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K-Theory and Traces

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Abstract. It is shown that for a unital C*-algebra, what is sometimes referred to as the Elliott invariant—loosely speaking, K-theory and traces—i.e., the order-unit K_0-group, the K_1-group, and the trace simplex, paired in the natural way with K_0, can be expressed purely in terms of K-theory, with the trace simplex and its pairing with K_0 recoverable in a simple way (using polar decomposition) from algebraic K_1, defined as in the purely algebraic context using invertible elements rather than just unitaries.

Résumé. L'invariant naïf d'Elliott, qui est à la base de la classification complète récente d'une énorme classe de C*-algèbres simples (celles qui sont de dimension nucléaire finie, qui sont séparables, et qui satisfont à l'UCT), peut s'exprimer entièrement dans le cadre de K-théorie algébrique.

1. It is a consequence of the Hahn-Banach theorem that the dual of the Banach space quotient of a (real or complex) Banach algebra by the closed linear subspace generated by the additive commutators (elements $[a, b] = ab - ba$) is (isometrically) isomorphic to the Banach space of continuous linear functionals on the algebra which are zero on additive commutators—namely, bounded traces.

In the case of a (complex) C*-algebra, it is customary to consider positive traces, which form a cone, whether they are finite-valued—in which case they are necessarily bounded—, or extended positive real-valued, as considered for example in [8].

In the case of a simple C*-algebra, the cone of positive traces that are densely defined (finite on some non-zero positive element), together with the norm function (possibly infinite-valued) is an important part of the invariant for the (now) complete classification of well-behaved (i.e., separable, finite nuclear dimension, UCT) simple C*-algebras—as is the pairing of this cone with K_0 (see Theorem 8 and Corollary 9, below). (See [16], [19], [12], [13], [7], [4], [23], [5], [6], [10], and [11]; see also [1] and [3] as the amenable C*-algebra and von Neumann algebra classifications are closely related.)

Again in the case of a simple C*-algebra, the cone of densely defined traces—or, rather, the lower semicontinuous regularizations of these—is isomorphic to the cone of bounded (i.e., continuous) traces on any non-zero hereditary sub-C*-algebra of the Pedersen ideal—of course, not in a way preserving the norm function, which may be considered as additional data, but in any case preserving

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the natural topology (pointwise convergence on the Pedersen ideal—for the traces in question, this coincides with the topology of [8]).

Once one restricts to bounded traces, one might as well look at the whole linear space of these (as is true for arbitrary bounded functionals—see [14]—there is a canonical linear decomposition into positive ones). Then, as recalled above, one might just as well look at the quotient of the algebra itself by the closed subspace generated by the additive commutators. (As long as one keeps track of the weak* topology on a Banach space arising from a given predual, or even just the restriction of this to the unit ball, the predual is uniquely determined.)

In [15] and [22], the Hausdorffized algebraic K_1-group of a Banach algebra (with the inductive limit topology from finite matrix algebras—a subset open if open at each stage) was shown to be determined by this quotient Banach space, together with Banach algebra K_0 and K_1. (To be historically precise, for the part of the calculation carried out in [22], only the C*-algebra setting was considered, and so the full statement only obtained for the version of algebraic K_1 involving unitaries rather than invertibles, but the proof is valid in the Banach algebra context, with invertible elements.) (See Theorem 2, Corollary 4, and Remark 6, below.)

In the present note, in an application of the remarkable results of Robert in [20] concerning the exponential map (involving a careful analysis of the Baker-Campbell-Hausdorff theorem), an analogous calculation of the entire (in general non-Hausdorff) algebraic K_1-group of a Banach algebra (equivalently, of its connected component) is obtained (Theorem 2 and Corollary 4, below), from which the determination of the Hausdorffized algebraic K_1 of [15] and [22] follows immediately (Remark 6, below).

In the case of a C*-algebra, not only, as shown in [22] in the Hausdorffized setting, does the skew-self-adjoint part of the algebra give rise via the exponential map (after unitization, if necessary) to a description of (the connected component) of the unitary analogue of the purely algebraic K_1-group (based on invertible elements), but also, the self-adjoint part of the algebra, again directly through exponentiation, gives rise to a description of a complementary subgroup of the connected component of the purely algebraic K_1-group, namely, the image in the K_1-group (the abelianization of the group of all invertible matrices over the algebra) of the subgroup generated by invertible positive elements of the algebra (matrices not necessary!). This result (Theorem 5, below) holds in both the non-Hausdorff and Hausdorffized settings, and in the latter setting amounts to a natural embedding of the tracial data of a C*-algebra—more precisely, the topological group of continuous real-valued affine functions on the cone of bounded positive traces, with addition, and uniform convergence on bounded sets—in the K-theory data—more precisely, as a direct summand of the connected component of the topological group of invertible elements of the algebra, modulo its commutator subgroup—matrices are not necessary in this case!—which may be characterized as the subgroup of self-adjoint elements of this group with respect to the natural involutive structure inherited from the C*-algebra. (The subgroup of skew-self-adjoint elements of this involutive abelian group, a complementary subgroup of the connected component, is isomorphic to the topological group of purely imaginary-valued continuous affine functions (zero at zero) on the cone of bounded positive traces, modulo the closure of the natural image of K_0—i.e., the subgroup corresponding to $2\pi i$ times the set of projections in matrix algebras.)

(The pairing between K_0 and traces is recoverable from the natural map into the self-adjoint direct summand of this involutive abelian group, via the usual map of K_0 into real-valued continuous affine functions on the tracial cone of the C*-algebra.) (This is assuming a unit, but the non-unital case, with just densely defined traces, is not far behind—see Corollary 9, below.)

