-blend.*

PROOF. Notice that $e\lambda(a)f$ vanishes on the orthogonal complement of $f(M)$. It is therefore enough to prove that

$$\sum_{i\in I} \|eaf\eta_i\|^2 \leq \|E(a^*a)\|,$$

for any (and hence all) orthonormal basis $\{\eta_i\}_{i\in I}$ of $f(M)$. Since the left hand side is defined to be the supremum of the sums over finite subsets of I, it is enough to prove that

$$\sum_{i=1}^{n} \|eaf\eta_i\|^2 \leq \|E(a^*a)\|, \tag{10.11}$$

for any finite orthonormal set $\{\eta_i\}_{i=1}^n$ contained in the range of f. By 10.7 one has that $f(M)$ is the closure of $C\xi$ in M so, for each i we may write

$$\eta_i = \lim_{k\to\infty} c_i^k \xi,$$

where $\{c_i^k\}_{k\in\mathbb{N}}$ is a sequence in C. Let μ^k be the $n \times n$ scalar matrix with $\mu_{ij}^k = \langle c_i^k\xi, c_j^k\xi\rangle$, as in Lemma 10.7. For all k one then has that

$$\sum_{i=1}^n \|eac_i^k\xi\|^2 \leq \|E(a^*a)\|\ \|\mu^k\|.$$

Since the η_i form an orthonormal set, we have that μ^k converges to the identity matrix, so (10.11) follows by taking the limit as $k \to \infty$. □

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Departamento de Matemática, Universidade Federal de Santa Catarina, 88040-970 Florianópolis SC, Brazil.
e-mail: exel@mtm.ufsc.br