# Geometrization of Postcritically-Finite Branched Coverings

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Abstract. We study canonical decompositions of postcritically-finite branched coverings of the 2-sphere, as defined by K. Pilgrim. We show that every hyperbolic cycle in the decomposition does not have a Thurston obstruction. It is thus Thurston equivalent to a rational map.

Résumé. Nous étudions les décompositions canoniques de revêtements ramifiés de la sphère, avec ensembles post-critiques finis, ainsi que K. Pilgrim les a définies. Nous montrons qu'aucun cycle hyperbolique dans la décomposition n'a d'obstruction de Thurston. Par conséquent, un tel cycle est équivalent au sens de Thurston à une application rationnelle.

*Foreword.* A proof of the main result was announced by Nikita Selinger at the Workshop "Holomorphic Dynamics around Thurston's Theorem" which took place at Roskilde University from September 27–October 1, 2010 (a preprint [10] soon appeared). At the same conference, we independently proposed a different approach to the proof, which is presented here. We are very grateful to Selinger for several useful discussions, particularly for pointing out an error in a previous version of this paper and helping us to correct it in the proof. Our argument bears a certain ideological similarity to that of [10]. However, it is based on a specific geometric surgery construction, rather than the more abstract concept of augmented Teichmüller space used by Selinger.

## 1. Introduction and statement of the result.

**Thurston maps and multicurves.** In this section we recall the basic setting of Thurston's characterization of rational functions. Let $f\colon S^2 \to S^2$ be an orientation-preserving branched covering map of the two-sphere. We define the *postcritical set $P_f$* by

$$P_f := \bigcup_{n>0} f^{\circ n}(\Omega_f),$$

where $\Omega_f$ is the set of critical points of $f$. When the postcritical set $P_f$ is finite, we say that $f$ is a *Thurston mapping.*

Two Thurston maps $f$ and $g$ are *Thurston equivalent* if there are homeomorphisms $\phi_0, \phi_1 \colon S^2 \to S^2$ such that

(1) the maps $\phi_0, \phi_1$ coincide on $P_f$, send $P_f$ to $P_g$ and are isotopic rel $P_f$;

(2) the diagram

$$
\begin{array}{ccc}
S^2 & \xrightarrow{\ \phi_1\ } & S^2 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
S^2 & \xrightarrow{\ \phi_0\ } & S^2
\end{array}
$$

commutes.

Given a Thurston map $f \colon S^2 \to S^2$, we define a function $N_f \colon S^2 \to \mathbb{N} \cup \infty$ as follows:

$$
N_f(x) = \begin{cases} 1 & \text{if } x \notin P_f, \\ \infty & \text{if } x \text{ is in a cycle containing a critical point,} \\ \mathrm{lcm}_{f^k(y)=x} \deg_y(f^{\circ k}) & \text{otherwise.} \end{cases}
$$

The pair $(S^2, N_f)$ is called the *orbifold of $f$*. The *signature* of the orbifold $(S^2, N_f)$ is the set $\{N_f(x)$ for $x$ such that $1 < N_f(x) < \infty\}$. The *Euler characteristic* of the orbifold is given by

$$
\chi(S^2, N_f) := 2 - \sum_{x \in P_f} \left(1 - \frac{1}{N_f(x)}\right).
$$

One can prove that $\chi(S^2, N_f) \le 0$. In the case where $\chi(S^2, N_f) < 0$, we say that the orbifold is *hyperbolic*. Observe that most orbifolds are hyperbolic: indeed, as soon as the cardinality $|P_f| > 4$, the orbifold is hyperbolic.

We recall that a simple closed curve $\gamma \subset S^2 - P_f$ is *non-essential* if it bounds a disk, and is *peripheral* if it bounds a punctured disk. We call a homotopy class of simple closed curves $[\gamma]$ *trivial* if it is either non-essential or peripheral.

DEFINITION 1.1.   A *multicurve* $\Gamma$ on $(S^2, P_f)$ is a set of disjoint, nonhomotopic, essential, nonperipheral simple closed curves on $S^2 - P_f$. A multicurve $\Gamma$ is *$f$-stable* if for every curve $\gamma \in \Gamma$, each component $\alpha$ of $f^{-1}(\gamma)$ is either trivial or homotopic rel $P_f$ to an element of $\Gamma$.

To any $f$-stable multicurve is associated its *Thurston linear transformation* $f_\Gamma \colon \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$, best described by the following transition matrix

$$
M_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f : \alpha \to \delta)}
$$

where the sum is taken over all the components $\alpha$ of $f^{-1}(\delta)$ that are isotopic rel $P_f$ to $\gamma$. Since this matrix has nonnegative entries, it has a leading eigenvalue $\lambda(\Gamma)$ that is real and nonnegative (by the Perron–Frobenius theorem).

We can now state Thurston's theorem.

