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The Bundle of KMS State Spaces for Flows on a Unital AF C*-algebra

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Abstract. It is shown that, for any unital simple infinite-dimensional AF algebra, the KMS-state bundle for a one-parameter automorphism group is isomorphic to an arbitrary proper simplex bundle over the real line with (as is necessary) fibre at (inverse temperature) zero isomorphic to the trace simplex.

Résumé. On démontre que, pour toute C*-algèbre AF simple à élément unité et à dimension infinie, le faisceau d'états KMS pour un groupe d'automorphismes à un paramètre est isomorphe à un faisceau de simplices propre arbitraire sur la ligne réelle tel que (nécessairement) le fibre sur la température inverse zéro est isomorphe au simplex tracial.

1. **Introduction** The collection of KMS state spaces for a flow on a unital C*-algebra can be thought of as a bundle of simplices over the real line. This may seem like a far-fetched analogy to other more established notions of bundles because the fibers of the bundle may be empty or they may all be mutually non-isomorphic, but we will show here that it is in fact a well-behaved and useful concept. Specifically, we use it to show that for any given unital separable infinite-dimensional simple AF algebra A and for any configuration of KMS state spaces which occurs for a flow on a unital separable C*-algebra and has the property that the simplex of 0-KMS states is affinely homeomorphic to the tracial state space of A, there is a flow on A with the same configuration. In particular, it follows that for any given closed subset F of real numbers containing 0 there are flows on A whose KMS spectrum is F. This removes the lower boundedness condition which occurs in a recent work by the second author, [25].

Since we deal with unital AF algebras, there are always 0-KMS states present. For flows on infinite C*-algebras this is not the case, and in a joint work with Y. Sato we have shown that for any unital, nuclear, purely infinite, simple, separable C*-algebra A in the UCT class and with torsion-free K_1 group, and for any configuration of KMS state spaces which occurs for a flow on a unital separable C*-algebra without trace states, there is also a flow on A with the same configuration; see [11]. In both cases we depend on results from the classification of simple C*-algebras.

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While the work in [25] was based on ideas from [1] and [2], in the present paper the underlying ideas are closer to those presented by Bratteli, Elliott, and Kishimoto in [3]. In particular, the idea of considering the configuration of KMS simplices as a bundle originates in [3].

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2. Proper Simplex Bundles

Let S be a second countable locally compact Hausdorff space and $\pi : S \to \mathbb{R}$ a continuous map. If the inverse image $\pi^{-1}(t)$, equipped with the relative topology inherited from S, is homeomorphic to a compact metrizable Choquet simplex for all $t \in \mathbb{R}$ we say that (S, π) is a *simplex bundle*. We emphasize that π need not be surjective, and we consider therefore also the empty set as a simplex. When (S, π) is a simplex bundle we denote by $\mathcal{A}(S, \pi)$ the set of continuous functions $f : S \to \mathbb{R}$ with the property that the restriction $f|_{\pi^{-1}(t)}$ of f to $\pi^{-1}(t)$ is affine for all $t \in \mathbb{R}$.

Definition 2.1. (Compare [3].) A simplex bundle (S, π) is a *proper simplex bundle* when

(1) π is proper; that is $\pi^{-1}(K)$ is compact in S when $K \subseteq \mathbb{R}$ is compact, and
(2) $\mathcal{A}(S, \pi)$ separate points on S; that is for all $x \neq y$ in S there is an $f \in \mathcal{A}(S, \pi)$ such that $f(x) \neq f(y)$.

Two proper simplex bundles (S, π) and (S', π') are *isomorphic* when there is a homeomorphism $\phi : S \to S'$ such that $\pi' \circ \phi = \pi$ and $\phi : \pi^{-1}(\beta) \to {\pi'}^{-1}(\beta)$ is affine for all $\beta \in \mathbb{R}$.

2.1. Proper simplex bundles from flows In this paper all C*-algebras are assumed to be separable and all traces and weights on a C*-algebra are required to be non-zero, densely defined and lower semi-continuous. Let A be a C*-algebra and θ a flow on A. Let $\beta \in \mathbb{R}$. A β-KMS weight for θ is a weight ω on A such that $\omega \circ \theta_t = \omega$ for all t, and

$$\omega(a^*a) \ = \ \omega\left(\theta_{-\frac{i\beta}{2}}(a)\theta_{-\frac{i\beta}{2}}(a)^*\right) \quad \forall a \in D(\theta_{-\frac{i\beta}{2}}) \ . \tag{2.1}$$

In particular, a 0-KMS weight for θ is a θ-invariant trace. A bounded β-KMS weight is called a β-KMS functional and a β-KMS state when it is of norm one. For states alternative formulations of the KMS condition can be found in [4].

Assume that A is unital. For each $\beta \in \mathbb{R}$ let S^θ_β be the (possibly empty) set of β-KMS states for θ. Let $E(A)$ be the state space of A, a compact convex set in the weak* topology. Set

$$S^\theta = \left\{(\omega, \beta) \in E(A) \times \mathbb{R} : \ \omega \in S^\theta_\beta\right\} \ ,$$

and equip S^θ with the relative topology inherited from the product topology of $E(A)\times\mathbb{R}$. Since S^θ is a closed subset of $E(A)\times\mathbb{R}$ by Proposition 5.3.23 of [4], it follows that S^θ is a second countable locally compact Hausdorff space. Denote by $\pi^\theta : S^\theta \to \mathbb{R}$ the projection to the second coordinate. Since the inverse image ${\pi^\theta}^{-1}(\beta)$ is homeomorphic to S^θ_β, which is a Choquet simplex by Theorem 5.3.30 of [4], the pair (S^θ,π^θ) is a simplex bundle. An obvious application of Proposition 5.3.23 of [4], using the compactness of $E(A)$, shows that π^θ is proper. Note that every self-adjoint element $a \in A$ gives rise to an element $\hat{a} \in \mathcal{A}(S^\theta,\pi^\theta)$ such that $\hat{a}(\omega,\beta)=\omega(a)$. A continuous function $f:\mathbb{R}\to\mathbb{R}$ also gives rise to an element of $\mathcal{A}(S^\theta,\pi^\theta)$:

$$S^\theta \ni (\omega,\beta) \ \mapsto \ f(\beta).$$

It follows that $\mathcal{A}(S^\theta,\pi^\theta)$ separates the points of S^θ, showing that (S^θ,π^θ) is a proper simplex bundle, which we shall call the *KMS bundle* of the flow. In general, ${\pi^\theta}^{-1}(0)$ is the set of θ-invariant trace states and hence non-empty if and only if A has trace states. When A is AF it is the simplex of all trace states of A.

Remark 2.2. Let (S,π) be a proper simplex bundle. Denote by $\mathcal{A}_\mathbb{R}(S,\pi)$ the subset of $\mathcal{A}(S,\pi)$ consisting of the elements that have a limit at infinity. This is a separable real Banach space (in the supremum norm) containing the constant function 1, and its state space

$$\begin{aligned}&E(\mathcal{A}_\mathbb{R}(S,\pi)) := \\ &\left\{\omega \in \mathcal{A}_\mathbb{R}(S,\pi)^* : \ |\omega(f)| \le \sup_{x\in S}|f(x)| \ \ \forall f \in \mathcal{A}_\mathbb{R}(S,\pi), \ \omega(1)=1\right\}\end{aligned}$$

is a metrizable compact convex set in the weak* topology. For $x \in S$, let $\mathrm{ev}_x \in E(\mathcal{A}_\mathbb{R}(S,\pi))$ denote evaluation at x. For each $\beta \in \mathbb{R}$, the set

$$K_\beta := \left\{\mathrm{ev}_x : \ x \in \pi^{-1}(\beta)\right\}$$

is a closed convex subset of $E(\mathcal{A}_\mathbb{R}(S,\pi))$. With $K = E(\mathcal{A}_\mathbb{R}(S,\pi))$, the system $K_\beta, \beta \in \mathbb{R}$, has the properties required in Theorem 2.1 of [2]. Thus, there is a unital, simple, separable, nuclear C*-algebra A equipped with a 2π-periodic flow θ such that $\pi^{-1}(\beta)$ is affinely homeomorphic to the simplex of β-KMS states for all $\beta \in \mathbb{R}$. In the construction of A in [2] the algebra appears to depend on (S,π) and on the many choices made in the process of its construction, but it was shown in [11], based on the Kirchberg-Phillips classification result, that when $\pi^{-1}(0)$ is empty one can take A to be any given separable, simple, nuclear, purely infinite C*-algebra in the UCT class and with torsion free K_1 group. It follows from the main result we describe next that when $\pi^{-1}(0)$ is not empty one can take A to be any infinite dimensional unital simple AF algebra whose tracial state space is affinely homeomorphic to $\pi^{-1}(0)$.

