C. R. Math. Rep. Acad. Sci. Canada Vol. **34** (4) 2012, pp. 105–111

Rationality of Dedekind Sums in Finite Fields

Dedicated to Professor Tadashi Yamazaki

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Presented by Eckhard Meinrenken, FRSC

Abstract. In [1], we introduced the Dedekind–Zagier sum in finite fields. It is defined by a lattice Λ. The objective of this paper is to present a criterion for the rationality of our Dedekind–Zagier sum. For this purpose, we establish a connection between the field of definition of the exponential function for Λ and the field of definition of the Dedekind–Zagier sum for Λ.

Résumé. Dans [1], nous avons introduit la somme de Dedekind–Zagier dans des corps finis. La somme est définie à partir d'un réseau Λ. L'objectif de ce travail est de présenter un critère de la rationalité de notre somme de Dedekind–Zagier. Pour le but, nous éstablissons la connexion entre le corps de définition de la fonction exponentielle pour Λ et le corps de définition de notre somme de Dedekind–Zagier pour Λ.

1. **Introduction.** For relatively prime integers $c > 0$ and a, the classical Dedekind sum is defined as

$$s(a,c) = \frac{1}{4c}\sum_{k=1}^{c-1}\cot\Big(\frac{\pi k}{c}\Big)\cot\Big(\frac{\pi k a}{c}\Big).$$

It satisfies a well-known relation called the reciprocity law

$$s(a,c) + s(c,a) = \frac{a^2+c^2+1-3ac}{12ac}$$

when a, c are coprime positive integers. For details, refer to the book by Rademacher and Grosswald [7]. A generalization of Dedekind sums to higher dimensions was presented by Zagier [8]. Let a_0 be a positive integer, and let $a_1, \dots, a_d$ be integers relatively prime to a_0. Zagier defined the higher-dimensional Dedekind sum:

$$d(a_0; a_1, \dots, a_d) := \frac{(-1)^{d/2}}{a_0}\sum_{k=1}^{a_0-1}\cot\Big(\frac{\pi k a_1}{a_0}\Big)\cdots\cot\Big(\frac{\pi k a_d}{a_0}\Big).$$

Received by the editors on April 9, 2012.
AMS Subject Classification: Primary: 11F20; secondary: 12E20.
Keywords: Dedekind sums, lattices, finite fields.

For pairwise coprime positive integers $a_0, a_1, \ldots, a_d$, this sum satisfies the reciprocity law

$$\sum_{j=0}^{d} d(a_j; a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_d) = 1 - \frac{l_d(a_0, a_1, \ldots, a_d)}{a_0 a_1 \cdots a_d},$$

where each $l_d(a_0, a_1, \ldots, a_d)$ is a polynomial in $a_0, a_1, \ldots, a_d$. These classical Dedekind sums are rational numbers, and they can be applied to number theory (quadratic reciprocity, Farey fractions) and combinatorial theory (the enumeration of lattice points in certain triangles, tetrahedrons and parallelepipeds). See [2, 7] for their results.

In our previous paper [1], we introduced higher-dimensional Dedekind sums in finite fields, defined by a lattice Λ. In this paper, we introduce a concept of "rational" for our Dedekind sums, as an analog of rational Dedekind sums. In contrast with the classical case, our Dedekind sums are not necessarily "rational". By considering their applications to function field arithmetic, the rationality of our Dedekind sums requires investigation. The objective of this paper is to present a criterion for the rationality of our Dedekind sums. For this purpose, we establish a connection between the field of definition of the exponential function for Λ and the field of definition of the higher-dimensional Dedekind sum associated with Λ.

2. Lattices and exponential functions.

We recall some facts about lattices and their exponential functions. We refer to Gekeler's paper [4] for details.

