

NOTE ON POINCARÉ L^p TYPE INEQUALITY FOR DIFFERENTIAL FORMS ON SEMIALGEBRAIC SETS

Dedicated to my advisor Pierre Milman, on the occasion of the fourth anniversary of his working seminar for graduate students and postdocs.

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ABSTRACT. We study local and global Poincaré type L^p inequalities on a compact semialgebraic subset of \mathbb{R}^n for $p \gg 1$. As a consequence, we obtain an isomorphism between L^p cohomology and singular cohomology of a normal compact semialgebraic set. The global inequality is derived from the local one, while the local inequality is proved by means of a semialgebraic Lipschitz deformation retraction with estimates on its derivatives.

RÉSUMÉ. On étudie les inégalités locales et globales de type L^p de Poincaré sur un sous-ensemble compact semialgébrique de \mathbb{R}^n pour $p \gg 1$. Par conséquent, nous obtenons un isomorphisme entre la cohomologie L^p et la cohomologie singulière d'un ensemble normal compact semialgébrique. L'inégalité globale est dérivée de la locale, tandis que l'inégalité locale est prouvée au moyen d'une rétraction de déformation semialgébrique Lipschitz avec des estimations sur ses dérivés.

1. Introduction. Let $X \subset \mathbb{R}^n$ be a pure dimensional compact semialgebraic set and ω a smooth k -form on X_{reg} , the regular (= smooth) part of X . We say that ω is L^p bounded when

$$\|\omega\|_{L^p} := \left(\int_{X_{\text{reg}}} |\omega(x)|^p d\text{Vol}(x) \right)^{1/p} < \infty,$$

where $|\omega(x)|$ is defined in Section 2. Assume ω is a closed L^p bounded smooth k -form on X_{reg} . We prove that if all of the integrals of ω on the cycles in X vanish (see Section 6 for the precise definition of an integral of an L^p bounded form on a cycle in X) then there exists a smooth $(k-1)$ -form ξ such that $\omega = d\xi$ and, moreover,

$$(1.1) \quad \|\xi\|_{L^p(X)} \leq C \|\omega\|_{L^p(X)}$$

holds for $p \gg 1$, where C depends only on the set X and p . Of course for a contractible semialgebraic set X , there are no cycles in X . Consequently there is a form ξ such that $\omega = d\xi$, and inequality (1.1) holds on X for $p \gg 1$ (cf. [S1]).

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The main technical tool used in the proof of (1.1) is a Lipschitz deformation retraction r to a single point with estimates on its derivatives (Theorem 3.2). Roughly, Theorem 3.2 says that for a semialgebraic set X and $a \in X$ there exist a neighborhood $U \subset X$ of a and a deformation retraction $r: U \times [0, 1] \rightarrow U$ with the following properties.

- (1) $r_0(x) = 0$, $r_1(x) = x$,
- (2) $|\det Dr_t| \geq C_0 t^\mu$, for some $\mu \geq 0$, $C_0 > 0$,
- (3) $\|Dr_t\| \leq C_1 t^\lambda$ for some $\lambda > 0$, $C_1 > 0$,

where $r_t(x) := r(x, t)$, Dr_t is the tangent map of r_t , $\|Dr_t\|$ denotes the operator max-norm of the tangent map and $|\det Dr_t|$ represents the infinitesimal change in volume of the map r_t .

By means of the latter deformation retraction we define a homotopy operator $R: \Omega_{L^p}^k(U \cap X_{\text{reg}}) \rightarrow \Omega_{L^p}^{k-1}(U \cap X_{\text{reg}})$ (see Section 2 for the definition of $\Omega_{L^p}^k$) such that for a closed form ω we have $\omega = dR\omega$, and our estimates on the derivatives of the deformation retraction r allow us to prove that for $p \gg 1$ our homotopy operator R is an L^p bounded operator. Consequently we conclude that $\xi := R\omega$ is a solution to inequality (1.1) on a neighborhood of a point in X .