THURSTON'S THEOREM. *Let $f\colon S^2 \to S^2$ be a Thurston map with hyperbolic orbifold. Then $f$ is Thurston equivalent to a rational function $g$ if and only if $\lambda(\Gamma) < 1$ for every $f$-stable multicurve $\Gamma$. The rational function $g$ is unique up to conjugation with an automorphism of $\mathbb{P}^1$.*

When a stable multicurve $\Gamma$ has a leading eigenvalue $\lambda(\Gamma) \geq 1$, we call it a *Thurston obstruction.*

**Pilgrim's canonical obstructions.** Below we describe a particular type of Thurston obstructions, which were defined by K. Pilgrim in [8]. Let us assume that a Thurston map $f$ has a hyperbolic orbifold. Let us denote $\mathcal{T}_f$ the Teichmüller space of the punctured sphere $S \equiv S^2 \setminus P_f$, and $d_{\mathcal{T}_f}(\cdot, \cdot)$ the Teichmüller distance; $\mathcal{M}_f$ will denote the moduli space of $S^2 \setminus P_f$; $p_f\colon \mathcal{T}_f \to \mathcal{M}_f$ will be the covering map. Further, for a choice of the complex structure $\tau$ on $S$, we let $\rho_\tau$ denote the hyperbolic metric on the Riemann surface $S_\tau \equiv (S, \tau)$, length$_\tau$ the hyperbolic length, and $d_\tau$ the hyperbolic distance. Similarly, for a general hyperbolic Riemann surface $W$ we denote by $\rho_W$, $d_W$, and length$_W$ the hyperbolic metric, distance, and length on $W$; by $\mathcal{T}_W$ the Teichmüller space, *etc.*

DEFINITION 1.2. For a non-trivial homotopy class of simple closed curves $[\gamma]$ on $S$ we let $\ell_\tau([\gamma])$ denote the length of the unique geodesic representative of $[\gamma]$ in $S_\tau$.

The map $f$ induces an analytic mapping on $\mathcal{T}_f$:

$$\sigma_f\colon \mathcal{T}_f \to \mathcal{T}_f, \quad \text{where } \sigma_f([\tau]) = [f^*\tau].$$

The map $\sigma_f$ does not increase Teichmüller distance. Douady and Hubbard [3] show that the amount by which $\sigma_f$ contracts $d_{\mathcal{T}_f}$ at a point $[\tau] \in \mathcal{T}_f$ depends only on $p_f(\tau)$ and a *finite* amount of additional information. More specifically:

PROPOSITION 1.1 (Lemma 5.2 of [3]). *There exists a tower*

$$\mathcal{T}_f \xrightarrow{\tilde{p}_f} \widetilde{\mathcal{M}}_f \xrightarrow{\bar{p}_f} \mathcal{M}_f$$

*of covering spaces, such that $\bar{p}_f$ is a finite cover, and a map $\tilde{\sigma}_f\colon \widetilde{\mathcal{M}}_f \to \mathcal{M}_f$, such that the diagram below commutes:*

$$
\begin{array}{ccc}
\mathcal{T}_f & \xrightarrow{\ \sigma_f\ } & \mathcal{T}_f \\
\downarrow{\scriptstyle \tilde{p}_f} & & \downarrow{\scriptstyle p_f} \\
\widetilde{\mathcal{M}}_f & \xrightarrow{\ \tilde{\sigma}_f\ } & \mathcal{M}_f
\end{array}
$$

*The Teichmüller norm $\|D_{[\tau]}\sigma_f\|$ of the differential of $\sigma_f$ depends only on the projection $\tilde{p}_f([\tau])$.*

There exists a rational mapping $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is Thurston equivalent to $f$ if and only if there is a fixed point $[\tau_*] = \sigma_f[\tau_*]$. In the absence of a Thurston obstruction, since $\sigma_f^2$ is strictly contracting [3], for any choice of the starting point $[\tau_0] \in \mathcal{T}_f$, the iterates $[\tau_n] \equiv \sigma_f^n([\tau_0])$ converge to $[\tau_*]$ geometrically fast.

Proposition 1.1 and contracting properties of $\sigma_f$ easily imply the following.

PROPOSITION 1.2 (Proposition 5.1 of [3]).   *The iterates $[\tau_n] \equiv \sigma_f^n([\tau_0])$ converge in $\mathcal{T}_f$ to $[\tau_*]$ which is a fixed point of $\sigma_f$ if and only if the sequence $\{p_f[\tau_n]\}$ is pre-compact in $\mathcal{M}_f$.*

Pilgrim showed that a presence of an obstruction implies that the sequence $[\tau_n]$ diverges to infinity in $\mathcal{T}_f$ in the following specific sense.

THEOREM 1.3 ([8]).   *Suppose $f$ is obstructed. Then*
 (i) *there exists a class $[\gamma]$ such that*

$$\ell_{\tau_n}([\gamma]) \to 0;$$

 (ii) *for a non-trivial homotopy class $[\gamma]$ the above property is independent of the starting point $[\tau_0] \in \mathcal{T}_f$;*
 (iii) *the union of all classes $[\gamma]$ as above forms a Thurston obstruction $\Gamma_c$.*

Pilgrim calls $\Gamma_c$ the *canonical* Thurston obstruction. Thus, the existence of an obstruction implies that the canonical obstruction exists (that is, $\Gamma_c \neq \varnothing$).

Pilgrim further showed the next theorem.

THEOREM 1.4 ([8]).   *Let $[\tau_0] \in \mathcal{T}_f$. There exists a constant $E = E([\tau_0])$ such that for every non-trivial simple closed curve $\gamma \notin \Gamma_c$ we have*

$$\inf \ell_{\sigma_f^n \tau_0}([\gamma]) > E.$$

**Pilgrim's decompositions and combinations of Thurston maps.**   What follows is a very brief review; the reader is referred to K. Pilgrim's book [9] for details. We adhere to the notation of [9], for ease of reference.