For $K = \mathbb{F}_q$, the finite field with q elements, $\overline{K}$ denotes a fixed algebraic closure of K. Let Λ be a subset of $\overline{K}$. We call Λ a *lattice* if it is a linear K-subspace of $\overline{K}$, with finite dimension. For such a lattice Λ, define the product

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

The map $e_\Lambda \colon \overline{K} \to \overline{K}$ satisfies the following properties:

- e_Λ is K-linear and Λ-periodic.
- If $\dim_K \Lambda = r$, then $e_\Lambda(z)$ has the form

$$e_\Lambda(z) = \sum_{i=0}^{r} \alpha_i(\Lambda) z^{q^i}, \tag{2.1}$$

  where $\alpha_0(\Lambda) = 1$, $\alpha_r(\Lambda) \neq 0$.
- $e_\Lambda(z)$ has simple zeros at the points of Λ, and no other zeros.
- $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$. Hence, we have

$$\frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}.$$

For a positive integer k,

$$E_k(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-k}$$

is called the *Eisenstein series of weight* k for Λ. We use the convention $E_0(\Lambda) = -1$. The function $z/e_\Lambda(z)$ is expressed as a formal series by

$$\frac{z}{e_\Lambda(z)} = -\sum_{k=0}^{\infty} E_k(\Lambda) z^k. \tag{2.2}$$

Let $K(\Lambda)$ be the field generated by Λ over K. For $a \in \overline{K} \setminus K(\Lambda)$, define

$$\Lambda\langle a\rangle = e_\Lambda(a\Lambda),$$

which has dimension $\dim_K \Lambda\langle a\rangle = \dim_K \Lambda$.

Example 2.1. Let $e_\Lambda(z)$ be given by (2.1). Write $\alpha_i = \alpha_i(\Lambda)$ $(i = 1, \ldots, r)$ for simplicity.
Case $r = 1$: $e_{\Lambda\langle a\rangle}(z) = z + \frac{\alpha_1}{(a-a^q)^{q-1}} z^q$.
Case $r = 2$: $e_{\Lambda\langle a\rangle}(z) = z + \beta_1 z^q + \beta_2 z^{q^2}$, where

$$\beta_1 = \frac{\alpha_1}{(a-a^{q^2})^{q-1}} - \alpha_1 \frac{(a-a^q)^q}{(a-a^{q^2})^q} + \frac{\alpha_1^{q^2}}{\alpha_2^q} \cdot \frac{(a-a^q)^{q^3}}{(a-a^{q^2})^q},$$

$$\beta_2 = \frac{\alpha_2}{(a-a^{q^2})^{q^2-1}} - \alpha_1^q \frac{(a-a^q)^{q^2}}{(a-a^{q^2})^{q^2}} + \frac{\alpha_1^{q^2+q}}{\alpha_2^q} \cdot \frac{(a-a^q)^{q^3}}{(a-a^{q^2})^{q^2}}.$$

3. Higher-dimensional Dedekind sums in finite fields.

In our previous paper [1], we introduced a finite field analog of the higher-dimensional Dedekind sum of Zagier. We recall its definition and properties. We consider a lattice Λ. Select $a_0, a_1, \ldots, a_d \in \overline{K} \setminus \{0\}$ such that

$$a_i/a_0 \notin K(\Lambda) \quad (i = 1, \ldots, d). \tag{3.1}$$

We define the *higher-dimensional Dedekind sum* by

$$s_\Lambda(a_0; a_1, \ldots, a_d) = \frac{(-1)^d}{a_0} \sum_{\lambda \in \Lambda \setminus \{0\}} e_\Lambda(a_1\lambda/a_0)^{-1} \cdots e_\Lambda(a_d\lambda/a_0)^{-1}.$$

THEOREM 3.1 ([1]). *The Dedekind sum $s_\Lambda(a_0; a_1, \ldots, a_d)$ has properties similar to those of Zagier's Dedekind sum:*
(i) *$s_\Lambda(a_0; a_1, \ldots, a_d)$ depends only on $a_i + a_0\Lambda$.*
(ii) *$s_\Lambda(a_0; a_1, \ldots, a_d)$ is symmetric in $a_1, \ldots, a_d$.*

(iii) $s_\Lambda(a_0; \zeta a_1, \ldots, a_d) = \zeta^{-1} s_\Lambda(a_0; a_1, \ldots, a_d)$ *for any* $\zeta \in K \setminus \{0\}$.
(iv) $s_\Lambda(a_0; \zeta a_1, \ldots, \zeta a_d) = s_\Lambda(a_0; a_1, \ldots, a_d)$ *for any* $\zeta \in K \setminus \{0\}$.
(v) $s_\Lambda(a_0; a_1, \ldots, a_d)$ *satisfies the reciprocity law. Thus, for* $a_0, a_1, \ldots, a_d \in \overline{K} \setminus \{0\}$ *such that*

$$a_i/a_j \notin K(\Lambda) \quad (i \neq j),$$

we have

$$\sum_{i=0}^{d} s_\Lambda(a_i; a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) = \sum_{\substack{i_0+\cdots+i_d=d \\ i_0,\ldots,i_d \geq 0}} \frac{a_0^{i_0} \cdots a_d^{i_d}}{a_0 \cdots a_d} E_{i_0}(\Lambda) \cdots E_{i_d}(\Lambda).$$