3. The Main Result and Applications

Theorem 3.1. *Let (S, π) be a proper simplex bundle and let A be a unital infinite-dimensional simple AF algebra with a tracial state space affinely homeomorphic to $\pi^{-1}(0)$. There is a 2π-periodic flow on A whose KMS bundle is isomorphic to (S, π).*

By definition the KMS spectrum of a flow θ on a unital C*-algebra is the set of real numbers β for which θ has a β-KMS state. When the KMS bundle of θ is isomorphic to a proper simplex bundle (S, π), the KMS spectrum of θ is the range $\pi(S)$ of π.

To exhibit possible ways to work with proper simplex bundles and to illustrate how Theorem 3.1 can be applied let us use it to prove the following statement.

Corollary 3.2. *Let A be a unital infinite-dimensional simple AF algebra and let F be a closed subset of real numbers containing* 0.

- *There is a 2π-periodic flow on A whose KMS spectrum is F and such that there is a unique β-KMS state for all $\beta \in F\backslash\{0\}$.*
- *There is a 2π-periodic flow θ on A whose KMS spectrum is F and such that S^θ_β is not affinely homeomorphic to $S^\theta_{\beta'}$ when $\beta, \beta' \in F\backslash\{0\}$ and $\beta \neq \beta'$.*

Proof. Given a proper simplex bundle (S, π), set $S_F = \pi^{-1}(F)$ and denote by π_F the restriction of π to $\pi^{-1}(F)$. The pair (S_F, π_F) is again a proper simplex bundle.

For the first item, let $T(A)$ be the tracial state space of A and fix an element $\omega_0 \in T(A)$. Let S be the subset

$$(T(A) \times \{0\}) \cup \{(\omega_0, t) : \ t \in \mathbb{R}\backslash\{0\}\}$$

of the topological product $T(A) \times \mathbb{R}$. Let $\pi : S \to \mathbb{R}$ be the canonical projection. Then (S, π) is a proper simplex bundle such that $\pi^{-1}(\beta) = \{(\omega_0, \beta)\}$ when $\beta \neq 0$ and such that $\pi^{-1}(0)$ is a copy of $T(A)$. The existence of the desired flow follows from Theorem 3.1 by applying it to the bundle (S_F, π_F).

For the second item, note that the KMS bundle of the flow described in Theorem 1.1 of [24] is a proper simplex bundle (S', π') such that $\pi'^{-1}(0)$ contains only one point, $\pi'(S) = \mathbb{R}$ and such that $\pi'^{-1}(\beta)$ is not affinely homeomorphic to $\pi'^{-1}(\beta')$ when $\beta \neq \beta'$. With (S, π) the bundle defined above set

$$S'' = \{(x, y) \in S' \times S : \ \pi'(x) = \pi(y)\} \ .$$

Define $\pi'' : S'' \to \mathbb{R}$ by $\pi''(x, y) = \pi(y)$. Then (S'', π'') is a proper simplex bundle and an application of Theorem 3.1, this time to (S''_F, π''_F), gives the desired flow. □

4. Proof of the Main Result

4.1. Tools The following lemma is an immediate consequence of Lemma 3.1 of [25] and Theorem 2.4 of [23].

Lemma 4.1. *Let D be a C^*-algebra. Denote by $\rho \in \operatorname{Aut}(D)$ an automorphism of D and let $q \in D$ a projection in D which is full*[1] *in $D \rtimes_\rho \mathbb{Z}$. Let $\hat{\rho}$ denote the restriction to $q(D \rtimes_\rho \mathbb{Z})q$ of the dual action on $D \rtimes_\rho \mathbb{Z}$ considered as a 2π-periodic flow. Let $P : D \rtimes_\rho \mathbb{Z} \to D$ denote the canonical conditional expectation. For each $\beta \in \mathbb{R}$, the map $\tau \mapsto \tau \circ P|_{q(D \rtimes_\rho \mathbb{Z})q}$ is an affine homeomorphism from the set of traces τ on D that satisfy*

$$\tau \circ \rho = e^{-\beta}\tau \text{ and } \tau(q) = 1, \tag{4.1}$$

onto the simplex of β-KMS states for $\hat{\rho}$.

In this lemma the topology on the set of traces of D with the properties (4.1) is given by pointwise convergence on elements from the corner qDq of D.

When D is an AF algebra it is well-known that the set of its traces can be identified, via the map $\tau \mapsto \tau_*$, with the set $\operatorname{Hom}^+(K_0(D), \mathbb{R})$ of non-zero positive homomorphisms $\phi : K_0(D) \to \mathbb{R}$; a fact stated as Lemma 3.5 in [25]. In the setting of Lemma 4.1 this implies that when D is an AF algebra the KMS spectrum and the structure of the KMS states for $\hat{\rho}$ can be determined directly from the pair $(K_0(D), \rho_*)$. To see how, we note that by Remark 3.3 in [17] every β-KMS state τ for $\hat{\rho}$ on $q(D \rtimes_\rho \mathbb{Z})q$ extends uniquely to a β-KMS weight $\hat{\tau}$ for the dual action. Since D is the fixed point algebra for the dual action the restriction of $\hat{\tau}$ to D is a trace on D, yielding a map

$$\tau \mapsto \left(\hat{\tau}|_D\right)_* \tag{4.2}$$

from the set of β-KMS states τ for $\hat{\rho}$ to $\operatorname{Hom}^+(K_0(D), \mathbb{R})$. Therefore Lemma 4.1 has the following consequence.

Corollary 4.2. *In the setting of Lemma* 4.1 *assume that D is an AF algebra. For each $\beta \in \mathbb{R}$ the map* (4.2) *is an affine homeomorphism from the set of β-KMS states for $\hat{\rho}$ on $q(D \rtimes_\gamma \mathbb{Z})q$ onto the set of positive homomorphisms $\phi \in \operatorname{Hom}^+(K_0(D), \mathbb{R})$ that satisfy*

$$\phi \circ \rho_* = e^{-\beta}\phi \text{ and } \phi([q]) = 1. \tag{4.3}$$

Here the topology on the elements from $\operatorname{Hom}^+(K_0(D), \mathbb{R})$ with the properties (4.3) is given by pointwise convergence on $\{x \in K_0(D) : \ 0 \leq x \leq [q]\}$.

Corollary 4.2 will be complemented by the following lemma which helps to control the Elliott invariant of $q(D \rtimes_\gamma \mathbb{Z})q$, and to ensure that it is classified by it. It follows from Lemma 3.4 of [25], which is based on arguments from [22], [18] and [19].

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[1] Recall that a projection q in a C*-algebra A is full when $\overline{AqA} = A$. This is automatic when $q \neq 0$ and A is simple.

Lemma 4.3. *Let D be a stable AF algebra such that $K_0(D)$ has large denominators.*[2] *For any order automorphism $\alpha \in \mathrm{Aut}(K_0(D))$ of $K_0(D)$ there is an automorphism $\gamma \in \mathrm{Aut}(D)$ of D such that*

(a) *$\gamma_* = \alpha$ on $K_0(D)$,*
(b) *the restriction map $\mu \mapsto \mu|_D$ is a bijection from traces μ on $D \rtimes_\gamma \mathbb{Z}$ onto the γ-invariant traces on D, and*
(c) *$D \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable; that is, $(D \rtimes_\gamma \mathbb{Z}) \otimes \mathcal{Z} \simeq D \rtimes_\gamma \mathbb{Z}$ where $\mathcal{Z}$ denotes the Jiang-Su algebra, [15].*

Given a proper simplex bundle (S,π) and a closed subset $F \subseteq \mathbb{R}$ we denote by (S_F,π_F) the proper simplex bundle with $S_F = \pi^{-1}(F)$ and π_F the restriction of π to S_F. The following lemma relates $\mathcal{A}(S_F,\pi_F)$ to $\mathcal{A}(S,\pi)$ and will be a crucial tool in what follows.