REMARK 1.1. The construction of a homotopy operator given a deformation retraction is classical in algebraic topology; see *e.g.* [BT, Chapter 1, Section 4].

Fortunately, ‘globalization’ of the local L^p inequality (1.1) to a semialgebraic set can be carried out essentially just like for a smooth manifold in [S1]. The basic two facts that are needed for proving this global version is the validity of the local version of inequality (1.1) and the existence of a partition of unity (with locally bounded differentials), which semialgebraic sets admit.

One of the most important applications of the local version of inequality (1.1) is in the theory of L^p cohomology on semialgebraic sets. To define L^p cohomology we consider a differential complex consisting of the L^p bounded forms with L^p bounded weak exterior derivatives on the regular part of the set in question. L^p cohomology is defined as the factor space of closed L^p bounded forms by the exact L^p bounded forms. Of course for compact semialgebraic sets, L^p cohomology is an invariant of the induced metric. But the question of finiteness of the latter in general (for any $p \leq \infty$) was open. The L^p cohomology theory is addressed in several special cases by various authors (see, *e.g.*, [Ch], [Y], [HP], [GKS], [GKS2], [GKS3], [Gr], [O]).

In this article we show that, as a consequence of inequality (1.1), the L^p Poincaré lemma is valid for $p \gg 1$. Hence the L^p cohomology coincides with the singular cohomology of a compact (normal) semialgebraic set.

The purpose of this article is to announce our main results. For proofs of our results, see [S2].

2. Notations and basic definitions. Let $X \subset \mathbb{R}^n$ be a semialgebraic set. Denote by X_{reg} the subset of X consisting of points where X is a smooth manifold and set $X_{\text{sing}} := X - X_{\text{reg}}$. Note that X_{reg} is a Riemannian manifold with metric

induced from \mathbb{R}^n . Denote by \bar{X} the closure of X .

- $(\Omega^\bullet(X_{\text{reg}}), d)$ denotes the complex of smooth k -forms on X_{reg} with exterior derivative $d: \Omega^k(X_{\text{reg}}) \rightarrow \Omega^{k+1}(X_{\text{reg}})$.
- For a form $\omega \in \Omega^k(X_{\text{reg}})$ define the L^p norm by

$$\|\omega\|_{L^p} := \left(\int_{X_{\text{reg}}} |\omega(x)|^p d\text{Vol}(x) \right)^{1/p} < \infty,$$

where $|\omega(x)|$ is the pointwise norm of ω at the point $x \in X_{\text{reg}}$ defined by

$$\sup_{v \in \wedge^k(T_x X_{\text{reg}})} \frac{|\omega(x; v)|}{|v|},$$

where $|v|$ is the norm of the element v .

Suppose that X_{reg} is of dimension n and ω is a locally integrable k -form on X_{reg} . A locally integrable $k+1$ -form γ on X_{reg} is said to be *the weak exterior derivative* of ω if for every point $p \in X_{\text{reg}}$ there exists a neighborhood U such that for every smooth $(n-k-1)$ -form ϕ compactly supported in U we have

$$\int_U \omega \wedge d\phi = (-1)^{k+1} \int_U \gamma \wedge \phi.$$

The weak exterior derivative of ω is denoted by $\bar{d}\omega$. Let $(\Omega_{L^p}^\bullet(X_{\text{reg}}), \bar{d})$ be the complex of L^p bounded forms with L^p bounded weak exterior derivatives, *i.e.*, forms ω with

$$\|\omega\|_{L^{p,1}} := \|\omega\|_{L^p} + \|\bar{d}\omega\|_{L^p} < \infty.$$

3. Lipschitz deformation retraction theorem. Deformation retractions play an important role in De Rham theory. For instance, a standard proof of the classical Poincaré lemma on a star-shaped domain $U \subset \mathbb{R}^n$ uses a smooth deformation retraction $r: U \times [0, 1] \rightarrow U$ to construct a primitive of a closed form ω in the following way. Let us assume for simplicity that U is star-shaped (from $0 \in \mathbb{R}^n$). Let $r_t(x) = r(x, t) := tx$. Assume that ω is a closed form, then we have the following (unique) decomposition of the pull back of ω by r :