4. Rationality for $s_\Lambda(a_0; a_1, \ldots, a_d)$.

From the definition of our Dedekind sum, it is easy to see that $s_\Lambda(a_0; a_1, \ldots, a_d) \in K(\Lambda)(a_0, a_1, \ldots, a_d)$.

DEFINITION 4.1. The Dedekind sum $s_\Lambda(a_0; a_1, \ldots, a_d)$ is called *rational* if it is contained in $K(a_0, a_1, \ldots, a_d)$.

Remark 4.2. As is well known, $\pi \cot \pi z$ is expressed by

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big).$$

This implies that the classical Dedekind sums are related to $\mathbb{Z}$, and they are rational in the sense of Definition 4.1 because $\mathbb{Q}(\mathbb{Z})(a_0, a_1, \ldots, a_d) = \mathbb{Q}$.

Example 4.3. (1) Let d be odd. By Theorem 3.1 (iii) and (iv), we have

$$(-1)^d s_\Lambda(a_0; a_1, \ldots, a_d) = s_\Lambda(a_0; a_1, \ldots, a_d).$$

Thus, if $\mathrm{Char}\, \mathbb{F}_q \neq 2$, then the sum is zero, and hence, rational. In this case, the field $K\big(s_\Lambda(a_0; a_1, \ldots, a_d)\big)$ is K.

(2) Let $\Lambda = K = \mathbb{F}_2$, $d = 1$. Suppose $a_1/a_0 \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then a_1/a_0 is a primitive element of $\mathbb{F}_{q^2}$ and $a_1/a_0 - a_1^q/a_0^q = a_1/a_0 + a_1^q/a_0^q \in \mathbb{F}_q \setminus \{0\}$. Hence,

$$s_K(a_0; a_1) = \frac{1}{a_0\big(\frac{a_1}{a_0} - \frac{a_1^q}{a_0^q}\big)} \in K(a_0, a_1),$$

which implies that $s_\Lambda(a_0; a_1)$ is rational. We take a primitive element a_0 of $\mathbb{F}_{q^4}$ and $a_1 = t a_0$, where t is a primitive element of $\mathbb{F}_{q^2}$. Then, the field $K\big(s_\Lambda(a_0; a_1)\big)$ is given by $K(a_0) = K(a_0, a_1)$.

We have a rationality criterion for $s_\Lambda(a_0; a_1, \ldots, a_d)$ as follows.

THEOREM 4.4. *Assume that the dimension of* Λ *over* K *is* r. *Then the following properties are equivalent:*

(i) $e_{\Lambda\langle a\rangle}(z) \in K(a)[z]$ *for all* $a \in \overline{K} \setminus K(\Lambda)$.
(ii) $e_\Lambda(z) \in K[z]$.
(iii) $s_\Lambda(a_0; a_1, \ldots, a_{q^i-1})$, $i = 1, \ldots, r$, *are rational.*

This theorem is our main result.

Remark 4.5. Let the notations be identical to those in Theorem 4.4. By (2.2) and Theorem 4.4, if $e_\Lambda(z) \in K[z]$, then the reciprocity law for $s_\Lambda(a_0; a_1, \ldots, a_{q^i-1})$ holds in $K(a_0, a_1, \ldots, a_{q^i-1})$ $(i = 1, \ldots, r)$.

Remark 4.6. (1) Hickerson [6] proved that the values of the classical Dedekind sum $s(a, c)$ stated in the Introduction are dense in $\mathbb{R}$. Let Λ be a lattice such that $e_\Lambda(z)$ is contained in $K[z]$. Fix a finite field $\mathbb{F}_{q^n}$. It is interesting to investigate the distribution of the values of $s_\Lambda(a_0; a_1, \ldots, a_d)$ for $a_0 \in \overline{K}\setminus\{0\}$ and $a_1, \ldots, a_d \in \overline{K}$ satisfying (3.1).