2. Theorem. *Let A be a unital complex Banach algebra. The exponential map, from A to the connected component of the group of invertible elements of A,*

$$A \ni a \mapsto e^a \in (A^{-1})_0,$$

gives rise to an isomorphism between the topological groups consisting of, on the one hand, the additive group A modulo the subgroup generated by the additive commutators in A, together with $2\pi i$ times the sum of the diagonal matrix entries of the idempotents in $M_n(A)$, $n = 1, 2, 3\ldots$, and, on the other hand, the multiplicative group $(A^{-1})_0$ modulo its commutator subgroup—or, rather, modulo the commutator subgroup of $(M_n(A)^{-1})_0$, i.e., that part of this which lies in the image of $(A^{-1})_0$ in the upper left-hand corner, $n = 1, 2, 3, \ldots$ (if this is larger)—the topology in both cases being the quotient topology arising from the norm metric on A. (The proof, although not the statement, refers to the natural extension of this topology to $M_n(A)$ but not to any particular norm.)

Proof. The theorem is a simple consequence of the remarkable algebraic analysis in [20] of the Baker–Campbell–Hausdorff theorem (as made more explicit by Dynkin and others), together with the Bott periodicity theorem.

By Lemma 5.3 of [20] (note that the proof of this lemma, stated in the case of a C*-algebra, is valid in a Banach algebra, as the proof of Theorem 5.2 of [20] is), the map $a \mapsto e^a$ is a group homomorphism from A to $(A^{-1})_0$ modulo (multiplicative) commutators, i.e., gives rise to a homomorphism from A into the abelianization of the group $(A^{-1})_0$.

By Theorem 5.4(a) of [20] (also valid for a Banach algebra, as it is deduced directly from 5.2 and 5.3), the map $a \mapsto e^a$ takes additive commutators in A into products of (multiplicative) commutators in $(A^{-1})_0$, i.e., gives rise to a homomorphism of the (additive) group A modulo additive commutators into the abelianization of $(A^{-1})_0$.

Let p be an idempotent element of $M_n(A)$ for some $n = 1, 2, \ldots$. Then $e^{2\pi i p} = 1_n$, as is seen by the same arithmetic as in the case $p = 1 \in \mathbb{C}$, or just by realizing that $e^{2\pi i p} - 1_n \in \mathbb{C}p \subseteq pAp$. (Here, 1_n denotes the unit of $M_n(A)$.) It is an elementary fact (well known—the zeroth level of Morita invariance of cyclic, or Hochschlld, homology—seen by just considering the (additive) commutators of a given matrix with the matrix units—see next paragraph) that, modulo additive commutators in $M_n(A)$, every element of $M_n(A)$ is equal to the sum of its diagonal entries, an element of A, considered, say, as embedded in the upper left-hand corner. Hence by Lemma 5.3 and Theorem 5.4 of [20]—as refined slightly in the next paragraph—, the exponential of every element of $M_n(A)$ is equal modulo commutators in $(M_n(A)^{-1})_0$ to the exponential of the sum of its diagonal entries, an element of $(A^{-1})_0$, considered, say, as embedded in the upper left-hand corner. In particular, the exponential of the sum of the diagonal entries of $2\pi ip$ (since $e^{2\pi ip} = 1_n$), as an element of $(A^{-1})_0$ in the upper left-hand corner, is a product of commutators in $(M_n(A)^{-1})_0$, as asserted.

(To see that an element of $M_n(A)$ with diagonal entries equal to zero (or to a sum of commutators in A) is a sum of additive commutators, it is enough to consider, first, the product of an element a of A and a single matrix unit, which is the commutator of two matrix units, one of them times a, and, second, the product of $a \in A$ and the difference of two diagonal matrix units, which again is the commutator of two matrix units, one of them times a. The converse is also true although we don't need it: the sum of the diagonal entries of a commutator in $M_n(A)$ is a sum of commutators in A. Again, by additivity, it is enoough to consider the case of the commutator of two products: an element $a \in A$ times one matrix unit, and an element $b \in A$ times a second matrix unit; in every case the sum of the diagonal entries is the commutator $[a, b] \in A$.)

Note that Lemma 5.3 and Theorem 5.4 of [20] are indeed stated for the commutator subgroup of $(A^{-1})_0$, not of A^{-1}. In fact, the proofs of these two results show that these two commutator subgroups are equal. Indeed, the proof of Lemma 5.3 shows that the conclusion of Lemma 5.3 is true with the smaller group. In the proof of Theorem 5.4, just replace the equation $[A, A] = [(A^{-1})_0, A]$ by $[A, A] = [A^{-1}, A]$, noting that $a + 2\|a\| \in (A^{-1})_0$, $a \in A$.

It remains to prove that the map between the stipulated quotients of the groups A and $(A^{-1})_0$ thus obtained (by simply exponentiating elements of A) is an isomorphism of topological groups. It is clearly continuous (as the exponential map $A \to (A^{-1})_0$ carried through to quotients with the quotient topology). It is also clearly surjective, as every element of $(A^{-1})_0$ is a finite product of exponentials (these form a connected subgroup, open because any element close enough to 1 has a logarithm by the holomorphic functional calculus)—and at the level of quotients the map is a group homomorphism.

To prove injectivity, let $a \in A$ be such that e^a belongs to the subgroup of $(A^{-1})_0$ that we are dividing out by—the subgroup generated by the commutators in $(M_n(A)^{-1})_0$ that belong to $(A^{-1})_0$ (embedded in the upper left corner)—it seems not to be known if these are in fact commutators in $(A^{-1})_0$. (Note that, as shown for instance in [17], purely algebraically, the commutator subgroup of A^{-1}, as opposed to (although equal to) that of $(A^{-1})_0$, is equal to the intersection with A^{-1} of the subgroup of $(M_2(A)^{-1})_0$ generated by transvections (elementary matrices). Thus if we look at the union of $A \subseteq M_2(A) \subseteq \cdots$, and consider the union of the subgroups generated by commutators whether in $M_n(A)^{-1}$ or in $(M_n(A)^{-1})_0$, or by transvections, we see that all three of these subgroups are the same, and define the abelianization of the union of $M_n(A)^{-1}$, in other words the algebraic K_1-group $\mathrm{K}_1^{\mathrm{alg}}(A)$ of A.)

We wish to show that a must belong to the subgroup of A that we are dividing out by, i.e., must be zero in the quotient (which, not incidentally, may be described as the quotient of the zeroth level of cyclic homology of A by the canonical image of $\mathrm{K}_0(A)$ by the Connes–Chern character—see [2], [3]).