As a motivation, consider that for the canonical Thurston obstruction $\Gamma_c \ni \gamma$, there is a choice of complex structure $\tau$ for which $\ell_\tau([\gamma])$ is arbitrarily small, and remains small under pull-back by $f$. It is thus natural to think of the punctured sphere $S^2 \setminus P_f$ as pinching along the homotopy classes $[\gamma] \in \Gamma_c$; instead of a single sphere we then obtain a collection of spheres, interchanged by a map $f$.

More specifically, let $f$ be a Thurston map, and $\Gamma = \bigcup \gamma_j$ an $f$-stable multi-curve. Consider also a finite collection of disjoint annuli $A_{0,j}$ whose core curves are the respective $\gamma_j$. For each $A_{0,j}$, consider only non-trivial preimages; these form a collection of annuli $A_{1,k}$ each of which is homotopic to one of the curves in $\Gamma$. Pilgrim says that the pair $(f, \Gamma)$ is in a *standard form* (see Figure 1) if there exists a collection of annuli $A_{0,j}$ as above such that the following properties hold:

(a) for each curve $\gamma_j$ the annuli $A_{1,k}$ in the same homotopy class are contained inside $A_{0,j}$;

(b) moreover, the two outermost annuli $A_{1,k}$ as above share their outer boundary curves with $A_{0,j}$;

(c) finally, restricted to a boundary curve $\chi$ of $A_{0,j}$, the map $f\colon \chi \to f(\chi)$ is, up to a homeomorphic change of coordinates in the domain and the range, given by $z \mapsto z^d\colon S^1 \to S^1$, for some $d \geq 1$.
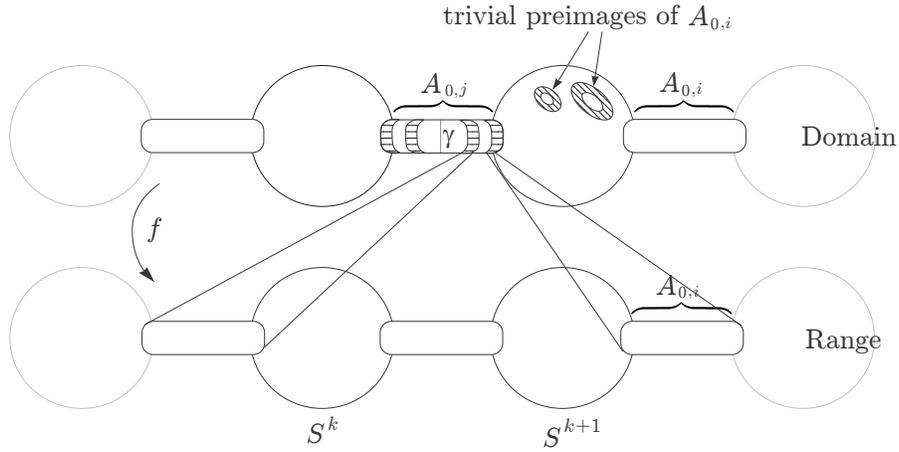


*Figure 1*: Pilgrim's decomposition of a Thurston mapping.

A Thurston map with a multicurve in a standard form can be decomposed as follows. First, all annuli $A_{0,j}$ are removed, leaving a collection of spheres with holes, denoted $S_0(j)$. For each $j$, there exists a unique connected component $S_1(j)$ of $f^{-1}\big(\bigcup S_0(j)\big)$ which has the property $\partial S_0(j) \subset \partial S_1(j)$. Any such $S_1(j)$ is a sphere with holes, with boundary curves being of two types: boundaries of removed annuli, or boundaries of trivial preimages of the removed annuli.

The holes in $S_0(j) \subset S^2$ can be filled using the property (c) above. Namely, let $\chi$ be a boundary curve of a component $D$ of $S^2 \setminus S_0(j)$. Let $k \in \mathbb{N}$ be the first return $f^k\colon \chi \to \chi$, if it exists. For each $0 \leq i \leq k-1$ the curve $\chi_i \equiv f^i(\chi)$ bounds a component $D_i$ of $S^2 \setminus S_0(m_i)$ for some $m_i$. Denote by $d_i$ the degree of $f\colon \chi_i \to \chi_{i+1}$. Select homeomorphisms

$$h_i\colon \bar{D}_i \to \bar{\mathbb{D}} \quad \text{so that } h_{i+1} \circ f \circ h_i^{-1}(z) = z^{d_i}.$$

Set $\tilde{f} \equiv f$ on $\bigcup S_0(j)$. Define new punctured spheres $\tilde{S}(j)$ by adjoining punctured caps $D_i^* \equiv h_i^{-1}(\bar{\mathbb{D}} \setminus \{0\})$ to $S_0(j)$. Extend the map $\tilde{f}$ to each $D_i^*$ by setting

$$\tilde{f}(z) = h_{i+1}^{-1} \circ \big(h_i(z)\big)^d.$$

We have thus replaced every hole by a cap with a single puncture.