(2) As seen in Example 4.3 (2), $K\big(s_\Lambda(a_0; a_1, \ldots, a_d)\big)$, the field generated by the rational Dedekind sum $s_\Lambda(a_0; a_1, \ldots, a_d)$ over K, depends on the value of the Dedekind sum above. It is also interesting to study the property of $K\big(s_\Lambda(a_0; a_1, \ldots, a_d)\big)$.

5. Proof of Theorem 4.4.

(i) ⇒ (ii). Put $e_\Lambda(z) = \alpha_0 z + \alpha_1 z^q + \cdots + \alpha_r z^{q^r}$ $(\alpha_0 = 1)$, which belongs to $K(\Lambda)[z]$. Take $a \in \overline{K} \setminus K(\Lambda)$ satisfying

$$[K(\Lambda)(a) : K(\Lambda)] > q^r, \tag{5.1}$$

$$[K(a) : K] \text{ is a prime number.} \tag{5.2}$$

Take any $\sigma \in \mathrm{Gal}\big(K(a)(\Lambda)/K(a)\big)$ and $\lambda \in \Lambda \setminus \{0\}$. Since $e_{\Lambda\langle a\rangle}(z) \in K(a)[z]$, $\sigma\big(e_\Lambda(a\lambda)\big)$ is a root of $e_{\Lambda\langle a\rangle}(z)$. Hence, we have $\sigma\big(e_\Lambda(a\lambda)\big) = e_\Lambda(a\lambda')$ for a certain $\lambda' \in \Lambda$. It follows that

$$\sum_{i=0}^{r} a^{q^i}\left(\sigma(\alpha_i\lambda^{q^i}) - \alpha_i{\lambda'}^{q^i}\right) = 0. \tag{5.3}$$

Since σ is given by $x \mapsto x^{q^n}$ for a positive integer n, all $\sigma(\alpha_i\lambda^{q^i}) - \alpha_i{\lambda'}^{q^i}$ are elements of $K(\Lambda)$. From (5.1) and (5.3), we get $\sigma(\lambda) = \lambda'$ and $\sigma(\alpha_i) = \alpha_i$ for $i = 1, \ldots, r$. This implies that $\alpha_1, \ldots, \alpha_r \in K(a)$. From (5.1), $K(a)$ is not contained in $K(\Lambda)$. Thus, we have $K(a) \cap K(\Lambda) \neq K(a)$. Using (5.2), we see that $K(a) \cap K(\Lambda) = K$, which gives $\alpha_1, \ldots, \alpha_r \in K$.

(ii) ⇒ (iii). Since $e_\Lambda(z)$ is in $K[z]$, $s_\Lambda(a_0; a_1, \ldots, a_{q^i-1})$ is invariant under the action of $\mathrm{Gal}\big(K(a_0, a_1, \ldots, a_{q^i-1})(\Lambda)/K(a_0, a_1, \ldots, a_{q^i-1})\big)$. Hence, $s_\Lambda(a_0; a_1, \ldots, a_{q^i-1})$ belongs to $K(a_0, a_1, \ldots, a_{q^i-1})$.

(iii) $\Rightarrow$ (i). By assumption, for any $a \in \overline{K} \setminus K(\Lambda)$,

$$s_\Lambda(1; \overbrace{a, \dots, a}^{q^i - 1}) = (-1)^{q^i - 1} \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{e_\Lambda(a\lambda)^{q^i-1}} = (-1)^{q^i-1} E_{q^i-1}(\Lambda\langle a \rangle) \in K(a) \tag{5.4}$$

for $i = 1, \dots, r$. Since $(E_{q^i-1}(\Lambda\langle a \rangle))^{q^j} = E_{q^{i+j}-q^j}(\Lambda\langle a \rangle)$, we have $E_{q^k-q^l}(\Lambda\langle a \rangle) \in K(a)$ for $1 \le l < k \le r$. We recall Newton's formula for the power sums of the zeros of a given polynomial.