By Theorem 5.4(a) of [20], e^a, considered as embedded in the upper left matrix corner, is the product of the exponentials of additive commutators of matrices with entries in A. Let us rearrange this equation so that one has $e^{a_1}e^{a_2}\cdots e^{a_k}e^a = 1_n$, where $a_1, a_2, \ldots, a_k$ are additive commutators in $M_n(A)$, and regard this as a closed path in $(M_n(A)^{-1})_0$, beginning with the logarithmically linear segments $[0,1] \ni t \mapsto e^{ta_1}$ and $[0,1] \ni t \mapsto e^{a_1}e^{ta_2}$, and continuing to the last segment $[0,1] \ni t \mapsto e^{a_1}\cdots e^{a_n}e^{ta}$. By Bott periodicity (see, e.g., [21]) this closed path (from $1_n \in M_n(A)^{-1}$ to 1_n) is homotopic through a path of closed paths in $M_{n'}(A)^{-1}$ for some $n' \geq n$, all beginning and ending now at $1_{n'} \in M_{n'}(A)^{-1}$, to a single component closed path

$$[0,1] \ni t \mapsto e^{2\pi i t a'} \in M_{n'}(A)^{-1}.$$

Moreover (as shown in [21]), a' may be chosen to be the difference of two orthogonal idempotents (i.e., with product zero)—interestingly, in any case, as shown in Remark 3, below, necessarily (as $e^{2\pi i a'} = 1_{n'}$), a' is an integral combination of orthogonal idempotents.

To complete the proof of injectivity, again by the Morita invariance of (zeroth level) cyclic homology (see [2], [3]), it is enough to show that a' is equal to a (embedded in the upper left corner of $M_{n'}(A)$) modulo additive commutators in $M_{n'}(A)$—a will then be equal to the sum of the diagonal entries of a'—the zeroth component of the Connes–Chern character of a'—modulo additive commutators from A, as desired.

It is enough to note that the homotopy of paths may be replaced by a new homotopy of paths (close to the known one—but that is immaterial—we do not even need the new a' to be the same!) consisting of a finite sequence of logarithmically piecewise linear small detours (from the given logarithmically piecewise linear closed path). More explicitly, we may clearly refine the given closed path, and the final one, into sufficiently finely divided logarithmically piecewise linear paths (replace a and each a_i by a finite number of small multiples), with the same number of segments, and break up sufficiently many of the intervening closed paths into segments, each approximated by a logarithmically linear segment, to obtain a finite number of logarithmically piecewise linear approximations to closed paths along the homotopy of closed paths, always with the same number of segments, such that each of the minimal piecewise logarithmically linear triangular perturbations required to go in finitely many steps from each of the perturbed closed paths in the homotopy to the next one (close to it) is small enough that it comes under the purview of Theorem 5.1 of [20]: it is close enough to the unit that the logarithm as a holomorphic function takes products into sums modulo additive commutators. Thus, since the product of the logarithmically linear three steps arising from replacing one step—or two adjacent steps—in one closed path by the associated two-step detour—or one-step shortcut—is equal to the unit, the sum of the three logarithms, which is the only change to the sum of the logarithms in the replaced short segment or segments, is zero modulo (matrix) additive commutators. In sum, the sum of a and the matrix additive commutators $a_1, a_2, \ldots, a_k$ (in $M_n(A)$) is equal to a', an integral combination of matrix idempotents, modulo matrix commutators in $M_{n'}(A)$, as asserted.

To verify continuity of the inverse of this (continuous) group isomorphism, which may still be visualized as the exponential map $a \mapsto e^a$, suppose that e^a, when embedded in the upper left corner (with the unit of A elsewhere on the diagonal, as usual), is close to a finite product of commutators in $(M_n(A)^{-1})_0$, which sits (or even possibly does not) in the upper left corner. We must prove that a is close to the specified subgroup of A.

On the principle that continuity of the inverse map should (as in differential equation theory) be closely related to its existence in the first place, we might expect to have to repeat the proof of existence—but in fact we can just use it. Namely, by standard use of the holomorphic functional calculus (in this case the logarithm, in the neighbourhood of $1 \in \mathbb{C}$), there exists a small element b of A such that $e^a e^b$ is (exactly) a finite product of matrix commutators. Then by the homomorphism property of the exponential map modulo commutators (Lemma 5.3 of [21]—of course already referred to), the single exponential e^{a+b} is (exactly) a finite product of (matrix) commutators. By injectivity of the exponential map (in the context of the specified quotients of A and $(A^{-1})_0$), $a+b$ belongs to the specified subgroup of A; that is, a itself is close to this subgroup, as desired.

3. Remark. Let A be a unital complex Banach algebra, and let $a \in A$ be such that $e^{2\pi i a} = 1$. Then a is an integral combination of mutually orthogonal (mutually annihilating) idempotents. (The converse is well known, and reduces easily to the case $A = \mathbb{C}$.) (Although this observation is referred to in the proof of Theorem 2, it is not used in an essential way. It would be interesting to see it actually used, say in a proof of Bott periodicity.)

This is a fairly straightforward consequence of the spectral mapping theorem—which implies immediately that the spectrum of a consists of integers—and the holomorphic functional calculus, which applied with a holomorphic partition of unity on a neighbourhood of the spectrum, extending the canonical (discrete) partition of unity on the spectrum itself, reduces the question to the case that the spectrum is a single integer—or, on subtracting this, that the spectrum is the point 0. Consider now the power series calculation

$$\begin{aligned} 0 = e^a - 1 &= a + \frac{a}{2!} + \frac{a^3}{3!} + \cdots \\ &= a\left(1 + \frac{a}{2!} + \frac{a^2}{3!} + \cdots\right). \end{aligned}$$

Again, by the spectral mapping theorem, the spectrum of the sum $1+\frac{a}{2!}+\frac{a^2}{3!}+\cdots$ is the point $1+0+0+\cdots=1$, and in particular this sum is invertible, whence the factor a on the right-hand side of the equation is 0. (And so before the last reduction a was an integral multiple of the unit, and so the original a a finite integral combination of orthogonal idempotents.)