Lemma 4.4. *Let (S,π) be a proper simplex bundle and $F \subseteq \mathbb{R}$ a closed subset.*

(1) *The map $\mathcal{A}(S,\pi) \to \mathcal{A}(S_F,\pi_F)$ given by restriction is surjective.*
(2) *Let $f_1, f_2, g_1, g_2 \in \mathcal{A}(S,\pi)$ such that $f_i(x) < g_j(x)$ for all $x \in S$ and all $i,j \in \{1,2\}$. Assume that there is an element $h^F \in \mathcal{A}(S_F,\pi_F)$ such that*

$$f_i(x) < h^F(x) < g_j(x), \quad x \in S_F,\ i,j \in \{1,2\}.$$

There is an element $h \in \mathcal{A}(S,\pi)$ such that $h(y) = h^F(y)$ for all $y \in S_F$ and

$$f_i(x) < h(x) < g_j(x), \quad x \in S,\ i,j \in \{1,2\}.$$

Proof. (1) Let $h \in \mathcal{A}(S_F,\pi_F)$. For each $n \in \mathbb{N}$ the pair $(S_{[-n,n]},\pi_{[-n,n]})$ is a compact simplex bundle in the sense of [3] and it follows from Lemma 2.2 in [3] that there are elements $f_n \in \mathcal{A}(S_{[-n,n]},\pi_{[-n,n]})$ such that the restriction $f_n|_{S_{F\cap[-n,n]}}$ of f_n to $S_{F\cap[-n,n]}$ agrees with $h|_{S_{F\cap[-n,n]}}$. For $n \in \mathbb{N}$ choose a continuous function $\chi_n : \mathbb{R} \to [0,1]$ such that $\chi_n(t) = 1$ for $t \leq n - \frac{1}{2}$ and $\chi_n(t) = 0$ for $t \geq n$. Define $f'_n : S_{[-n,n]} \to \mathbb{R}$ recursively by

$$f'_1(x) = (1 - \chi_1(|\pi(x)|))f_2(x) + \chi_1(|\pi(x)|)f_1(x)\ ,$$

and then f'_n for $n \geq 2$ such that $f'_n(x) = f'_{n-1}(x)$ when $x \in \pi^{-1}([-n+1,n-1])$ and $f'_n(x) = (1-\chi_n(|\pi(x)|))f_{n+1}(x) + \chi_n(|\pi(x)|)f_n(x)$ when $x \in \pi^{-1}([n-1,n] \cup [-n,-n+1])$. Then $f'_n|_{S_{F\cap[-n,n]}} = h|_{S_{F\cap[-n,n]}}$ and since f'_{n+1} extends f'_n for all n, there is an element $f \in \mathcal{A}(S,\pi)$ such that $f|_{S_{[-n,n]}} = f'_n$. This element f extends h.

The assertion (2) follows in a similar way on using Lemma 2.3 in [3]. □

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[2] An ordered group (G,G^+) has large denominators when the following condition holds: For any $a \in G^+$ and any $n \in \mathbb{N}$ there are an element $b \in G$ and an $m \in \mathbb{N}$ such that $nb \leq a \leq mb$; see [20].

4.2. Dimension groups from proper simplex bundles Given a Choquet simplex Δ, as usual denote by $\operatorname{Aff}\Delta$ the set of real-valued continuous and affine functions on Δ. Fix a proper simplex bundle (S,π) with $\pi^{-1}(0)$ non-empty. Let H be a torsion free abelian group and $\theta : H \to \operatorname{Aff}\pi^{-1}(0)$ a homomorphism. We assume

(1) there is an element $u \in H$ such that $\theta(u) = 1$, and
(2) $\theta(H)$ is dense in $\operatorname{Aff}\pi^{-1}(0)$.

Set

$$H^+ = \left\{h \in H : \ \theta(h)(x) > 0, \quad x \in \pi^{-1}(0)\right\} \cup \{0\} \ .$$

Then (H, H^+) is a simple dimension group; see [7].

It follows from (1) of Lemma 4.4 that the map $r : \mathcal{A}(S,\pi) \to \operatorname{Aff}\pi^{-1}(0)$ given by restriction is surjective and we can therefore choose a linear map $L : \operatorname{Aff}\pi^{-1}(0) \to \mathcal{A}(\pi, S)$ such that $r \circ L = \mathrm{id}$. We arrange, as we can, that $L(1) = 1$. Define $\hat{L} : \bigoplus_{\mathbb{Z}} H \to \mathcal{A}(S,\pi)$ by

$$\hat{L}\left((h_n)_{n\in\mathbb{Z}}\right)(x) = \sum_{n\in\mathbb{Z}} L(\theta(h_n))(x) e^{n\pi(x)} \ .$$

Let $\mathbb{Q}[e^{-\pi}, 1-e^{-\pi}]$ denote the $\mathbb{Q}$-linear span of the functions defined on $S\backslash\pi^{-1}(0)$ by

$$x \mapsto e^{n\pi(x)}(1 - e^{-\pi(x)})^l \tag{4.4}$$

for some $n, l \in \mathbb{Z}$. For each $k \in \mathbb{N}$, choose continuous functions $\psi_k^0, \psi_k^\pm : \mathbb{R} \to [0,1]$ such that

$\psi_k^0(t) = 1, \ -\frac{1}{2k} \leq t \leq \frac{1}{2k}$,
$\psi_k^-(t) = 1, \ t \leq -\frac{1}{k}$,
$\psi_k^+(t) = 1, \ t \geq \frac{1}{k}$, and
$\psi_k^-(t) + \psi_k^0(t) + \psi_k^+(t) = 1$ for all $t \in \mathbb{R}$.

Consider the countable subgroup of $\mathcal{A}(S,\pi)$

$$G_k := \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]\psi_k^- \circ \pi + \hat{L}(\bigoplus_{\mathbb{Z}} H)\psi_k^0 \circ \pi + \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]\psi_k^+ \circ \pi \ .$$

In what follows we denote the support of a real-valued function f by $\operatorname{supp} f$. Let $\mathcal{A}_{00}(S,\pi)$ denote the set of elements f from $\mathcal{A}(S,\pi)$ for which $\operatorname{supp} f$ is compact and contained in $S\backslash\pi^{-1}(0)$. Since the topology of S is second countable we can choose a countable subgroup G_{00} of $\mathcal{A}_{00}(S,\pi)$ with the following density property:

Property 4.5. For all $N \in \mathbb{N}$, all $\epsilon > 0$ and all $f \in \mathcal{A}_{00}(S,\pi)$ with $\operatorname{supp} f \subseteq \pi^{-1}(]-N,N[\backslash\{0\})$, there is $g \in G_{00}$ such that

$$\sup_{x\in S} |f(x) - g(x)| < \epsilon$$

and $\operatorname{supp} g \subseteq \pi^{-1}(]-N,N[\backslash\{0\})$.