$$r^*\omega = \omega_0 + dt \wedge \omega_1,$$

where the differential forms ω_0 and ω_1 do not contain any terms involving dt . Set

$$\gamma(x) := \int_0^1 \omega_1(x, t) dt.$$

Now $d\gamma = \omega$. Indeed, since $d\omega = 0$ and d commutes with r^* we have

$$0 = dr^*\omega = d_x\omega_0 + dt \wedge \left(\frac{\partial\omega_0}{\partial t} - d_x\omega_1 \right),$$

where d_x represents the exterior derivative with respect to x . Therefore, $\frac{\partial \omega_0}{\partial t} = d_x \omega_1$ and hence

$$d\gamma(x) = \int_0^1 d_x \omega_1 dt = \int_0^1 \frac{\partial \omega_0}{\partial t} dt = \omega_0(x, t)|_{t=0}^{t=1} = r_1^* \omega(x) - r_0^* \omega(x) = \omega(x).$$

However, in this article we deal with semialgebraic sets X , which need not have star-shaped neighborhoods of every point. Therefore, to extend the Poincaré lemma to our setting we will have to construct Lipschitz semialgebraic deformation retractions with controlled growth of their derivatives. The main techniques of our construction are based on Lipschitz semialgebraic geometry theory developed in [V1].

DEFINITION 3.1. A *cell* in \mathbb{R}^n is defined by induction on n . For $n = 1$, a cell is either a point or an open interval. For $n > 1$ a cell is either a graph of a continuous semialgebraic function over a cell in \mathbb{R}^{n-1} or a band, *i.e.*, a set of the form $\{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \xi_1(x) < y < \xi_2(x), x \in C'\}$ where ξ_1 and ξ_2 are continuous semialgebraic functions over a cell C' in \mathbb{R}^{n-1} .

A *cell subdivision* of \mathbb{R}^n is a finite subdivision of \mathbb{R}^n into a disjoint collection of cells.

Let U be a closed semialgebraic set in \mathbb{R}^n . A *stratification* Σ of U is a partition of U into smooth manifolds called strata such that the boundary of each stratum is a union of strata from Σ . The pair (U, Σ) is called a *stratified set*. A stratification Σ of U is said to be compatible with $A \subset U$ if A is a union of strata from Σ .

The main result of this section is the following theorem.

THEOREM 3.2 (Lipschitz deformation retraction theorem). *Let Σ_0 be a stratification of \mathbb{R}^n , $X = \bigcup X_j$, $X_j \in \Sigma_0$, $0 \in \overline{X_j} \cap X$, $j = 1, \dots, m$. There exists a stratified neighborhood (U, Σ_U) of 0 in \mathbb{R}^n with Σ_U a cell subdivision such that $\Sigma_U \prec \Sigma \cap U$ and a Lipschitz semialgebraic deformation retraction $r: U \times [0, 1] \rightarrow U$ such that*

- (1) $r_0(x) = 0$, $r_1(x) = x$,
- (2) $r|_{S \times (0, 1]}$ is smooth for $S \in \Sigma_U$,
- (3) $|\det Dr_t| \geq C_0 t^\mu$, for some $\mu \geq 0$, $C_0 > 0$,
- (4) $\|Dr_t\| \leq C_1 t^\lambda$ for some $\lambda > 0$, $C_1 > 0$,

where the determinant $\det Dr_t$ is taken with respect to the coordinates of the respective cell of Σ_U and $\|Dr_t\|$ denotes the operator max-norm of the tangent map.

For a proof of this theorem, see [S2].