4. Corollary. *Let A be as in Theorem 2 but without (necessarily having) a unit element. Then what might be called the quasi-exponential map, from A to the group of quasi-invertible elements of A,*

$$A \ni a \mapsto e^a - 1 = a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots \in (A^{-1})_0$$

(the last symbol still, somewhat abusively, denoting the connected component of the group of quasi-invertible elements of A), gives rise to an isomorphism between the two topological groups consisting of, first, the additive group A modulo the subgroup generated by the additive commutators in A, together with $2\pi i$ times the canonical image of $K_0(A)$ in A by the Connes–Chern character (see [2], [3]—the differences of pairs of idempotents in $A^\sim$, the algebra A with unit adjoined, which are equal in the canonical quotient $\mathbb{C}$, and the sums of the diagonal entries of the differences of pairs of idempotents in $M_n(A^\sim)$, $n=1,2,\ldots$, which are equivalent in the canonical quotient $M_n(\mathbb{C})$), and second, the multiplicative group $(A^{-1})_0$, modulo its commutator subgroup—or, rather, modulo the (possibly larger?) subgroup generated by the commutators in $(M_n(A)^{-1})_0$ which belong to the upper left-hand corner, $n=1,2,3,\ldots$—the topologies being those arising from the norm metric on A.

Proof. The conclusion is immediate on adjoining a unit e to A (with the norm, for instance, $\|a+\lambda e\| = \|a\|+|\lambda|,\ \lambda\in\mathbb{C}$), and applying Theorem 2.

5. Theorem. *Let A be a C^*-algebra. The isomorphism of Theorem 2 and Corollary 4 (the non-unital case) arising from exponentiation (the map $a\mapsto e^a$), decomposes as the direct sum of its restriction to the self-adjoint part $A_{\text{s.a.}}$ of A and the skew-self-adjoint part $iA_{\text{s.a.}}$, modulo additive commutators and the sums of the diagonal entries of $2\pi i$ times idempotents (equivalently, projections) in matrix algebras. More precisely, the (internal) direct sum decomposition $A = A_{\text{s.a.}} \dot{+} iA_{\text{s.a.}}$ passes to a direct sum decomposition of the quotient of A by the subgroup specified in Theorem 2 and Corollary 4, which is naturally isomorphic to the external direct sum of, first, the quotient of the group $A_{\text{s.a.}}$ by the subgroup generated by the (self-adjoint) additive commutators $[a,b]=ab-ba$ where a is self-adjoint and b is skew-self-adjoint and, second, the quotient of $iA_{\text{s.a.}}$ by the subgroup generated by the (skew-self-adjoint) additive commutators $[a,b]=ab-ba$ where both a and b are self-adjoint, together with $2\pi i$ times the canonical image of K_0 in A (see statement of Corollary 4). The isomorphism of Theorem 2 and Corollary 4 takes the second quotient, as a topological group, isomorphically* *onto a direct summand of its codomain naturally identifiable as the quotient of the connected component $U(A)_0$ of the group of quasi-unitary elements of A by its commutator subgroup—or, rather, as before, by the (possibly larger?) subgroup of $U(A)_0$ (the connected component of the group of quasi-unitaries), considered as embedded in the upper left-hand matrix corner of $U(M_n(A))_0$, $n = 1, 2, \ldots$, consisting of those elements that are products of commutators in $U(M_n(A))_0$. It takes the first quotient, as a topological group, isomorphically onto the subgroup of $(A^{-1})_0$ generated by the image of $A_{\text{s.a.}}$ (i.e., by the set, to be denoted by $(A^{-1})^+$, of elements $b - 1$ where b is a positive invertible element of $A^\sim$, the unitization of A, considered modulo its intersection with the subgroup of $(A^{-1})_0$ specified in Theorem 2 and Corollary 4—including the commutator subgroup but possibly larger to include those elements arising as products of matrix commutators).*

To summarize briefly, the isomorphism is compatible with the automorphism of order two of the two abelian groups under consideration, based on the involutive algebra A. (Question: In the description of the first direct summand is it already enough to consider commutators of elements from inside the subgroup of $(A^{-1})_0$ generated by $(A^{-1})^+$ (or from the analogous matrix group), as it is for the second direct summand, the quasi-unitary setting? Or perhaps this is the same thing?)

Proof. There are two things to check: first, that the independent subgroups $A_{\text{s.a.}}$ and $iA_{\text{s.a.}}$ of A are still independent in the specified quotient of A, and their images in this quotient are equal (isomorphic in the natural way) to the two individual quotients described, and, second, that the images of these two subgroups by (what might be called) the exponential isomorphism are the two subgroups mentioned of the specified quotient of $(A^{-1})_0$, and these are equal (isomorphic in a natural way) to the two individual quotients described.

The first statement, once one knows that the independent subgroups consisting of the intersections of the additive commutator subgroup of A with $A_{\text{s.a.}}$ and $iA_{\text{s.a.}}$ are the subgroups described, is a consequence of the following general isomorphism statement for pairs of inclusions of abelian groups $G \subseteq E$ and $H \subseteq F$: the quotient $E \oplus F)/(G \oplus H)$ is isomorphic in the natural way with $E/G \oplus F/H$. To see that a sum of additive commutators from A belonging to $iA_{\text{s.a.}}$ is a sum of additive commutators of self-adjoint elements, note that, on subtracting the adjoint of this sum from it, and dividing by two (which yields the same element), it is enough to consider the single sum (difference) $[a, b] - [a, b]^*$, which (by a variant of polarization) is equal to one half of

$$[a + a^*, b + b^*] + [i(a - a^*), i(b - b^*)],$$

a sum of commutators of self-adjoint elements of A, as desired. The analogous statement for $A_{\text{s.a.}}$ follows in a similar way from the identity

$$2([a, b] + [a, b]^*) = [a + a^*, b - b^*] - [b + b^*, a - a^*].$$

The image of $iA_{\text{s.a.}}$ by the exponential map (in the unital case or $a \mapsto e^a - 1$ in the general case) clearly generates the group $U(A)_0$, again by the holomorphic (or in this case the continuous) functional calculus. So, much as above in the additive setting, it is sufficient to show that a product of commutators from $(M_n(A)^{-1})_0$ which belongs to $U(A)_0$ in fact is a product of commutators in $U(M_n(A))_0$, or in $U(M_{n'}(A))_0$ for some $n' \geq n$.