When $f_1 \in \hat{L}(\bigoplus_{\mathbb{Z}} H)$ and $f_2^{\pm} \in \mathbb{Q}[e^{-\pi}, 1-e^{-\pi}]$ the difference between

$$f_2^- \psi_k^- \circ \pi + f_1 \psi_k^0 \circ \pi + f_2^+ \psi_k^+ \circ \pi$$

and

$$f_2^- \psi_{k+1}^- \circ \pi + f_1 \psi_{k+1}^0 \circ \pi + f_2^+ \psi_{k+1}^+ \circ \pi$$

is an element of $\mathcal{A}_{00}(S, \pi)$, and so by enlarging G_{00} we can ensure that

$$G_k + G_{00} \subseteq G_{k+1} + G_{00} \ . \tag{4.5}$$

Let $\alpha_0 \in \operatorname{Aut} \mathcal{A}(S, \pi)$ be defined by

$$\alpha_0(f)(x) = e^{-\pi(x)} f(x) \ .$$

Since $\alpha_0(\mathcal{A}_{00}) = \mathcal{A}_{00}$ and since $(1-e^{-\pi})^{-1}\mathcal{A}_{00}(S,\pi) \subseteq \mathcal{A}_{00}(S,\pi)$, we can enlarge G_{00} further to achieve that

$$\alpha_0(G_{00}) = G_{00} \tag{4.6}$$

and that

$$(\mathrm{id} - \alpha_0)(G_{00}) = G_{00}. \tag{4.7}$$

We define

$$G = \bigcup_{k=1}^{\infty} (G_k + G_{00}).$$

Let $\sigma \in \operatorname{Aut}\left(\bigoplus_{n \in \mathbb{Z}} H\right)$ denote the shift:

$$\sigma\left((h_n)_{n\in\mathbb{Z}}\right) = (h_{n+1})_{n \in \mathbb{Z}} \ .$$

Then $\alpha_0 \circ \hat{L} = \hat{L} \circ \sigma$, which implies that $\alpha_0(G_k) = G_k$ and hence

$$\alpha_0(G) = G.$$

Set

$$\mathcal{A}(S,\pi)^+ = \{f \in \mathcal{A}(S,\pi) : \ f(x) > 0 \ \forall x \in S\} \cup \{0\}$$

and

$$G^+ = G \cap \mathcal{A}(S,\pi)^+ \ .$$

LEMMA 4.6. *The pair (G, G^+) has the following properties.*

(1) $G^+ \cap (-G^+) = \{0\}$.
(2) $G = G^+ - G^+$.
(3) (G, G^+) *is unperforated, i.e.,* $n \in \mathbb{N}\backslash\{0\}, \ g \in G, \ ng \in G^+ \Rightarrow g \in G^+$.

(4) *(G, G^+) has the strong Riesz interpolation property, i.e. if $f_1, f_2, g_1, g_2 \in G$ and $f_i < g_j$ in G for all $i, j \in \{1, 2\}$, then there is an element $h \in G$ such that*

$$f_i < h < g_j$$

for all $i, j \in \{1, 2\}$.

PROOF. (1) and (3) are obvious. (2): Let $f \in G$. Then $f \in G_k + G_{00}$ for some $k \in \mathbb{N}$ and we can write

$$f = h^- \psi_k^- \circ \pi + f_0 \psi_k^0 \circ \pi + h^+ \psi_k^0 \circ \pi + g$$

where $h^\pm \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$, $f_0 \in \hat{L}(\bigoplus_{\mathbb{Z}} H)$ and $g \in G_{00}$. By definition of $\mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ there are $n, m \in \mathbb{N}$ and $M > 0$ such that

$$h^-(x) \leq e^{-n\pi(x)}, \quad x \in \pi^{-1}(]-\infty, -M]) \ ,$$

and

$$h^+(x) \leq e^{m\pi(x)}, \quad x \in \pi^{-1}([M, \infty[) \ .$$

Since $f_0 \psi_k^0 \circ \pi$ and g are compactly supported there is $K \in \mathbb{N}$ such that $f(x) < g(x)$ for all $x \in S$, where

$$g = K \left(e^{-n\pi} \psi_k^- \circ \pi + \psi_k^0 \circ \pi + e^{m\pi} \psi_k^+ \circ \pi \right) \ \in \ G^+ \ .$$

Note that $f = g - (g - f) \in G^+ - G^+$.

(4): Since $\theta(H)$ has the Riesz interpolation property for the strict order by Lemma 3.1 in [7], there is $h_0 \in \theta(H)$ such that $f_i(x) < h_0(x) < g_j(x)$ for all i, j and all $x \in \pi^{-1}(0)$. We claim that there are elements $h^\pm \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ and $K^\pm \in \mathbb{N}$ such that $f_i(x) < h^-(x) < g_j(x)$ when $\pi(x) \leq -K^-$ and $f_i(x) < h^+(x) < g_j(x)$ when $\pi(x) \geq K^+$. To find h^+ and K^+, note that we may assume that $\pi(S)$ contains arbitrarily large positive numbers; otherwise we take K^+ larger that $\sup \pi(S)$ and $h^+ = 0$. By definition of G there is $N \in \mathbb{N}$ so large that there are polynomials p_1, p_2, q_1, q_2 with rational coefficients such that

$$\left(e^{\pi(x)} (e^{\pi(x)} - 1) \right)^N f_i(x) = p_i(e^{\pi(x)})$$

and

$$\left(e^{\pi(x)} (e^{\pi(x)} - 1) \right)^N g_j(x) = q_j(e^{\pi(x)})$$

for all i, j and all large x. Then $p_i(y) < q_j(y)$ for all large elements y of $e^{\pi(S)}$ and it follows therefore from Lemma 4.7 that there is a polynomial h' with rational coefficients such that $p_i(x) < h'(x) < q_j(x)$ for all i, j and all large x. Set

$$h^+(x) = \left(e^{\pi(x)} (e^{\pi(x)} - 1) \right)^{-N} h'(e^{\pi(x)}) \ .$$

Then $h^+ \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ and if K^+ is large enough $f_i(x) < h^+(x) < g_j(x)$ for all i, j and all $x \in \pi^{-1}([K^+, \infty))$. The pair h^-, K^- is constructed in a similar way: Multiply each of the functions from $\{f_1, f_2, g_1, g_2\}$ with the same function of the form

$$(e^{-\pi})^N (e^{-\pi} - 1)^N$$

to get them into the form $x \mapsto p(e^{-\pi(x)})$ for p a polynomial with rational coefficients and apply Lemma 4.7.

Having $h^\pm$ and $K^\pm$ we use the statement (2) of Lemma 4.4 to find $H \in \mathcal{A}(S, \pi)$ such that $H(x) = h^-(x)$ when $\pi(x) \leq -K^-$, $H(x) = h^+(x)$ when $\pi(x) \geq K^+$, $H(x) = h_0(x)$ when $x \in \pi^{-1}(0)$, and $f_i(x) < H(x) < g_j(x)$ for all $x \in S$ and all i, j. Set

$$H'(x) = H\psi_k^- \circ \pi + L(h_0)\psi_k^0 \circ \pi + H\psi_k^+ \circ \pi.$$

If k is large enough we have

$$f_i(x) < H'(x) < g_j(x)$$

for all $x \in S$ and all i, j. Set

$$H''(x) = H'(x) - L(h_0)\psi_k^0 \circ \pi - h^-\psi_k^- \circ \pi - h^+\psi_k^+ \circ \pi,$$

and note that $\sup H'' \subseteq \,] - K^-, K^+[\backslash\{0\}$. Set $K = \max\{K^-, K^+\}$ and let $\delta > 0$ be given, smaller than $g_j(x) - H'(x)$ and $H'(x) - f_i(x)$ for all i, j and all $x \in \pi^{-1}([-K, K])$. By Property 4.5 there is an element $g' \in G_{00}$ such that

$$\operatorname{supp} g' \subseteq \pi^{-1}(] - K, K[\backslash\{0\})$$

and $\sup_{x \in S} |g'(x) - H''(x)| < \frac{\delta}{2}$. Then

$$h = g' + L(h_0)\psi_k^0 \circ \pi + h^-\psi_k^- \circ \pi + h^+\psi_k^+ \circ \pi \ \in \ G$$

and $f_i(x) < h(x) < g_j(x)$ for all i, j and all $x \in S$. □

LEMMA 4.7. *Let $p_i, q_j,\ i, j \in \{1, 2\}$, be polynomials with rational coefficients. Assume that there is a sequence $\{x_n\}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = \infty$ and such that $p_i(x_n) < q_j(x_n)$ for all i, j, n. It follows that there is a polynomial h with rational coefficients and a $K > 0$ such that*

$$p_i(x) < h(x) < q_j(x)$$

for all $i, j \in \{1, 2\}$ and all $x \geq K$.