By construction, the map

$$\tilde{f}\colon \bigcup \tilde{S}(j) \to \bigcup \tilde{S}(j)$$

contains a finite number of periodic cycles of punctured spheres. For every periodic cycle of spheres, pick a representative $\tilde{S}(j)$, and denote by $\mathcal{F}$ the first return map $f^{k_j}\colon \tilde{S}(j) \to \tilde{S}(j)$. This is again a Thurston map. The collection of maps $\mathcal{F}$ and the combinatorial information required to glue the spheres $S_0(j)$ back together is what Pilgrim calls a *decomposition* of $f$.

Pilgrim shows the following theorem.

THEOREM 1.5. *For every obstructed Thurston map $f$ there exists an equivalent map $g$, whose canonical obstruction we denote $\Gamma_c^g$, such that $(g, \Gamma_c^g)$ is in a standard form, and thus can be decomposed.*

**Statement of the geometrization result.** It is natural to ask whether the canonical decomposition described above has the maximality property: that is, whether the restrictions of the return map $\mathcal{F}$ to spheres $\tilde{S}(j)$, which have topological degrees greater than one, and a hyperbolic orbifold, are *unobstructed*. In view of Thurston's theorem, this would imply that for every such $\tilde{S}(j)$ there is a unique, up to a normalization, rational map $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is equivalent to $\mathcal{F}\colon \tilde{S}(j) \to \tilde{S}(j)$. This constitutes the main conjecture posed by Pilgrim in [9].

In our main result we prove Pilgrim's conjecture. Thus, Pilgrim's decomposition of an obstructed Thurston mapping canonically breaks it into unobstructed, and thus *geometrizable*, pieces.

MAIN THEOREM. *Let $\mathcal{F}\colon \bigcup \tilde{S}(j) \to \bigcup \tilde{S}(j)$ be given by Pilgrim decomposition of an obstructed Thurston map along the canonical obstruction. For every $j$ such that $\mathcal{F}\colon \tilde{S}(j) \to \tilde{S}(j)$ has a topological degree greater than 1 and a hyperbolic orbifold, there does not exist a Thurston obstruction in $\tilde{S}(j)$.*

## 2.  Preliminaries.

*Some notation.* Let $\tilde{S}(j)$ be as in the statement of the main theorem. It is obtained by adding caps $D_i$ to a sphere with holes $S_1(j) \subset S$. The restriction of the first return map $\mathcal{F}$ to $\tilde{S}(j)$ is an iterate $\tilde{f}^N$, so that we have a cycle of punctured spheres

$$(2.1) \qquad S^0 \equiv \tilde{S}(j) \xrightarrow{\tilde{f}} S^1 \xrightarrow{\tilde{f}} \cdots \xrightarrow{\tilde{f}} S^{N-1} \xrightarrow{\tilde{f}} S^0.$$

*Collar Lemmas.* Let us denote by $s(x)$ the function

$$s(x) = \sinh^{-1}\big(1/\sinh(x/2)\big).$$

Note that $s(x)$ decreases from $\infty$ to $0$ as $x$ increases from $0$ to $\infty$. The *collar* around a simple closed hyperbolic geodesic $\gamma$ on a hyperbolic Riemann surface $W$ is the neighborhood

$$\mathcal{C}(\gamma) \equiv \big\{ z \in W \mid d_W(z, \gamma) < s\big(\mathrm{length}_W(\gamma)\big) \big\}.$$

The following is known as Collar Lemma (*cf.* [2]).

THEOREM 2.1 (Collar Lemma for closed geodesics). *The collar $\mathcal{C}(\gamma)$ is an annulus. Further, if $\gamma$ and $\delta$ are two disjoint simple closed geodesics on a hyperbolic Riemann surface $W$, then*

$$\mathcal{C}(\gamma) \cap \mathcal{C}(\delta) = \varnothing.$$

We also recall a limiting version of Collar Lemma for cusps (see [5]).

LEMMA 2.2 (Collar Lemma for cusps). *Let us denote $\mathcal{C}$ the quotient of the region*

$$\{ z : \mathrm{Im}(z) > 1 \} \subset \mathbb{H}$$

*by the translation $z \mapsto z + 2$. For every cusp $\kappa$ of a hyperbolic surface $X$ there is an isometry*

$$\iota_\kappa \colon \mathcal{C} \to C(\kappa)$$

*between $\mathcal{C}$ and a collar neighborhood $C(\kappa) \subset X$. Furthermore, the collars about different cusps are disjoint, and $C(\kappa)$ is disjoint from the collar $C(\gamma)$ about any simple geodesic $\gamma$ on $X$.*

For $\Delta > 1$ we denote $\mathcal{C}_\Delta \subset \mathcal{C}$ the quotient of $\{ z : \mathrm{Im}(z) > \Delta \} \subset \mathbb{H}$ by $z \mapsto z + 2$, and call

$$C_\Delta(\kappa) \equiv \iota_\kappa(\mathcal{C}_\Delta)$$

the $\Delta$-*neighborhood of the cusp $\kappa$.*

*Teichmüller spaces.* Here we describe a key technical tool that we will use, which is due to Y. Minsky (see [7]).