This can be established by recapitulating the proof of Theorem 2 (and Corollary 4) in the (quasi-)unitary setting. (See also next paragraph.) Given that the results of [20] (Theorems 5.1, 5.2, and 5.4(a), and Lemma 5.3) used in the proof of Theorem 2 (and Corollary 4), in the setting of invertible elements, are also valid (as pointed out explicitly in the statement of Theorems 5.1 and 5.2) in the (quasi-)unitary setting, the only part of the proof the formulation of which diverges from the purely (quasi-)unitary setting is the homotopy construction in the proof of injectivity. The same phenomenon arises in the polynomial approximation proof of Bott periodicity (see [21]). To homotope a given closed curve of unitaries to the one determined by an orthogonal linear combination of projections, in a matrix algebra, with coefficients in $2\pi i\mathbb{Z}$, it is necessary first to approximate the path by a (trignometric) polynomial one (as an algebra-valued function on the circle). But once the homotopy is accomplished, on passing to the unitary parts in the polar decompositions of all invertible (matrix) algebra elements involved, a new homotopy purely through unitary paths is obtained. This may again be triangulated into small directed triangles, for each of which the (skew-self-adjoint) logarithms of the three directed sides (each vertex being canonically an exponential close to the unit times either of the other vertices—we are implicitly assuming the unital context, that a unit has been adjoined) add up to zero modulo additive commutators of self-adjoint elements, so that modulo such commutators the sum of the logarithmic increments over small partitions of the original and final closed paths coincide. Since the final sum is $2\pi i$ times an integral combination of projections (self-adjoint idempotents), the desired conclusion of injectivity, in the unitary context, is obtained.

Alternatively, as pointed out to me by Leonel Robert, one can deduce the desired conversion of a unitary product of (matrix) commutators to a product of unitary (matrix) commutators more or less directly from the statement of Theorem 2. Let $u \in U(A)_0$ be a product of (multiplicative) commutators in $(M_n(A)^{-1})_0$, and let us show that u is a product of commutators in $U(M_{n'}(A))_0$ for some $n' \geq n$. We may assume that $u = e^{ih}$ for a single $h \in A_{\mathrm{s.a.}}$. (A priori, $u = e^{ih_1} \cdots e^{ih_k}$ with $h_1, \ldots, h_k \in A_{\mathrm{s.a.}}$, but by Lemma 6.1 of [20], $e^{ih_1} \cdots e^{ih_k} = e^{i(h_1 + \cdots + h_k)}$ modulo commutators in $U(Z)_0$.) By Theorem 2 ih is a sum of additive commutators in A and $2\pi i$ times the sum of the diagonal matrix entries of idempotents in various matrix algebras over A. Given that (as Kaplansky showed—see Lemma 11.2.7 of [21]) an idempotent in a C*- algebra is similar to a projection, modifying the sum of additive commutators we may assume that the idempotents are self-adjoint—although note that the additive commutators introduced by the simialrity transformations on idempotents ($ses^{-1} - e = [se, s^{-1}]$) belong to matrix algebras. As above, since the sum of additive commutators is now skew-self-adjoint, we may express it as a sum of commutators of self-adjoint elements. As it happens, though, we don't need to do this as by Theorem 6.2(b) of [20] it is enough that the sum itself of the commutators is skew-self-adjoint—for its exponential to be a product of (multiplicative) commutators in $U(M_{n'}(A))_0$, for suitable n'. Again, as mentioned in the proof of Theorem 2, the sum of the diagonal entries of a matrix projection (as for any matrix) over A, sitting in the upper left corner, differs by a sum of commutators from the given matrix, and in particular, after being multiplied by $2\pi i$, is skew-self-adjoint. We may therefore absorb a further contribution into the skew-self-adjoint sum of commutators, and assume that ih is the sum of two skew-self-adjoint terms, the first a sum of commutators, and the second a sum of $2\pi i$ times projections—which, although this is not necessary, we may assume mutually orthogonal by adjusting the similarity transformation introduced above. As mentioned, the exponential of the first term is a product of commutators in $U(M_{n'}(A))_0$, as is the second (it is equal to the identity matrix!), and by Lemma 6.1 of [20] the exponential of the sum of the two terms, i.e., of ih, is the product of these two elements of the commutator subgroup of $U(M_{n'}(A))_0$, modulo this commutator subgroup—as desired.

The statement of the theorem concerning the image of the direct summand $A_{\text{s.a.}}$ of A is much weaker, and does not require further verification.

(As a matter of fact, this statement is best possible, since, for instance, in the C*-algebra $A = M_n(\mathbb{C})$, the subgroup of A^{-1} generated by the commutators of elements of $(A^{-1})^+$ is equal to the entire commutator subgroup of A^{-1}. Indeed, this subgroup is normalized both by $U(A)$ and by $(A^{-1})^+$, and hence by polar decomposition by the whole of A^{-1}, but of course its elements have determinant only equal to 1, so—since it contains a non-trivial unitary—e.g. that of a non-trivial polar decomposition $e^{a_1}e^{a_2} = ue^{a_3}$—it must be equal to the only non–trivial normal subgroup of A^{-1}. Conceivably this is true in any C*-algebra.[1]) (Note: this shows that the statement of Theorem 5.1 of [20], which is asserted for skew-self-adjoint elements as well as for general ones, does not hold for self-adjoint elements.)

6. Remark. It is an immediate consequence of bicontinuity of the group isomorphisms of Theorem 2, Corollary 4, and Theorem 5 (see below), that they carry over to the Hausdorffizations of the topological groups under consideration, which in each case simply consist of the quotient by the closure of the zero element. (Alternatively, the quotient of, respectively, A, $A_{\text{s.a.}}$, $iA_{\text{s.a.}}$, $U(A)_0$, and $(A^{-1})^+$ by the closure of the specified subgroup—in the case of $(A^{-1})^+$, just a set, the image of this modulo the closure of the specified subgroup of $(A^{-1})_0$.)