4. Local L^p inequality on a semialgebraic set. Let $X \subset \mathbb{R}^n$ be a compact semialgebraic set with $a \in X$. We say that X admits a *local L^p estimate* near $a \in X$ if there is a neighborhood U of a in X such that for every closed smooth

L^p bounded k -form ω , $k \geq 1$, defined in U there is a smooth form ξ , defined in U , such that

$$(4.1) \quad \begin{cases} \omega = d\xi \text{ in } U, \\ \|\xi\|_{L^p(U)} \leq C\|\omega\|_{L^p(U)}, \end{cases}$$

where $C > 0$ is independent of ω .

We show in this section that X admits local L^p estimate for $p \gg 1$. The main technical tool is our Lipschitz deformation retraction of Theorem 3.2.

4.1. Homotopy Operator. Let (U, Σ) be a stratified neighborhood and let $r: U \times I \rightarrow U$, $I := [0, 1]$, be the Lipschitz semialgebraic deformation retraction obtained by applying Theorem 3.2 to the set X and any stratification of \mathbb{R}^n that is compatible with X . Let $\varepsilon > 0$. We associate a homotopy operator R_ε with the deformation retraction r as follows: Let α be an L^p bounded smooth k -form on X_{reg} . The pull back $r^*\alpha$ is a form on $U \times I$ and can be represented as $\alpha_0 + dt \wedge \alpha_1$ where t is the coordinate in I . Define an operator

$$P: \Omega_{L^p}^k(U) \rightarrow \Omega_{L^p}^{k-1}(U \times I), \quad P\alpha := \alpha_1.$$

Set

$$R_\varepsilon\alpha := \int_\varepsilon^1 \alpha_1(x, t) dt.$$

Observe that $R_\varepsilon\alpha$ is defined almost everywhere on every stratum of Σ that is contained in U . The next theorem establishes the crucial bounds on the L^p norm of the homotopy operator.

THEOREM 4.1 (Local L^p inequality theorem). *Suppose that ω is a smooth L^p bounded k -form, $k \in \mathbb{N}$, defined on X_{reg} near $0 \in X$. Then there is a neighborhood U of $0 \in X$ such that*

- (i) $\|R_\varepsilon\omega\|_{L^p(U)} \leq C\|\omega\|_{L^p(U)}$,
- (ii) $R_\varepsilon\omega \rightarrow R_0\omega$ in L^p ,
- (iii) $\|r_\varepsilon^*\omega\|_{L^p(U)} \rightarrow 0$ as $\varepsilon \rightarrow 0$,

where $p \gg 1$ and $C > 0$ depend only on the set X .

REMARK 4.2. In Theorem 4.1 (i) and (ii), a lower bound on p is $\frac{\mu}{1+(k-1)\lambda}$ and in part (iii) is $\frac{\mu}{\lambda k}$, where μ and λ are from Theorem 3.2.

Next, we show that R_ε satisfies the classical homotopy identity.

PROPOSITION 4.3. *Suppose that V is a stratified neighborhood of $0 \in X$ provided by Theorem 3.2 and consequently the operator R_ε is defined on V . The homotopy operator R_ε satisfies the following homotopy identity:*

$$\bar{d}R_\varepsilon\alpha + R_\varepsilon d\alpha = \alpha - r_\varepsilon^*\alpha$$

for any smooth L^p bounded form α defined on $U := X_{\text{reg}} \cap V$.

It follows from Proposition 4.3 that in the case that ω is a closed k -form one has $\bar{d}R_\varepsilon\omega = \omega - r_\varepsilon^*\omega$. According to Theorem 4.1 for $p \gg 1$, $\|r_\varepsilon^*\omega\|_{L^p(U)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and $R_\varepsilon\omega \rightarrow \xi := R_0\omega$ and by (i) of Theorem 4.1 the form ξ satisfies $\|\xi\|_{L^p(U)} \lesssim \|\omega\|_{L^p(U)}$.

The form ξ is a limit in L^p space and thus is not necessarily smooth. However, by using smoothing techniques we obtain the following proposition.