PROOF. Since polynomials only have finitely many zeros there is a $K' > 0$ such that $p_i(x) < q_j(x)$ for all i, j and all $x \geq K'$. Write $q_j(x) = a_{0,j} + a_{1,j}x + a_{2,j}x^2 + \cdots + a_{N,j}x^N$ and $p_i(x) = b_{0,j} + b_{1,j}x + b_{2,j}x^2 + \cdots + b_{N,j}x^N$ for some N larger than the degree of any of the four given polynomials, and set

$$\xi_i = (b_{N,i}, b_{N-1,i}, \cdots, b_{0,i}) \in \mathbb{Q}^{N+1}$$

and

$$\eta_j = (a_{N,j}, a_{N-1,j}, \cdots, a_{0,j}) \in \mathbb{Q}^{N+1}.$$

Since $p_i(x) < q_j(x)$ for all large x, we have that $\xi_i <_{lex} \eta_j$ for all i, j with respect to the lexicographic order $<_{lex}$. Since $\mathbb{Q}^{N+1}$ is totally ordered in the lexicographic order there is an element $c = (c_N, c_{N-1}, c_{N-2}, \cdots, c_0) \in \mathbb{Q}^{N+1}$ such that $\xi_i <_{lex} c <_{lex} \eta_j$ for all i, j. By changing c_0 by a small amount we can arrange that $c \notin \{\xi_1, \xi_2, \eta_1, \eta_2\}$. Then the polynomial

$$h(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N$$

will have the desired property. $\square$

Consider the subset Γ of $(\bigoplus_{\mathbb{Z}} H) \oplus G$ consisting of the elements $(\xi, g) \in (\bigoplus_{\mathbb{Z}} H) \oplus G$ with the property that there is an $\epsilon > 0$ such that

$$\hat{L}(\xi)(x) = g(x), \quad x \in \pi^{-1}(]-\epsilon, \epsilon[) \ . \tag{4.8}$$

Γ is a subgroup of $(\bigoplus_{\mathbb{Z}} H) \oplus G$.

LEMMA 4.8. *The projection $\Gamma \to G$ is surjective.*

PROOF. Let $g \in G$. By definition of G there is an element $\xi \in \bigoplus_{\mathbb{Z}} H$ and $k \in \mathbb{N}$ such that $g(x) = \hat{L}(\xi)$ on $\pi^{-1}(]-\frac{1}{2k}, \frac{1}{2k}[$. $\square$

Set

$$\Gamma^+ = \left\{(\xi, g) \in \Gamma : \ g \in G^+ \backslash \{0\}\right\} \cup \{0\} \ .$$

By combining Lemma 4.8 and Lemma 4.6 above with Lemma 3.1 and Lemma 3.2 in [7] we conclude that (Γ, Γ^+) is a dimension group.

Given an element $h \in H$ we denote in what follows by $h^{(0)}$ the element of $\bigoplus_{\mathbb{Z}} H$ defined by $(h^{(0)})_0 = h$ and $(h^{(0)})_n = 0$ when $n \neq 0$. Define $\Sigma : \bigoplus_{\mathbb{Z}} H \to H$ by

$$\Sigma\left((h_n)_{n \in \mathbb{Z}}\right) = \sum_{n \in \mathbb{Z}} h_n \ .$$

LEMMA 4.9. *(Γ, Γ^+) has large denominators; that is for all $x \in \Gamma^+$ and $m \in \mathbb{N}$ there is an element $y \in \Gamma^+$ and an $n \in \mathbb{N}$ such that $my \leq x \leq ny$.*

PROOF. Let $x = (\xi, g) \in \Gamma^+ \backslash \{0\}$ and $m \in \mathbb{N}$ be given. Then $\Sigma(\xi) \in H^+ \backslash \{0\}$ and since H has large denominators by [20], there is an element $b \in H^+$ such that $mb < \Sigma(\xi) < nb$ for some $n \in \mathbb{N}, \ n > m + 2$. Since $L(\theta(\Sigma(\xi)))$ agrees with g on $\pi^{-1}(0)$, there is a compact neighborhood U of 0 such that

$$mL(\theta(b))(x) < g(x) < nL(\theta(b))(x)$$

for all $x \in \pi^{-1}(U)$. There is also a $K \in \mathbb{N}$ such that $U \subseteq \,]-K, K[$ and functions $f^{\pm} \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ such that $g(x) = f^-(x), \ x \leq -K$, and $g(x) = f^+(x), \ x \geq K$. It follows from (2) of Lemma 4.4 that there is an element $a \in \mathcal{A}(S, \pi)$ such that

$\frac{1}{n}g(x) < a(x) < \frac{1}{m}g(x), x \in S$,
$a(x) = L(\theta(b))(x)$ for all $x \in \pi^{-1}(U)$,
$a(x) = \frac{1}{m+1}f^-(x)$ for $x \leq -K$, and
$a(x) = \frac{1}{m+1}f^+(x)$ for all $x \geq K$.

Choose $k \in \mathbb{N}$ so large that $[-\frac{1}{k}, \frac{1}{k}] \subseteq U$ and note that

$$a(x) = L(\theta(b))(x)\psi_k^0 \circ \pi(x) + a(x)\psi_k^+ \circ \pi(x) + a(x)\psi_k^- \circ \pi(x).$$

Then the function

$$a' := a - L(\theta(b))\psi_k^0 \circ \pi - \frac{1}{m+1}f^+\psi_k^+ \circ \pi - \frac{1}{m+1}f^-\psi_k^- \circ \pi$$

is supported in $]-K, K[\backslash\{0\}$. Let $\delta > 0$ be smaller than $\frac{1}{m}g(x) - a(x)$ and $a(x) - \frac{1}{n}g(x)$ for all $x \in \pi^{-1}([-K, K])$. By Property 4.5 we can find $c \in G_{00}$ such that $\operatorname{supp} c \subseteq]-K, K[\backslash\{0\}$ and $|c(x) - a'(x)| < \delta$ for all $x \in S$. Then

$$g' := c + L(\theta(b))\psi_k^0 \circ \pi + \frac{1}{m+1}f^+\psi_k^+ \circ \pi + \frac{1}{m+1}f^-\psi_k^- \circ \pi \ \in \ G$$

and $mg'(x) < g(x) < ng'(x)$ for all $x \in S$. It follows that $y = (b^{(0)}, g) \in \Gamma^+$ has the desired property. □

Since $\hat{L} \circ \sigma = \alpha_0 \circ \hat{L}$ we can define $\alpha \in \operatorname{Aut}\Gamma$ by

$$\alpha = \sigma \oplus \alpha_0.$$

LEMMA 4.10. *The only order ideals I in Γ such that $\alpha(I) = I$ are $I = \{0\}$ and $I = \Gamma$.*

PROOF. Recall that an order ideal I in Γ is a subgroup such that

(a) $I = I \cap \Gamma^+ - I \cap \Gamma^+$, and
(b) when $0 \leq y \leq x$ in Γ and $x \in I$, then $y \in I$.

Let I be a non-zero order ideal such that $\alpha(I) = I$. Since $I \cap \Gamma^+ \neq \{0\}$ there is an element $g \in G^+\backslash\{0\}$ and an element $\xi \in \bigoplus_{n\in\mathbb{Z}} H$ such that $(\xi, g) \in I$. Set $h = \Sigma(\xi)$. By definition of $G^+\backslash\{0\}$ there are natural numbers $n, m, k, K \in \mathbb{N}$ such that the function

$$g' = L(\theta(h))\psi_k^0 \circ \pi + e^{n\pi}\psi_k^- \circ \pi + e^{-m\pi}\psi_k^+ \circ \pi$$

has the property that

$$0 < g'(x) < Kg(x), \quad x \in S.$$

It follows that $(h^{(0)}, g') \in I \cap \Gamma^+$. Note that

$$\alpha_0^l(g') = e^{-l\pi} L(\theta(h))\psi_k^0 \circ \pi + e^{(n-l)\pi}\psi_k^- \circ \pi + e^{-(m+l)\pi}\psi_k^+ \circ \pi$$

for all $l \in \mathbb{Z}$. Consider an arbitrary element $(\xi', f) \in \Gamma^+ \backslash \{0\}$. We can then find $l_1, l_2 \in \mathbb{Z}$ and $M \in \mathbb{N}$ such that

$$f(x) < M \left(\alpha_0^{l_1}(g')(x) + \alpha_0^{l_2}(g')(x)\right), \quad x \in S.$$

Since

$$\alpha^{l_1}((h^{(0)}, g')) + \alpha^{l_2}((h^{(0)}, g')) \in I \cap \Gamma^+,$$

it follows that $(\xi', f) \in I$ and we conclude therefore that $I = \Gamma$. $\square$

4.3. Some homomorphisms $\Gamma \to \mathbb{R}$ Note that the constant function 1 is in G and that, with $v := (u^{(0)}, 1)$, $v \in \Gamma^+$. Let $\beta \in \mathbb{R}$ and $\omega \in \pi^{-1}(\beta)$. As in Remark 2.2 we denote in the sequel by $\mathcal{A}_{\mathbb{R}}(S, \pi)$ the real Banach space consisting of the elements of $\mathcal{A}(S, \pi)$ that have a limit at infinity.