(To recap, continuity of the exponential map, $A \ni a \mapsto e^a \in A_0^{-1}$, and hence of the isomorphism in the forwards direction, is clear. To verify continuity of

[1] Added September 18, 2021: Leonel Robert has communicated to me that this is true—in any C*-algebra every multiplicative commutator is a product of invertible positive elements, and in fact of commutators of such elements. This makes the last statement of Theorem 5 stronger. (But it still involves commutators of invertible positive matrices.) The analogous statement for the closures of these groups is Theorem 2 of Robert's preprint arXiv: 2103.15238.

the inverse map, let w be an element of A_0^{-1} close to 1, and note that by the holomorphic functional calculus, $w = e^{a_0}$ with a_0 close to 0. But the inverse of any e^a by the isomorphism is trivially a modulo the subgroup of A in the statement of Theorem 2 (and Corollary 4 in the non-unital case, on working with the map $a \mapsto e^a - 1$), and in particular the inverse on $w = e^{a_0}$ (replace a by a_0) is close to the subgroup of A in question—this shows that the isomorphism is continuous in the backwards direction.)

In the particular setting that A is a C*- algebra, and one considers only the subgroup of unitary (or quasi-unitary) elements of A (and matrices over A), and the additive group of all purely imaginary-valued continuous linear functionals on the self-adjoint traces, this result was obtained by Thomsen in [22]. (See Lemma 3.1 and Theorem 3.2 of [22].)

The statement above on Hausdorffization, in the general Banach algebra setting of Theorem 2 and Corollary 4, is in fact a consequence of Proposition 4 of [15], which as it happens lies midway between Theorem 2 (and Corollary 4) and the statement above (for the Banach algebra A).

In more detail, Theorem 2 and Corollary 4 above are purely algebraic, and the statement in the first paragraph above is purely (Hausdorff) topological, dividing out by the closure of zero in both the domain and codomain of the isomorphism. In Proposition 4 of [15], just the subgroup of A generated by additive commutators is replaced by its closure in the quotient on the domain side—the subgroup consisting of $2\pi i$ times the canonical image of the set of idempotents in matrix algebras (in the unital case—in the general case, adjoin a unit) is unaltered. In Lemma 3(a) of [15], de la Harpe and Skandalis compute the kernel of their map Δ_{T} from A_0^{-1} to the quotient of A by the subgroup just described; namely, it is the image under the exponential map $A \ni a \mapsto e^a \in A_0^{-1}$ of the closure of the subgroup generated by the additive commutators; in Proposition 4 of [15] they prove that the closure of this image is the intersection with A_0^{-1} of the closure of the union of the commutator subgroups of the various $M_n(A)_0^{-1}$; this, together with continuity of Δ_{T} (which is trivial—cf. above), implies, as asserted, the statement in the first paragraph above.

7. Remark. As a consequence of Theorem 5 and Remark 6, the tracial part of what is sometimes termed the Elliott invariant of a unital C*-algebra, as well as, of course, the C*-algebra K_1 part, is included in the (Hausdorffized) algebraic K_1-group, $\mathrm{K}_1^{\mathrm{alg}}$. Namely, the space of real-valued affine continuous functions on the cone of positive traces on the C*-algebra A is embedded as the image in (Hausdorffized) $\mathrm{K}_1^{\mathrm{alg}}$ of the quotient of $A_{\mathrm{s.a.}}$ by the closure of the subgroup generated by additive commutators (by Theorem 5, commutators $i[a, b]$ with a and b self-adjoint). Furthermore, the canonical image of $\mathrm{K}_0(A)$ in the real-valued affine functions—as opposed to $2\pi i$ times this in the group of imaginary-valued affine functions, the quotient with respect to which (by Theorem 5) is the abelianization of the group of unitary matrices of arbitrary order over A (this convention for separating real and imaginary valued affine functions is what is consistent with the basic exponential map $a \mapsto e^a$ of Theorem 2)—, although this image is not an injective one if traces on A do not separate projections, does recapture the canonical pairing between traces and K_0. Thus, in the unital case—which the general case can be reduced to (which one has to do anyway to define traces on K_0; see Corollary 9 which even deals with densely defined traces)—one just maps a projection in A (or in a matrix algebra over A) into its exponential considered as an element of the group of invertible elements (or matrices), the abelianization of which breaks up by Theorem 5 as the direct sum of the images modulo matrix commutators of the group $iA_{\mathrm{s.a.}}$ (unitaries—here the commutators may as well be of unitaries) and the group $A_{\mathrm{s.a.}}$ (positive invertible elements—here modulo arbitrary matrix commutators—possibly exactly the commutator subgroup of the group of invertible elements generated by the image of $A_{\mathrm{s.a.}}$).

As pointed out in Proposition 2(d) of [15], this component of (Hausdorffized) algebraic K_1, the image in $\mathrm{K}_1^{\mathrm{alg}}(A)$ of $A_{\mathrm{s.a.}}$ under the exponential map, is exactly the Fuglede-Kadison determinant ([9]) in the case that A is a type II_1 factor. (In this case the other component, the image of $iA_{\mathrm{s.a.}}$, is zero, since the subgroup of A modulo additive commutators generated by $2\pi i$ times projections is exactly $i\mathbb{R} \subseteq \mathbb{C}$, and so the quotient is $\mathbb{R}$, and the range of the exponential map is $\mathbb{R}_{>0}$.)

8. Further to the remark in Section 1 concerning the pairing between densely defined lower semicontinuous traces on a C*-algebra and K_0 in the non-unital case, the underlying fact is as follows—pertinent as the Pedersen ideal of a C*-algebra, as the smallest dense two-sided ideal, is contained in the ideal of definition of every densely defined lower semicontinuous trace.

Theorem. *Let A be a C*-algebra. The canonical map*

$$\mathrm{K}_0(\mathrm{Ped}A) \to \mathrm{K}_0 A$$

from the (algebraic) K_0-group of the Pedersen ideal, PedA ([18]), to the K_0-group of A is an isomorphism of groups.