PROPOSITION 4.4. *In the setting of Proposition 4.3, assume ω is a smooth L^p bounded closed form defined on U . Then there exists a smooth L^p bounded form ξ solving problem (4.1).*

5. L^p -cohomology. In this section we consider an L^p cohomology theory of a normal compact semialgebraic set X . The L^p cohomology is the cohomology of the complex $(\Omega_{L^p}^\bullet(X), \bar{d})$ commonly defined by

$$H_{L^p}^k(X) := \frac{\text{Ker}(\bar{d}: \Omega_{L^p}^k(X_{\text{reg}}) \rightarrow \Omega_{L^p}^{k+1}(X_{\text{reg}}))}{\text{Im}(\bar{d}: \Omega_{L^p}^{k-1}(X_{\text{reg}}) \rightarrow \Omega_{L^p}^k(X_{\text{reg}}))}.$$

Let

$$\Lambda_{L^p}^k(X) := \Omega^k(X_{\text{reg}}) \cap \Omega_{L^p}^k(X_{\text{reg}}),$$

and denote the k -th cohomology group of $(\Lambda_{L^p}^\bullet, d)$ by $H^k(\Lambda_{L^p}^\bullet(X))$. By using smoothing techniques one has the following proposition.

PROPOSITION 5.1. $H^k(\Lambda_{L^p}^\bullet(X)) = H_{L^p}^k(X)$.

DEFINITION 5.2. We will refer to a k -dimensional subset $X \subset \mathbb{R}^n$ as *normal* if for any $x \in X$, there exists $\varepsilon > 0$ such that $S^{n-1}(x, \varepsilon) \cap X_{\text{reg}}$ is connected, where $S^{n-1}(x, \varepsilon)$ is an $(n-1)$ -sphere in \mathbb{R}^n centered at x with radius ε (see [GM, Section 4]).

Since we work with compact sets, L^p boundedness is a local property and hence germs of L^p bounded k -forms define a sheaf on X . Namely, for every open set $U \subset X$ we associate the set $\Omega_{L^p}^k(U \cap X_{\text{reg}})$ or $\Lambda_{L^p}^k(U \cap X_{\text{reg}})$. We denote the sheaf of L^p bounded k -forms by $\Omega_{L^p}^k$ and the sheaf of smooth L^p bounded forms by $\Lambda_{L^p}^k$. The sheaves $\Omega_{L^p}^k$ and $\Lambda_{L^p}^k$ are fine on a compact semialgebraic set $X \subset \mathbb{R}^n$. Indeed, every open cover of X can be extended to an open cover of a neighborhood of X in \mathbb{R}^n , on which existence of a partition of unity is evident.

According to Theorem 4.4, locally, smooth closed k -forms on a semialgebraic set X are exact for $p \gg 1$ and $k > 0$. If X is normal then closed 0-forms are locally constant functions. It follows from here that the sheaf complex $\Lambda_{L^p}^\bullet$ on a normal set X comprises a fine resolution of the constant sheaf \mathbb{R} on X and therefore, a standard argument from sheaf theory implies that the singular cohomology of X naturally coincides with the cohomology of $\Omega_{L^p}^\bullet(X)$.

THEOREM 5.3. *Let $X \subset \mathbb{R}^n$ be a normal compact semialgebraic set. There exists $p \gg 1$ such that the cohomology of the L^p complex $\Omega_{L^p}^\bullet(X)$ is isomorphic to the singular cohomology of X .*

We remark that for small values of p the isomorphism of Theorem 5.3 need not hold.

EXAMPLE. Let X be a semialgebraic set such that (X_{reg}, g) is a smooth Riemannian manifold diffeomorphic to $M \times (0, 1]$, where M is a smooth compact m -dimensional manifold with a Riemannian metric g_M . Suppose that $g = dr^2 + r^{2\alpha}g_M$, where r is the coordinate in $(0, 1]$ and $\alpha \geq 1$. Topologically, X is a cone over the manifold M . Let ω be a smooth, closed, nonexact and radially constant k -form on X_{reg} , *i.e.*, ω does not contain any terms of the form $dr \wedge \cdots$ and the coefficients of ω are independent of r . The volume form on X_{reg} is given by $dV = r^{\alpha m} dr \wedge dV_M$, where dV_M stands for the volume form on M . The pointwise norm of ω is given by

$$|\omega(x, r)| = r^{-k\alpha} |\omega(x, r)|_M, \quad (x, r) \in M \times (0, 1],$$

where $|\cdot|_M$ is the pointwise norm on M . Note that since ω is closed and radially constant, it is independent of r and, in abuse of notation, we will write $\omega(x)$ instead of $\omega(x, r)$.