Let $S$ be a surface of finite type, and $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a system of disjoint, homotopically distinct, non-trivial simple closed curves on $S$. Then we denote by $\mathrm{Thin}_\epsilon(S, \Gamma)$ the *thin part* of the Teichmüller space of $S$ given by

$$\mathrm{Thin}_\epsilon(S, \Gamma) = \{ [\tau] \in \mathcal{T}_s \text{ for which } \sup_i \ell_\tau([\gamma_i]) < \epsilon \}.$$

Denote $S_\Gamma$ the multiply connected surface obtained from $S$ by decomposing along the collection $\gamma_i$ (that is, cutting along $\gamma_i$, and capping every hole by a punctured disk). There is a natural homeomorphism

$$\Pi_\Gamma \colon \mathcal{T}_S \to \mathcal{T}_{S_\Gamma} \times \mathbb{H}_1 \times \cdots \times \mathbb{H}_k$$

defined as follows. First, we complete the family $\Gamma$ into a larger family of curves $\hat{\Gamma}$ decomposing $S$ into pairs of pants (possibly degenerate). Recall the definition of Fenchen–Nielsen coordinates

$$\big(\ell_\tau(\gamma), \mathrm{Twist}_\tau(\gamma)\big)_{\gamma \in \hat{\Gamma}} \quad \text{on } \mathcal{T}_S,$$

given by the homotopy classes of curves in $\hat{\Gamma}$. The projection $\mathcal{T}_S \mapsto \mathcal{T}_{S_\Gamma}$ is obtained by restriction of these coordinates to $\hat{\Gamma} \setminus \Gamma$. For each $\gamma_i \in \Gamma$, the map $\Pi_\Gamma$ sends

$$\big(\ell_\tau([\gamma_i]), \mathrm{Twist}_\tau([\gamma_i])\big) \longrightarrow \Big(\frac{1}{\ell_\tau([\gamma_i])}, \mathrm{Twist}_\tau([\gamma_i])\Big) \in \mathbb{H}_i.$$

   Minsky's result gives a distortion bound on this homeomorphism for small enough values of $\epsilon$.

THEOREM 2.3 ([7]).   *Endow the product $\mathcal{T}_{S_\Gamma} \times \mathbb{H}_1 \times \cdots \times \mathbb{H}_k$ with the maximum of the Teichmüller distances on the connected components of $\mathcal{T}_{S_\gamma}$, and the hyperbolic distances on each copy of $\mathbb{H}$. Then, for all $\epsilon > 0$ sufficiently small, the homeomorphism $\Pi_\Gamma$, restricted to $\mathrm{Thin}_\epsilon(S, \gamma)$, distorts distances by a bounded additive amount.*

*Decompositions.* To fix the ideas, for the remainder of this paper, the decomposition of a Riemann surface $S$ with a canonical multicurve $\Gamma_c$ on the level of Teichmüller spaces will be carried out by restricting the Fenchen–Nielsen projection

$$(2.2) \qquad\qquad \Pi_\Gamma \colon \mathcal{T}_S \to \mathcal{T}_{S_{\Gamma_c}} \times \mathbb{H}_1 \times \cdots \times \mathbb{H}_k$$

to the first coordinate, by post-composing with the forgetful map

$$(2.3) \qquad\qquad \mathcal{T}_{S_{\Gamma_c}} \times \mathbb{H}_1 \times \cdots \times \mathbb{H}_k \longrightarrow \mathcal{T}_{S_{\Gamma_c}}.$$

*Controlling hyperbolic lengths.* The following facts will help us in bounding the changes to hyperbolic lengths, and complex structures, when decomposing Riemann surfaces. First, we note a standard consequence of the comparison of the hyperbolic metric on a connected subdomain $\Omega$ of $\hat{\mathbb{C}}$ with $\frac{1}{d(z, \partial\Omega)}|dz|$ (see, e.g., [6]).

LEMMA 2.4.   *Let $\Omega$ denote the Riemann sphere $\hat{\mathbb{C}}$ with $p$ punctures ($p \geq 3$) with its Poincaré metric $\rho_\Omega$, and let $\gamma$ be a non-peripheral curve. For any constant $\epsilon > 0$ there exists $\delta > 0$ so that the following holds. Let $D_i$ be a collection of disks around the punctures with spherical radii at most $\delta$. Then*

$$\mathrm{length}_{\Omega \setminus (\bigcup D_i)}(\gamma) < \mathrm{length}_\Omega(\gamma) + \epsilon.$$

   Furthermore, we have the following lemma.

LEMMA 2.5. *For every $\epsilon > 0$, there exists $M > 0$ such that the following holds. Suppose $S$ is the Riemann sphere with $p \geq 3$ punctures, $A \subset S$ is an annulus such that $\mathrm{mod}(A) > M$, and $S_1$ and $S_2$ are the two connected components of $S \setminus A$. Then, for every pair of points $a, b$ in $S_1$, we have*

$$d_{S \setminus S_2}(a, b) < d_S(a, b) + \epsilon.$$

Finally, we state (*cf.* [4]):

LEMMA 2.6. *Let $X$ be a finite-type hyperbolic surface and denote $\sigma_0$ the standard complex structure in $X$.*
  (i) *Let $x \in X$, and $\mathcal{U} \subset \mathcal{T}_X$ any neighborhood of $[\sigma_0]$ in the Teichmüller space of $X$. Denote by $B_r(x)$ the hyperbolic ball with radius $r > 0$ around $X$. There exists $r_0 > 0$ such that for any complex structure $\sigma$ on $X$ with $\sigma \equiv \sigma_0$ outside $B_r(x)$ we have $[\sigma] \in \mathcal{U}$.*
  (ii) *Let $\kappa$ be a cusp in $X$, and $\mathcal{U} \subset \mathcal{T}_X$ any neighborhood of $[\sigma_0]$ in the Teichmüller space of $X$. There exists a cusp neighborhood $C_\Delta(\kappa)$ such that, for any complex structure $\sigma$ on $X$ with $\sigma \equiv \sigma_0$ outside $C_\Delta(\kappa)$, we have $[\sigma] \in \mathcal{U}$.*