Lemma 4.11. *Let* $f \in \mathcal{A}_{\mathbb{R}}(S, \pi)$ *and let* $\epsilon > 0$ *be given. There is an element* $g \in G$ *such that* $\sup_{x \in S} |f(x) - g(x)| \leq \epsilon$.

Proof. An initial approximation gives us an element $f_1 \in \mathcal{A}(S, \pi)$ which is compactly supported and a real number $r \in \mathbb{R}$ such that

$$\sup_{x \in S} |f(x) - f_1(x) - r| \leq \frac{\epsilon}{2} .$$

Let $q \in \mathbb{Q}$ such that $|q - r| < \frac{\epsilon}{6}$ and choose an element $h \in H$ such that $|\theta(h)(y) - f_1(y) - r| < \frac{\epsilon}{6}$ for all $y \in \pi^{-1}(0)$. There is a $k \in \mathbb{N}$ such that $|L(\theta(h))(x) - f_1(x) - r| < \frac{\epsilon}{6}$ for all $x \in \pi^{-1}([-\frac{1}{k}, \frac{1}{k}])$. Since $f_1\psi_k^+ \circ \pi + f_1\psi_k^- \circ \pi$ is compactly supported in $S \backslash \pi^{-1}(0)$ it follows from Property 4.5 that there is an element $g' \in G_{00}$ such that

$$\sup_{x \in S} \left|g'(x) - f_1(x)\psi_k^+ \circ \pi(x) - f_1(x)\psi_k^- \circ \pi(x)\right| \leq \frac{\epsilon}{6} .$$

Then

$$g = L(\theta(h))\psi_k^0 \circ \pi + q\psi_k^+ \circ \pi + q\psi_k^- \circ \pi + g' \in G$$

is an element with the desired property. $\square$

Let $\beta \in \mathbb{R}$. For each $\omega \in \pi^{-1}(\beta)$, define $\omega_\beta : \Gamma \to \mathbb{R}$ by

$$\omega_\beta(\xi, g) = g(\omega) .$$

Then $\omega_\beta(\Gamma^+) \subseteq [0, \infty)$, $\omega_\beta(v) = 1$, and $\omega_\beta \circ \alpha = e^{-\beta}\omega_\beta$.

Lemma 4.12. *Let $\phi : \Gamma \to \mathbb{R}$ be a positive homomorphism with the properties that $\phi(v) = 1$ and $\phi \circ \alpha = s\phi$ for some $s > 0$. Set $\beta = -\log s$. There is an element $\omega \in \pi^{-1}(\beta)$ such that $\phi = \omega_\beta$.*

Proof. The projection $p : \Gamma \to G$ is surjective by Lemma 4.8. Assume that $(\xi, g) \in \Gamma$ and $p(\xi, g) = g = 0$. Since (Γ, Γ^+) has large denominators by Lemma 4.9 there is for each $n \in \mathbb{N}$ an element $(\xi_n, g_n) \in \Gamma^+$ and a natural number k_n such that $n(\xi_n, g_n) \leq v \leq k_n(\xi_n, g_n)$. Then $\pm(\xi, g) \leq (\xi_n, g_n)$ in Γ and hence $\pm\phi(\xi, g) \leq \phi(\xi_n, g_n) \leq \frac{1}{n}$. It follows that $\phi(\xi, g) = 0$ and we conclude that there is a homomorphism $\phi' : G \to \mathbb{R}$ such that $\phi' \circ p = \phi$. Let $g \in G$ and assume that $g(x) \geq 0$ for all $x \in S$, and let $n \in \mathbb{N}$. There is $h_n \in H^+$ such that $0 < \theta(h_n) < \frac{1}{n}$, and then also a natural number $k \in \mathbb{N}$ such that $0 < L(\theta(h_n))(x)\psi^0_k \circ \pi(x) + \frac{1}{2n}\psi^+_k(\pi(x)) + \frac{1}{2n}\psi^-_k(\pi(x)) < \frac{1}{n}$ for all $x \in S$. Then

$$g'_n := L(\theta(h_n))\psi^0_k \circ \pi + \frac{1}{2n}\psi^-_k \circ \pi + \frac{1}{2n}\psi^+_k \circ \pi \in G^+$$

and $0 \leq n(h^{(0)}_n, g'_n) \leq v$ in Γ. Hence $0 \leq \phi'(g'_n) \leq \frac{1}{n}$. Let $\xi \in \bigoplus_{n\in\mathbb{Z}} H$ be an element such that $(\xi, g) \in \Gamma$. Then $(h^{(0)}_n + \xi, g'_n + g) \in \Gamma^+$ and hence $0 \leq \phi'(g'_n + g) \leq \phi'(g) + \frac{1}{n}$. Letting n tend to infinity we find that $\phi'(g) \geq 0$, proving that ϕ' is positive on G. Let $g \in G$ and $n, m \in \mathbb{N}$ satisfy $|g(x)| < \frac{n}{m}$ for all $x \in S$. Then $-n < mg(x) < n$ for all $x \in S$ and since $\phi'(1) = 1$ this leads to the conclusion that $|\phi'(g)| \leq \frac{n}{m}$. Combined with Lemma 4.11 it follows from the last estimate that ϕ' extends by continuity to a linear map $\phi' : \mathcal{A}_{\mathbb{R}}(S, \pi) \to \mathbb{R}$ such that $|\phi'(f)| \leq \sup_{x\in S} |f(x)|$. Using a Hahn-Banach theorem we extend ϕ' in a norm-preserving way to the space of all continuous real-valued functions on S with a limit at infinity. Since $\phi'(1) = 1$, the extension is positive. It follows that there is a bounded Borel measure m on S such that

$$\phi'(f) = \int_S f(x)\ \mathrm{d}m$$

for all $f \in \mathcal{A}_0(S, \pi)$, where $\mathcal{A}_0(S, \pi)$ denotes the space of elements in $\mathcal{A}_{\mathbb{R}}(S, \pi)$ that vanish at infinity. Let $C_c(\mathbb{R})$ denote the set of continuous real-valued compactly supported functions on $\mathbb{R}$ and note that $C_c(\mathbb{R})$ is mapped into $\mathcal{A}_0(S, \pi)$ by the formula $F \mapsto F \circ \pi$. Since $\phi \circ \alpha = s\phi$ by assumption it follows that the measure $m \circ \pi^{-1}$ on $\mathbb{R}$ satisfies

$$\int_{\mathbb{R}} e^{-t} F(t)\ \mathrm{d}m \circ \pi^{-1}(t) = s \int_{\mathbb{R}} F(t)\ \mathrm{d}m \circ \pi^{-1}(t) \quad \forall F \in C_c(\mathbb{R}).$$