Proof. While there is never a unital case, unless $\mathrm{Ped}A = A$, it is enough to prove that the canonical map between the unitizations,

$$\mathrm{K}_0((\mathrm{Ped}A)^\sim) \to \mathrm{K}_0(A^\sim),$$

is an isomorphism, since then the canonical commutative diagram connecting the two (vertical) quotient maps,

$$\begin{array}{ccc} \mathrm{K}_0(\mathrm{Ped}A)^\sim & \longrightarrow & \mathrm{K}_0 A^\sim \\ \downarrow & & \downarrow \\ \mathrm{K}_0\mathbb{C} & \longrightarrow & \mathrm{K}_0\mathbb{C}, \end{array}$$

determines an isomorphism between the kernels, by definition the canonical map

$$\mathrm{K}_0(\mathrm{Ped}A) \to \mathrm{K}_0 A.$$

The only properties of Ped$A$ as a subalgebra of A that we shall use are that it is a dense two-sided ideal, that it contains the hereditary sub-C*-algebra generated by any element (or, therefore, any finite subset—see [18]), and that these properties hold after passing to a matrix algebra (a consequence of the identity $\mathrm{Ped}(M_n(A)) = M_n(\mathrm{Ped}A)$, which follows from the bijective correspondence for any algebra between its two-sided ideals and those of a matrix algebra over it).

First, let e be a projection in $A^\sim$, and let us show that e belongs to $(\mathrm{Ped}A)^\sim$. If $e \in A$, then, as $e^2 = e$, by the construction of Ped$A$ in [18], $e \in \mathrm{Ped}A$—or, if one just wants to use the properties above, choose $a \in \mathrm{Ped}A$ close to e, with $a = a^* = eae$ as we may suppose, and then e belongs to the C*-algebra generated by a and so belongs to Ped$A$.

If, now, $e = e^* = e^2 \in A^\sim$ and $e \notin A$, then $e = 1$ (modulo A), i.e., $1 - e \in A$, and so $1 - e \in \mathrm{Ped}A$ as before, i.e., $e \in (\mathrm{Ped}A)^\sim$.

More generally, if $f = f^* = f^2$ is a projection in the multiplier C*-algebra of A, so that $A + \mathbb{C}f$ is a sub-C*-algebra of the multiplier algebra, and if e is a projection in $A + \mathbb{C}f$, then there exists a projection $e' \in \mathrm{Ped}A + \mathbb{C}f$ arbitrarily close to e. (Note that any multiplier of A also multiplies any two-sided ideal of A, so $\mathrm{Ped}A + \mathbb{C}f$ is an algebra.) To see this, consider as before the two cases $e = 0$ modulo A and $e = f$ modulo A. In the first case, i.e., $e \in A$, as before, $e \in \mathrm{Ped}A$. In the second case, $e = a + f$ with $a \in A$. If $a' = a'^* \in \mathrm{Ped}A$ is close to a, then $a' + f$ is close to being a projection, i.e., is approximately equal to its square. Consider the sub-C*-algebra B of A generated by a' and $a'f$. This is closed under multiplication by f and is contained in Ped$A$. Since $a' + f$ belongs to the C*-algebra $B + \mathbb{C}f$, and is close to being a projection, there is a continuous function h on $\mathbb{R}$ with values in the interval [0,1], such that $h(0) = 0$ and $h(1) = 1$, and h is equal to 0 or 1 on the spectrum of $a' + f$. Then $e' := h(a' + f)$ is a projection in $B + \mathbb{C}f \subseteq \mathrm{Ped}A + \mathbb{C}f$, close to e, as asserted.

In particular, the previous paragraph implies that if e is a projection in $M_n(A^\sim)$, $n = 2, 3, \ldots$, then there exists a projection e' in $M_n((\mathrm{Ped}A)^\sim)$ arbitrarily close to e. Indeed, apply the statement above with $f \in M_n(\mathbb{C}) \subseteq M_n(A^\sim)$ the image of e in the quotient $M_n(A^\sim)/M_n(A) \cong M_n(\mathbb{C})$, and with $M_n(A)$ in place of A, to obtain $e' \in \mathrm{Ped}M_n(A) + \mathbb{C}f \subseteq M_n(A) + \mathbb{C}f$ close to e, and recall from above that $\mathrm{Ped}M_n(A) = M_n(\mathrm{Ped}A)$. This implies that the canonical map $\mathrm{K}_0(\mathrm{Ped}A)^\sim \to \mathrm{K}_0 A^\sim$ is surjective. It remains to prove that it is injective.

In other words, we must prove that, if e_1 and e_2 are projections in $(\mathrm{Ped}A)^\sim$ (or in a matrix algebra over this algebra), and if e_1 and e_2 are equivalent in A (or in the matrix algebra under consideration), or, rather, if they are stably equivalent, i.e., equivalent after addition of some projection orthogonal to both (in a matrix algebra), then e_1 and e_2 are stably equivalent in $(\mathrm{Ped}A)^\sim$ (or in a matrix algebra). It is of course sufficient to show this with just equivalence in place of stable equivalence, and just for A rather than a matrix algebra. Suppose that e_1 and e_2 are projections in $(\mathrm{Ped}A)^\sim$ that are equivalent in $A^\sim$, and let us check that they must be equivalent in $(\mathrm{Ped}A)^\sim$. Let $v \in A^\sim$ be a partial isometry with $v^*v = e_1$ and $vv^* = e_2$. With $v' \in (\mathrm{Ped}A)^\sim$ close to v, and, as we may suppose, with $e_2v'e_1 = v'$ (replace v' by $e_2v'e_1$), v' is close to being a partial isometry, and indeed v'^*v' is close to e_1. With w the square root of the inverse of v'^*v' in the C*-algebra $e_1(\mathrm{Ped}A)^\sim e_1$, $v'' := v'w$ is a partial isometry in $(\mathrm{Ped}A)^\sim$ with $v''^*v'' = e_1$ and $v''^* = e_2$, as desired.