*Combinations of punctured surfaces.* For future reference, let us briefly describe the inverse of the above decomposition procedure. Pilgrim's *combination* applied to $\bigcup \tilde{S}(j)$ involves connecting the punctured spheres by annuli to reverse the decomposition construction. The end result is a branched covering which is equivalent to $f \colon S^2 \to S^2$. Let $S_1 \equiv \tilde{S}(j_1)$ and $S_2 \equiv \tilde{S}(j_2)$ be two punctured spheres to be combined at the pair of punctures $p_1 \in S_1$ and $p_2 \in S_2$. We will glue in an annulus by removing a small topological disk $D_i \subset S_i$ around each of the punctures $p_i$, and attaching the boundary curves of the annulus to the boundary circles $\partial D_i$. We again would like to perform the gluing in a hyperbolically rigid fashion.

Let $W_1, \ldots, W_N$ be a finite collection of punctured spheres, each one being a hyperbolic Riemann surface. For each puncture $p_j \in W_1 \sqcup \cdots \sqcup W_N$ we choose a disk around it, $D_{p_j}$, that contains only the puncture $p_j$.

DEFINITION 2.1 (Admissible pairing of caps). We say that a finite family of pairs $(p_i, p_j)$ is *an admissible pairing of punctures* if
  (i) for each pair $(p_i, p_j)$ the punctures $p_i$ and $p_j$ belong to distinct spheres;
  (ii) the connected sum $W_1 \# \cdots \# W_N$ obtained by gluing along the pairs $(p_i, p_j)$ is a punctured sphere with at least three punctures.

We can now describe combining $S_1$ and $S_2$, using Lemma 2.5.

LEMMA 2.7. *Let us consider two hyperbolic punctured spheres $S_1, S_2$ each one with a distinguished puncture $p_1 \in S_1$ and $p_2 \in S_2$. For every $\epsilon > 0$ there exists $\Delta > 1$ such that the following holds. Let $D_1 \ni p_1$, $D_2 \ni p_2$ be two topological disks in the respective $\Delta$-neighborhoods of $p_1$, $p_2$ respectively. Let $S$ be a combination of the surfaces $S_1' \equiv S_1 \setminus D_1$ and $S_2' \equiv S_2 \setminus D_2$. That is, there exist conformal embeddings*

$$\chi_i \colon S_i' \to S$$

*with disjoint images, such that*

$$A \equiv S \setminus \big( \chi_1(S_1') \cup \chi_2(S_2') \big)$$

*is an annulus with a core curve $\gamma$. Denote $\hat{S}_1$, $\hat{S}_2$ the decomposition of $S$ along $\Gamma = \{\gamma\}$. Then*

- *$\mathrm{length}_S(\gamma) < \epsilon$, and*
- *$d_{\mathcal{T}_{S_i}}(\hat{S}_i, S_i) < \epsilon$.*

By applying the above statement repeatedly, we obtain the following.

LEMMA 2.8.   *Let $S_1, \ldots, S_N$ be a collection of hyperbolic punctured spheres together with an admissible pairing of punctures $\{(p_{i_1}, p_{j_1}), \ldots, (i_K, p_{j_K})\}$.   For every $\epsilon > 0$ there exists $\Delta > 1$ such that the following holds. Let $D_i \subset C_\Delta(p_i)$ be a topological disk around $p_i$ and let $S_i' \equiv S_i \setminus \bigcup(D_j)$. Then the surfaces $S_i$ can be combined into a hyperbolic surface $S$ according to the above pairing of punctures so that*

- *each $S_i'$ is conformally embedded into $S$ with complementary annuli $A_{p_i,p_j}$;*
- *the length of the core curve $\gamma_{i,j}$ of each $A_{p_i,p_j}$ is less than $\epsilon$;*
- *denoting by $\bigcup \hat{S}_i$ the decomposition of $S$ along $\Gamma = \bigcup \gamma_{i,j}$, we have*

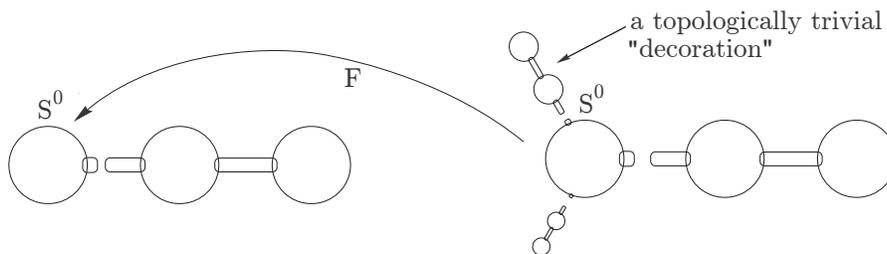$$d_{\mathcal{T}_{S_i}}(S_i, \hat{S}_i) < \epsilon$$

*for all $i$.*



*Figure 2*: Pull-back of a decomposition.