It follows that $m \circ \pi^{-1}$ is concentrated at the point $\beta = -\log s$ and hence that m is concentrated on $\pi^{-1}(\beta)$. We can therefore define a linear functional $\phi'' : \mathrm{Aff}\pi^{-1}(\beta) \to \mathbb{R}$ by

$$\phi''(f) = \phi'(\hat{f}) = \int_S \hat{f}(x)\ \mathrm{d}m(x)\ ,$$

where $\hat{f} \in \mathcal{A}_0(S,\pi)$ is any element with $\hat{f}|_{\pi^{-1}(\beta)} = f$, which exists by (1) in Lemma 4.4. If $f \geq 0$ it follows from (2) of Lemma 4.4 that $\hat{f}$ can be chosen such that $\hat{f} \geq -\epsilon$ for any $\epsilon > 0$ and we see therefore that ϕ'' is a positive linear functional. Since every state of $\operatorname{Aff}\pi^{-1}(\beta)$ is given by evaluation at a point in $\pi^{-1}(\beta)$ it follows in this way that there is an $\omega \in \pi^{-1}(\beta)$ and a number $\lambda \geq 0$ such that

$$\phi'(g) = \lambda g(\omega) \tag{4.9}$$

for all $g \in \mathcal{A}_0(S,\pi)$. In particular, this conclusion holds for all $g \in G \cap \mathcal{A}_0(S,\pi)$. A general element $f \in G$ can be write as a sum

$$f = f_- + f_0 + f_+,$$

where $f_\pm, f_0 \in G$, f_0 has compact support and there are natural numbers $n_\pm \in \mathbb{N}$ such that $e^{n_- \pi} f_- \in \mathcal{A}_0(S,\pi)$ and $e^{-n_+ \pi} f_+ \in \mathcal{A}_0(S,\pi)$. Then $\phi'(f_0) = \lambda f_0(\omega)$,

$$\phi'(f_-) = \phi'(\alpha^{n_-}(e^{n_-\pi} f_-)) = s^{n_-}\phi'(e^{n_-\pi} f_-) = s^{n_-}\lambda e^{n_-\pi(\omega)} f_-(\omega) = \lambda f_-(\omega),$$

and, similarly, $\phi'(f_+) = \lambda f_+(\omega)$. It follows that $\phi(f) = \lambda f(\omega)$. Inserting $f = 1$ we find that $\lambda = 1$ and the proof is complete. ☐

4.4. Application of the Pimsner-Voiculescu exact sequence Let B be a stable AF algebra with $(K_0(B), K_0(B)^+) = (\Gamma, \Gamma^+)$ and let γ be an automorphism of B such that $\gamma_* = \alpha$; see [8].

Additional properties 4.13. By Lemma 4.3 we can arrange that γ has the following additional properties:

(A) The restriction map $\mu \mapsto \mu|_B$ is a bijection from traces μ on $B \rtimes_\gamma \mathbb{Z}$ onto the γ-invariant traces on B, and
(B) $B \rtimes_\gamma \mathbb{Z}$ is $\mathcal{Z}$-stable; that is $(B \rtimes_\gamma \mathbb{Z}) \otimes \mathcal{Z} \simeq B \rtimes_\gamma \mathbb{Z}$ where $\mathcal{Z}$ denotes the Jiang-Su algebra, [15].

Set

$$C = B \rtimes_\gamma \mathbb{Z}\ .$$

It follows from Lemma 4.10 and [8] that B is γ-simple and hence from [9] (see also[16]) that C is simple. It follows from the Pimsner-Voiculescu exact sequence, [21], that we can identify $K_0(C)$, as a group, with the quotient

$$\Gamma/(\operatorname{id} - \alpha)(\Gamma),$$

in such a way that the map $\iota_* : K_0(B) \to K_0(C)$ induced by the inclusion $\iota : B \to C$ becomes the quotient map

$$q : \Gamma \to \Gamma/(\operatorname{id} - \alpha)(\Gamma).$$

Define $S_0 : \Gamma \to H$ such that

$$S_0(\xi, g) = \Sigma(\xi).$$

LEMMA 4.14. $\ker S_0 = (\mathrm{id} - \alpha)(\Gamma)$.

PROOF. Since $(\mathrm{id} - \alpha)(\Gamma) = (\mathrm{id} - \sigma) \oplus (\mathrm{id} - \alpha_0)$ and $\Sigma \circ (\mathrm{id} - \sigma) = 0$, we find that $(\mathrm{id} - \alpha)(\Gamma) \subseteq \ker S_0$. Let $(\xi, g) \in \Gamma$, and assume that $S_0(\xi, g) = \Sigma(\xi) = 0$. By Lemma 4.6 of [25] there is an element $\xi' \in \bigoplus_{\mathbb{Z}} H$ such that $(\mathrm{id} - \sigma)(\xi') = \xi$. By the definition of Γ there is $\epsilon > 0$ such that $\hat{L}(\xi)$ and g agree on $\pi^{-1}(] - \epsilon, \epsilon[)$, and so when $k \geq \epsilon^{-1}$ we have

$$g = \hat{L}(\xi)\psi_k^0 \circ \pi + h^- \psi_k^- \circ \pi + h^+ \psi_k^+ \circ \pi + g_0$$

for some $h^{\pm} \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ and some $g_0 \in G_{00}$. By the definition of $\mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ there are elements $f^{\pm} \in \mathbb{Q}[e^{-\pi}, 1 - e^{-\pi}]$ such that $h^{\pm} = (\mathrm{id} - \alpha_0)(f^{\pm})$ and by (4.7) there is an element $g' \in G_{00}$ such that $g_0 = (\mathrm{id} - \alpha_0)(g')$. Define

$$g'' := \hat{L}(\xi')\psi_k^0 \circ \pi + f^- \psi_k^- \circ \pi + f^+ \psi_k^+ \circ \pi + g' \ \in \ G.$$

Since

$$(\mathrm{id} - \alpha_0)(\hat{L}(\xi')\psi_k^0 \circ \pi) = \hat{L}((\mathrm{id} - \sigma)(\xi'))\psi_k^0 \circ \pi = \hat{L}(\xi)\psi_k^0 \circ \pi,$$

it follows that $g = (\mathrm{id} - \alpha_0)(g'')$ and hence that $(\xi, g) = (\mathrm{id} - \alpha)(\xi', g'')$. □

It follows from Lemma 4.14 that S_0 induces an isomorphism

$$S : K_0(C) = \Gamma/(\mathrm{id} - \alpha)(\Gamma) \to H$$

such that $S \circ q = S_0$.

LEMMA 4.15. $S(K_0(C)^+) = H^+$.

PROOF. Let $h \in H^+ \backslash \{0\}$. There is then a k so large that $L(\theta(h))(x) > 0$ for all $x \in \pi^{-1}([-\frac{1}{k}, \frac{1}{k}])$. Define

$$g := L(\theta(h))\psi_k^0 \circ \pi + \psi_k^- \circ \pi + \psi_k^+ \circ \pi \in G_k \ .$$

Then $(h^{(0)}, g) \in \Gamma^+$, $q((h^{(0)}, g)) \in K_0(C)^+$, and $S(q((h^{(0)}, g))) = h$. Hence, $S(K_0(C)^+) \supseteq H^+$. Consider an element $x \in K_0(C)^+ \backslash \{0\}$ and write $x = q(\xi, g)$ for some $(\xi, g) \in \Gamma$. Let $\omega \in \pi^{-1}(0)$. Since $\omega_0 \circ \alpha = \omega_0$, there is a γ-invariant trace τ_ω on B such that $\tau_{\omega *} = \omega_0$; see Lemma 3.5 in [25]. Denote by $P : C \to B$ the canonical conditional expectation and note that $\tau_\omega \circ P$ is a trace on C. Since $x \in K_0(C)^+ \backslash \{0\}$ and C is simple it follows that

$$(\tau_\omega \circ P)_*(x) > 0.$$

Since $(\tau_\omega \circ P)_*(x) = \Sigma(\xi)(\omega)$, and $\omega \in \pi^{-1}(0)$ was arbitrary, it follows that $S(x) = \Sigma(\xi) \in H^+ \backslash \{0\}$. Hence, $S(K_0(C)^+) \subseteq H^+$. □

LEMMA 4.16. $K_1(C) = 0$.

PROOF. To establish this from the Pimsner-Voiculescu exact sequence, [21], we must show that $\mathrm{id} - \alpha$ is injective. Let $(\xi, g) \in \Gamma$ and assume that $\alpha(\xi, g) = (\xi, g)$. Then $\sigma(\xi) = \xi$, implying that $\xi = 0$ and hence that $g|_{\pi^{-1}(0)} = 0$. Since $(1 - e^{-\pi(x)})g(x) = 0$ for all $x \in S$, it follows that $g = 0$. □

Let $e \in B$ be a projection such that $[e] = v$ in $K_0(B) = \Gamma$. Since eCe is stably isomorphic to C by [5] it follows that $(K_0(eCe), K_0(eCe)^+) = (K_0(C), K_0(C)^+)$.