9. Corollary. *Let A be a C*-algebra.The additive mapping*

$$\mathrm{K}_0A \to \mathrm{K}_0(\mathrm{Ped}A) \to \mathrm{Ped}A/[\mathrm{Ped}A, \mathrm{Ped}A],$$

the composition of the group isomorphism of Theorem 8 and the zeroth-level Connes-Chern character ([2], [3]) for the complex algebra PedA (the codomain of which is the vector space quotient of PedA by the linear subspace generated by the additive commutators $[a,b] = ab - ba$ in this algebra), defines a (continuous) pairing between K_0A and the (topological) convex cone TA of densely defined lower semicontinuous (positive) traces on A (with the topology of pointwise convergence on PedA).

Proof. Immediate from the fact that a trace in $\mathrm{T}A$ is by definiiton finite-valued on the positive elements of a dense two-sided ideal of A, which then necessarily contains $\mathrm{Ped}A$, and therefore extends from the positive cone of $\mathrm{Ped}A$ to a linear functional on all of $\mathrm{Ped}A$, zero on elements $x^*x - xx^*$ with $x \in \mathrm{Ped}A$, and hence by linearity zero on all of $[\mathrm{Ped}A, \mathrm{Ped}A]$, and thus passing to the quotient vector space, and hence via Connes-Chern to a functional on $\mathrm{K}_0(\mathrm{Ped}A)$, and thence (note this uses that the canonical map of Theorem 8 is invertible) to an additive functional on K_0A. (Continuity is clear.)

One should perhaps note that the Connes-Chern character, most often defined for a unital algebra, is defined naturally in the non-unital case, in a way parallel to the definition of K_0 (or for that matter of K_1), with the result that the Connes-Chern character of the non-unital algebra is just the restriction of that for the unitization: in the level-zero setting (much as above), it is the map between the kernels of the (vertical) quotient maps in the commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_0(A^\sim) & \longrightarrow & A^\sim/[A^\sim, A^\sim] \\
\downarrow & & \downarrow \\
\mathrm{K}_0\mathbb{C} & \longrightarrow & \mathbb{C}/[\mathbb{C}, \mathbb{C}] = \mathbb{C}.
\end{array}$$

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References

1. A. Connes, Classification of injective factors, cases II_1, II_∞, $\mathrm{III}_\lambda, \lambda \neq 1$, *Ann. of Math.* **104** (1976), 73–115.
2. A. Connes, Non-commutative differential geometry, *Publ. Math. Inst. Hautes Études Sci.*, No. 62 (1985), 41–144.
3. A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, CA, 1994.
4. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, On the classification of simple amenable C*-algebras with finite decomposition rank, II, preprint 2015. arXiv:1507.03437
5. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, Simple stably projectionless C*-algebras with generalized tracial rank one, *J. Noncommutative Geom.* **14** (2020), 251–347.
6. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, The classification of simple separable KK-contractible C*-algebras with finite nuclear dimension, *J. Geom. Physics* **158** (2020), article 103861, 51 pages. arXiv: 1712.09463
7. G. A. Elliott and Z. Niu, On the classification of simple amenable C*-algebras with finite decomposition rank, *Operator Algebras and Their Applications: A Tribute to Richard V. Kadison* (editors, R. S. Doran and E. Park), *Contemporary Mathematics* **671** (2016), pp. 117–125.
8. G. A. Elliott, L. Robert, and L. Santiago, The cone of lower semicontinuous traces on a C*-algebra, *Amer. J. Math.* **133** (2011), 969–1005.
9. B. Fuglede and R. V. Kadison, Determinant theory in finite factors, *Ann. of Math. (2)* **55** (1952), 520–530.
10. G. Gong and H. Lin, On classification of non-unital amenable simple C*-algebras, II, *J. Geom. Physics* **158** (2020), article 103865, 102 pages. arXiv:1702.01073
11. G. Gong and H. Lin, On classification of non-unital amenable simple C*-algebras, III: the range and the reduction, preprint 2021. arXiv:2010.01788v4
12. G. Gong, H. Lin, and Z. Niu, A classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, I: C*-algebras with generalized tracial rank one, *C. R. Math. Rep. Acad. Sci. Canada* **42** (2020), 63–450.
13. G. Gong, H. Lin, and Z. Niu, A classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, II: C*-algebras with rational generalized tracial rank one, *C. R. Math. Rep. Acad. Sci. Canada* **42** (2020), 451–539.
14. A. Grothendieck, Un résultat sur le dual d'une C*-algèbre, *J. Math. Pures Appl.* **36** (1957), 97–108.
15. P. de la Harpe and G. Skandalis, Déterminant associé à une trace sur une algèbre de Banach, *Ann. Inst. Fourier (Grenoble)* **34** (1984), 241–260.
16. E. Kirchberg and N. C. Phillips, Embedding of exact C*-algebras in the Cuntz algebra $\mathcal{O}_2$, *J. Reine Angew. Math.* **525** (2000), 17–53.
17. J. Milnor, *Introduction to Algebraic K-Theory*, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1971.
18. G. K. Pedersen, C*-algebras and their automorphism groups (Second edition, edited S. Eilers and D. Olesen), Academic Press, Elsevier, 2018.
19. N. C. Phillips, A classification theorem for nuclear purely infinite simple C*-algebras, *Doc. Math.* **5** (2000), 49–114.
20. L. Robert, Normal subgroups of invertibles and of unitaries in a C*-algebra, *J. Reine Angew. Math.* **756** (2019), 285–309.
21. M. Rørdam, F. Larsen, and N. J. Laustsen, *An Introduction to K-Theory for C*-Algebras*, London Mathematical Society Student Texts 49, Cambridge University Press, Cambridge, UK, 2000.
22. K. Thomsen, Traces, unitary characters, and crossed products by $\mathbb{Z}$, *Publ. Res. Inst. Math. Sci.* **31** (1995), 1011–1029.
23. A. Tikuisis, S. White, and W. Winter, Quasidiagonality of nuclear C*-algebras, *Ann. of Math. (2)* **185** (2017), 259–284.

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