**Idea of the proof of the Main Theorem.** When we pull back a complex structure $\tau_0$ by $f$, only the geodesics of the canonical restriction become arbitrarily short. If we restrict to a subsurface $S_1(j)$, then, with respect to the structure

$$\tau_n \equiv \sigma_f^n \tau_0,$$

all simple closed curves have hyperbolic length bounded from below independently of $n$ (Theorem 1.4). However, if we restrict $\tau_0$ to a complex structure $\mu_0$ on $\tilde{S}_0(j)$, and pull back by $\sigma_{\mathcal{F}}$, the result may *a priori* be different. As illustrated in Figure 2, the restrictions of $\tau_n$ to $S_0(j)$ will include "decorations": trivial preimages of the other subsurfaces. They are attached by very thin tubes, and will not create much difference for the first few pull-backs. However, without some uniform distortion bound on the change in the complex structure caused by restriction to the subsurface, we will not be able to control their impact on the length of geodesics. Here Minsky's Theorem 2.3 will be a key tool for us, as it produces a uniform bound on the change in complex structure caused by a decomposition.

3. **Proof of the Main Theorem.** Let us start by fixing $[\tau_0] \in \mathcal{T}_f$, and setting $\tau_n \equiv \sigma_f \tau_0$. Let $E = E(\tau_0)$ be given by Theorem 1.4. Further, let us consider the first return map

$$\mathcal{F} \colon \bigcup \tilde{S}(j) \to \bigcup \tilde{S}(j),$$

given by Pilgrim's decomposition.

Each $\tilde{S}(j)$ in the domain of $\mathcal{F}$ is a sphere with a finite number of punctures $P_j \subset \tilde{S}(j)$, the map $\mathcal{F} \colon \tilde{S}(j) \to \tilde{S}(j)$ is a Thurston map (2.1), and the postcritical set $P_{\mathcal{F}} \cap \tilde{S}(j) \subset P_j$. By construction, the extra punctures $P_j \setminus P_{\mathcal{F}}$ correspond to cuts in the Pilgrim's decomposition.

Let $S^0 = \tilde{S}(i)$ be as in the statement of the Main Theorem. The *period* of $S^0$ is the number $N$ of the punctured spheres in the cycle (2.1). By analogy with the previously adopted notation, we will denote by $\mathcal{T}_{\mathcal{F}}$ the Teichmüller space of $S^0$, and $\mathcal{M}_{\mathcal{F}}$ the moduli space. We further let

$$\mathcal{T}_{\mathcal{F}} \xrightarrow{\tilde{p}_{\mathcal{F}}} \widetilde{\mathcal{M}_{\mathcal{F}}} \xrightarrow{\bar{p}_{\mathcal{F}}} \mathcal{M}_{\mathcal{F}}$$

be as in Proposition 1.1.

Finally, for the Fenchen–Nielsen decomposition (2.2) we denote by $\pi_{\mathcal{T}_{\mathcal{F}}}$ the forgetful map

$$\pi_{\mathcal{T}_{\mathcal{F}}} \colon \mathcal{T}_{S_{\Gamma_c}} \times \mathbb{H}_1 \times \cdots \times \mathbb{H}_k \longrightarrow \mathcal{T}_{\mathcal{F}}.$$

We first prove the following lemma.

LEMMA 3.1. *There exists a compact set $K \Subset \widetilde{\mathcal{M}_{\mathcal{F}}}$ with the following property. Consider the decomposition of $(S, \tau_n)$ given by*

$$\mu_n \equiv \pi_{\mathcal{T}_{\mathcal{F}}} \circ \Pi_{\Gamma_c}([\tau_n]) \in \mathcal{T}_{\mathcal{F}}.$$

*Let $\tilde{\mu}_n \equiv \tilde{p}_{\mathcal{F}}(\mu_n) \in \widetilde{\mathcal{M}_{\mathcal{F}}}$. Then for all $n \in \mathbb{N}$ we have the projections $\tilde{\mu}_n \in K$.*

PROOF.    As before, let $\Gamma_c$ stand for the canonical obstruction of $f$. By Theorem 1.3, for every $\epsilon > 0$ there exists $n_0$ such that for all $n > n_0$ we have

$$[\tau_n] \in \mathrm{Thin}_\epsilon(S, \Gamma_c).$$

By the Collar Lemma, and Lemma 2.4, for every $\epsilon_1 > 0$ there exists $n_1 \in \mathbb{N}$ such that for every essential curve $\gamma \subset S_0(j)$

$$\|\ell_{\tau_n}([\gamma]) - \ell_{\tau_n|S_0(j)}([\gamma])\| < \epsilon_1.$$

By Theorem 1.4, for every simple closed curve $\gamma \subset \bigcup S_0(j)$ such that
(a) $[\gamma]$ is non-trivial in $S$, and
(b) $[\gamma] \notin \Gamma_c$,
we have $\ell_{\tau_n}([\gamma]) > E$. Thus, by the Mumford Compactness Theorem, the projections

$$p_{\mathcal{F}} \circ \pi_{\mathcal{T}_{\mathcal{F}}} \circ \Pi_{\Gamma_c}(\tau_n) = \tilde{p}_{\mathcal{F}}(\mu_n)$$

lie in a compact subset $\hat{K}$ of $\mathcal{M}_{\mathcal{F}}$. Since $\bar{p}_{\mathcal{F}}$ is a finite covering, the set

$$K \equiv (\bar{p}_{\mathcal{F}})^{-1}(\hat{K}) \supset \{\tilde{\mu}_n\}$$

is a compact subset of $\widetilde{\mathcal{M}}_{\mathcal{F}}$.                                                 $\square$

Let us continue using the notation

$$\mu_n \equiv \pi_{\mathcal{T}_{\mathcal{F}}} \circ \Pi_{\Gamma_c}([\tau_n]) \in \mathcal{T}_{\mathcal{F}}.$$

We prove the following.