4.5. Completing the proof of Theorem 3.1 *via classification theory* Let (S, π) and A be as in Theorem 3.1. With $H = K_0(A)$ and the assumed identification of the tracial state space $T(A)$ of A with $\pi^{-1}(0)$ we get the homomorphism $\theta : H \to \mathrm{Aff}\pi^{-1}(0)$ from the canonical map $K_0(A) \to \mathrm{Aff}T(A)$. It follows from Theorem 4.11 of [14] that $\theta(K_0(A))$ is dense in $\mathrm{Aff}\pi^{-1}(0)$ and that

$$K_0(A)^+ = \left\{h \in K_0(A) : \ \theta(h)(x) > 0, \ x \in \pi^{-1}(0)\right\} \cup \{0\}.$$

We can therefore apply the preceding analysis with $H = K_0(A), \ H^+ = K_0(A)^+$, and $u = [1]$.

Let τ be a trace state on eCe. Then $\tau_* \circ S^{-1} : H \to \mathbb{R}$ is a positive homomorphism such that $\tau_* \circ S^{-1}(u) = \tau_*(q(v)) = \tau(e) = 1$, and there is therefore a unique trace state τ' on A such that

$${\tau'}_* = \tau_* \circ S^{-1}$$

on $K_0(A) = H$.

LEMMA 4.17. *The map $\tau \to \tau'$ is an affine homeomorphism from $T(eCe)$ onto $T(A)$.*

PROOF. The map is clearly affine. To show that it is continuous, assume that $\{\tau_n\}$ is a convergent sequence in $T(eCe)$ and let $\tau = \lim_{n\to\infty} \tau_n$. Then $\lim_{n\to\infty} \tau_{n*} \circ S^{-1}(h) = \tau_* \circ S^{-1}(h)$ for all $h \in H$. Since A is AF this implies that $\lim_{n\to\infty} \tau'_n = \tau'$ in $T(A)$. To see that the map is surjective, let $\tau \in T(A)$. Then $\tau_* : H \to \mathbb{R}$ is given by evaluation at a point $\omega \in \pi^{-1}(0)$, and $\tau_1 = \tau_\omega \circ P$ is a trace state on eCe such that $\tau'_1 = \tau$. To see that the map is also injective, consider $\tau_1, \tau_2 \in T(eCe)$. If $\tau'_1 = \tau'_2$, it follows that $\tau_{1*} = \tau_{2*}$. Since $\tau_{1*} \circ \iota_* = \tau_{2*} \circ \iota_*$ and B is AF it follows that $\tau_1|_B = \tau_2|_B$. Thanks to (A) from Additional properties 4.13 this implies that $\tau_1 = \tau_2$. ☐

LEMMA 4.18. *eCe is $*$-isomorphic to A.*

PROOF. Since A is AF the K_1 group of A is trivial, and by Lemma 4.16 the same is true for eCe since eCe is stably isomorphic to C. The affine homeomorphism $\tau \to \tau'$ of Lemma 4.17 is compatible with the isomorphism of ordered groups $S : K_0(eCe) \to K_0(A)$ from Lemma 4.15 in the sense that ${\tau'}_* \circ S = \tau_*$, resulting in an isomorphism from the Elliott invariant of eCe onto that of A. Both algebras, A and eCe, are separable, simple, unital, nuclear and in the UCT class. It is well known that all infinite-dimensional unital simple AF algebras are approximately divisible and hence $\mathcal{Z}$-absorbing by Theorem 2.3 of [27]; in particular, A is $\mathcal{Z}$-absorbing. Since C is $\mathcal{Z}$-absorbing thanks to (B) in Additional properties 4.13, it follows from Corollary 3.2 of [27] that eCe is $\mathcal{Z}$-absorbing. Therefore eCe is isomorphic to A by Corollary D of [6], which in turn is based on [12], [13], [10] and [26]. (In the case where A is a UHF algebra there is an alternative route through the literature to the same effect. See Remark 4.12 in [25].) ☐

We consider the dual action on $C = B \rtimes_\gamma \mathbb{Z}$ as a 2π-periodic flow and we denote by θ the restriction of this flow to eCe.

Lemma 4.19. *The KMS bundle (S^θ, π^θ) of θ is isomorphic to (S, π).*

Proof. Let $(\omega, \beta) \in S^\theta$. By Corollary 4.2, $(\hat{\omega}|_B)_*$ is a positive homomorphism $\Gamma \to \mathbb{R}$ such that $(\hat{\omega}|_B)_*(v) = 1$ and $(\hat{\omega}|_B)_* \circ \alpha = e^{-\beta}(\hat{\omega}|_B)_*$. By Lemma 4.12, there is $\mu \in \pi^{-1}(\beta)$ such that $(\hat{\omega}|_B)_*(\xi, g) = g(\mu)$ for all $(\xi, g) \in \Gamma$. μ is unique since G separates the points of S by Lemma 4.11. We define $\Phi : S^\theta \to S$ by $\Phi(\omega, \beta) = \mu$. By combining Lemma 4.12 and Corollary 4.2 we conclude that Φ restricts to an affine bijection from ${\pi^\theta}^{-1}(\beta)$ onto $\pi^{-1}(\beta)$ for every $\beta \in \mathbb{R}$. It follows in particular that Φ is surjective. If $(\omega_i, \beta_i) \in S^\theta$, $i = 1, 2$, are such that $\Phi((\omega_1, \beta_1)) = \Phi((\omega_2, \beta_2))$, it follows that $\beta_1 = \pi\left(\Phi((\omega_1, \beta_1))\right) = \pi\left(\Phi((\omega_2, \beta_2))\right) = \beta_2$ and hence that $(\omega_1, \beta_1) = (\omega_2, \beta_2)$. Thus, Φ is a bijection. Since $\pi \circ \Phi = \pi^\theta$, and π and π^θ are both proper maps, it suffices to show that Φ^{-1} is continuous. Let therefore $\{\omega^n\}$ be a sequence in S such that $\lim_{n\to\infty} \omega^n = \omega$ in S. Set $\beta_n = \pi(\omega^n)$ and note that $\lim_{n\to\infty} \beta_n = \beta$, where $\beta = \pi(\omega)$. It follows that $\lim_{n\to\infty} \omega^n_{\beta_n}(x) = \omega_\beta(x)$ for all $x \in \Gamma$. Let τ^n and τ be the traces on B determined by the conditions that ${\tau^n}_* = \omega^n_{\beta_n}$ and $\tau_* = \omega_\beta$. Then $\Phi^{-1}(\omega^n) = (\tau^n \circ P|_{eCe}, \beta_n)$ and $\Phi^{-1}(\omega) = (\tau \circ P|_{eCe}, \beta)$. It suffices therefore to show that $\lim_{n\to\infty} \tau^n \circ P(exe) = \tau \circ P(exe)$ for all $x \in C$. Since $\tau^n \circ P(e) = \tau \circ P(e) = 1$, it suffices to check for x in a dense subset of C. If w is the canonical unitary in the multiplier algebra of C coming from the construction of C as a crossed product, it suffices to show that $\lim_{n\to\infty} \tau^n \circ P(ebw^k e) = \tau \circ P(ebw^k e)$ for all $k \in \mathbb{Z}$ and all $b \in B$. Since $P(ebw^k e) = 0$ when $k \neq 0$ it suffices to consider the case $k = 0$; that is, it suffices to show that $\lim_{n\to\infty} \tau^n(ebe) = \tau(ebe)$. By approximating ebe by a linear combination of projections from eBe it suffices to show that $\lim_{n\to\infty} \tau^n(p) = \tau(p)$ when p is a projection in eBe. This holds because

$$\lim_{n\to\infty} \tau^n(p) = \lim_{n\to\infty} \omega^n_{\beta_n}([p]) = \omega_\beta([p]) = \tau(p).$$

□

The proof of Theorem 3.1 is complete.

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