LEMMA 5.1 (Newton's formula *cf.* [3], [5]). *Let*

$$f(X) = X^n + c_1 X^{n-1} + \cdots + c_{n-1} X + c_n$$

be a polynomial over a field L, and let $\alpha_1, \dots, \alpha_n$ be the roots of $f(X)$. For each non-negative integer k, put $T_k = \alpha_1^k + \cdots + \alpha_n^k$. Then it holds that

$$T_k + c_1 T_{k-1} + \cdots + c_{k-1} T_1 + k c_k = 0 \quad (k \le n),$$
$$T_k + c_1 T_{k-1} + \cdots + c_{n-1} T_{k-n+1} + c_n T_{k-n} = 0 \quad (k \ge n).$$

We now return to the proof of the theorem. If $e_{\Lambda\langle a \rangle}(z) = \beta_0 z + \beta_1 z^q + \cdots + \beta_r z^{q^r}$ $(\beta_0 = 1)$, then we have

$$e_{\Lambda\langle a \rangle}(z^{-1}) z^{q^r} = z^{q^r-1} + \beta_1 z^{q^r-q} + \cdots + \beta_{r-1} z^{q^r-q^{r-1}} + \beta_r.$$

Since $\{1/x \mid x \in \Lambda\langle a \rangle \setminus \{0\}\}$ is the set of roots of this polynomial, using Newton's formula, it follows that

$$E_{q^i-1}(\Lambda\langle a \rangle) + \beta_1 E_{q^i-q}(\Lambda\langle a \rangle) + \cdots + \beta_{i-1} E_{q^i-q^{i-1}}(\Lambda\langle a \rangle) = \beta_i \tag{5.5}$$

for $i = 1, \dots, r$. Using this identity repeatedly, we deduce that $\beta_1, \dots, \beta_r \in K(a)$. Therefore, we conclude that $e_{\Lambda\langle a \rangle}(z)$ is included in $K(a)[z]$.

6. **Concluding remarks.** We fix d and take an r-dimensional lattice Λ. Even if $s_\Lambda(a_0; a_1, \dots, a_d)$ is rational, $e_\Lambda(z)$ is not necessarily contained in $K[z]$. Indeed, let us consider the case $d = q - 1$, $r \ge 2$. Let $a \in \overline{K} \setminus K(\Lambda)$. The Dedekind sum $s_\Lambda(1; a, \dots, a)$ is, up to an element of $\mathbb{F}_q$, $E_{q-1}(\Lambda\langle a \rangle)$ in the same way as (5.4). From (5.5), we have $E_{q-1}(\Lambda\langle a \rangle) = \beta_1$, where β_1 is the coefficient of z^q in $e_{\Lambda\langle a \rangle}(z) = z + \beta_1 z^q + \cdots + \beta_r z^{q^r}$. By assumption, $\beta_1 \in K(a)$. If β_2 does not belong to $K(a)$, then $e_{\Lambda\langle a \rangle}(z)$ is not in $K(a)[z]$. From Theorem 4.4, $e_\Lambda(z)$ is not in $K[z]$.

Next, we define an analog of the classical Dedekind sum $s(a, c)$ stated in the introduction.

DEFINITION 6.1. For $a, c \in \overline{K} \setminus \{0\}$ such that $a/c \notin K(\Lambda)$, the *inhomogeneous Dedekind sum* $s_\Lambda(a, c)$ is defined as

$$\begin{aligned} s_\Lambda(a, c) &= s_\Lambda(c; a, 1) \\ &= \frac{1}{c} \sum_{\lambda \in \Lambda \setminus \{0\}} e_\Lambda(a\lambda/c)^{-1} e_\Lambda(\lambda/c)^{-1}. \end{aligned}$$

From the above discussion, even if $s_\Lambda(a, c)$ is rational, $e_\Lambda(z)$ is not necessarily contained in $K[z]$.

REFERENCES

1. A. Bayad and Y. Hamahata, *Higher dimensional Dedekind sums in finite fields.* Finite Fields Appl. **18** (2012), 19–25.
2. M. Beck and S. Robins, *Computing the Continuous Discretely.* Springer, 2007.
3. E.-U. Gekeler, *On the coefficients of Drinfeld modular forms.* Invent. Math. **93** (1988), 667–700.
4. ———, *Finite modular forms.* Finite Fields Appl. **7** (2001), 553–572.
5. D. Goss, *The algebraist's upper half-planes.* Bull. Amer. Math. Soc. (N.S.) **2** (1980), 391–415.
6. D. Hickerson, *Continued fractions and density results for Dedekind sums.* J. Reine Angew. Math. **290** (1977), 113–116.
7. H. Rademacher and E. Grosswald, *Dedekind Sums.* The Mathematical Association of America, Washington, D.C., 1972.
8. D. Zagier, *Higher dimensional Dedekind sums.* Math. Ann. **202** (1973), 149–172.

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