PROPOSITION 3.2.    *Let $N$ be the period of $S^0$. There exists $\epsilon > 0$ such that the following holds. For every $m \in \mathbb{N}$ there exists $k_0 = k_0(m)$ such that for all $k \geq k_0$ the distance*

$$d_{k,m} \equiv d_{\mathcal{T}_{\mathcal{F}}}\left([\sigma_{\mathcal{F}}^m(\mu_k)], \left[\pi_{\mathcal{T}_{\mathcal{F}}} \circ \Pi_{\Gamma_c}\left(\sigma_{f^N}^m(\tau_k)\right)\right]\right) < \epsilon.$$

PROOF.    Recall that the surface $S^0$ is obtained by capping the surface with holes $S_1(j) \subset S_0(j)$.
Let us *combine* the truncated surface

$$S' \equiv \left(S \setminus S_0(j), \sigma_{f^N}^m(\tau_k)\right)$$

with the surface

$$\left(S^0, \sigma_{\mathcal{F}}^m(\mu_k)\right).$$

Using Lemma 2.7, we can combine the two surfaces into a surface $(S, \lambda_{k,m})$ of the same topological type as $S$, with the only difference being the complex structure on the component $S_0(j)$, which is now given by $\sigma_{\mathcal{F}}^m(\mu_k)$.

By Minsky's Theorem, for large enough $k$, up to a bounded additive error, the Teichmüller distance between $(S, \lambda_{k,m})$ and $(S, \sigma_{f^N}^m(\tau_k))$ is given by the distance $d_{k,m}$.

By the Collar Lemma and Lemma 2.6,

$$d_{\mathcal{T}_f}\Big([(S, \lambda_{k,m})], \big[(S, \sigma_{f^N}^m(\tau_k))\big]\Big) \xrightarrow[k \to \infty]{} 0.$$

By Minsky's Theorem 2.3, and Lemma 3.1, $d_{k,m}$ is uniformly bounded for large values of $k$. $\qquad\square$

Fix $\epsilon > 0$ as in Proposition 3.2, and let $K \Subset \widetilde{\mathcal{M}_{\mathcal{F}}}$ be as in Lemma 3.1. We can select a subsequence of the natural numbers $\{n_j\}$ with the property

$$\tilde{\mu}_{n_j} \to \tilde{\alpha} \in \widetilde{\mathcal{M}_{\mathcal{F}}}.$$

We will endow $\mathcal{M}_{\mathcal{F}}$ and $\widetilde{\mathcal{M}_{\mathcal{F}}}$ with the distance induced by the Teichmüller metric on $\mathcal{T}_{\mathcal{F}}$. Let $\bar{\alpha} = \bar{p}_{\mathcal{F}}(\tilde{\alpha}) \in \mathcal{M}_{\mathcal{F}}$, and consider a ball $B$ of radius $3\epsilon$ around $\bar{\alpha}$. Denote

$$\tilde{B} \equiv (\bar{p}_{\mathcal{F}})^{-1}(B) \Subset \widetilde{\mathcal{M}_{\mathcal{F}}}.$$

By Proposition 1.1, there exists $s \in (0,1)$ such that for every $[\tau] \in \tilde{p}_{\mathcal{F}}^{-1}(\tilde{B})$ the Teichmüller norm of $D_{[\tau]}\sigma_{\mathcal{F}}$ is bounded by $s$. Hence, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ the following holds. Let $\tilde{\zeta} \in \tilde{B}$ and let $[\zeta] \in \tilde{p}_{\mathcal{F}}^{-1}(\tilde{\zeta})$ realize the minimum of the Teichmüller distance from $[\mu_{n_j}]$. Let $m \in \mathbb{N}$ be such that $n_j + m \geq n_{j+k}$. The distance

$$d_{\mathcal{T}_{\mathcal{F}}}\big(\sigma_{\mathcal{F}}^m([\zeta]), [\mu_{n_j+m}]\big) < s^k d_{\mathcal{T}_{\mathcal{F}}}([\zeta], [\mu_{n_j}]).$$

Let $j$ be such that $\tilde{\mu}_{n_{j+k}}$ is $\frac{s}{3}$-close to $\tilde{\alpha}$ for all $k \geq 0$. Let us choose $k$ so that for $m = n_{j+k} - n_j$ we have $s^m < 1/3$. By Proposition 3.2, there exists a branch of $\tilde{\sigma}_{\mathcal{F}^m}$ which maps $\tilde{B}$ compactly inside $B$. Hence, there exists $[\lambda] \in p_{\mathcal{F}}^{-1}(B)$ such that $[\lambda]$ and $\sigma_{\mathcal{F}^m}([\lambda])$ lie over the same point in $B$. Thus, $\mathcal{F}^m$ is Thurston equivalent to a rational map. By Pilgrim's Theorem 1.3, the map $\mathcal{F}$ is not obstructed, and the proof of the Main Theorem is completed.

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