Clearly $\|\omega\|_{L^p(X)} < \infty$ if and only if $p < \frac{\alpha m + 1}{k\alpha}$:

$$\begin{aligned} \int_X |\omega(x)|^p dV &= \int_0^1 \int_M |\omega|^p r^{\alpha m} dr dV_M \\ &\approx \int_0^1 r^{-\alpha k p + \alpha m} dr < \infty. \end{aligned}$$

In fact, an L^p bounded closed (not exact) radially constant form defines a nontrivial cohomology class in $H_{L^p}^k$. Indeed, otherwise assume $\omega = d\xi$, $\|\xi\|_{L^p} < \infty$. Since ω is radially constant ξ must be radially constant as well. But then $\omega|_M = d\xi|_M$ which contradicts the assumption of ω not being exact. However, for $p \gg 1$ it follows from Theorem 5.3 that $H_{L^p}^k(X) = 0$ for $k > 0$. In our example, the minimal p for which $H_{L^p}^k(X) = 0$ equals $\frac{\alpha m + 1}{k\alpha}$.

6. Global L^p inequality on a semialgebraic set. For a closed form considered in Section 4 the problem of finding an antiderivative with the bound (4.1) can be generalized to a global problem on a compact semialgebraic set X . In the case that X is a compact smooth manifold, such problem is treated in [S1]. In general, due to the existence of topological obstructions, of course there are no antiderivatives for some closed forms. To overcome this obstacle, we derive a combinatorial condition under which closed forms are exact. In the case that X is a compact smooth manifold, a closed form on X is exact if and only if its integrals vanish on every cycle in X . We generalize this condition to closed L^p bounded forms by extending the notion of integration over cycles in X to the case of closed L^p bounded forms. In particular, our generalized condition for closed L^p bounded forms to be exact is the usual one (mentioned above) whenever X is a (nonsingular) manifold. Our definition of an integral of a closed

L^p bounded form over a cycle in X is placed in the forthcoming subsection and is of a combinatorial nature. In [S1] a combinatorial formula is derived for an integral of a closed form over a cycle in a manifold. It is constructed iteratively by means of a process of an application of the L^p inequality for forms on various contractible subsets. This process can be carried out for any class of forms that satisfy the L^p inequality for forms on a ‘good’ (or even ‘weakly good’) covering by contractible subsets. We show in the forthcoming subsection that closed L^p bounded forms satisfy the L^p inequality for forms on contractible sets, which would allow us to extend the notion of an integral over cycles to L^p bounded forms.

REMARK 6.1. In a paper by Gol’dshstein, Kuz’minov and Shvedov [GKS], the authors defined an integral of forms in $W_{p,q}^k$ over any k -dimensional manifold parametrized by a Lipschitz map. However, for our purposes, it suffices to define integrals of closed L^p bounded forms just over cycles.

6.1. *Definition of an integral of a closed L^p bounded form.* Assume $X \subset \mathbb{R}^n$ is a compact semialgebraic set.

DEFINITION 6.2. If $\mathcal{U} = \{U_i\}$ is a cover of X , we say that \mathcal{U} is a *weakly good cover* if each finite intersection of U_i ’s is contractible. The *nerve complex* of a cover \mathcal{U} is a simplicial complex $(C(\mathcal{U})_\bullet, \partial)$ with simplex $[I]$ associated with every nonempty intersection U_I . The boundary operator $\partial: C_l(\mathcal{U}) \rightarrow C_{l-1}(\mathcal{U})$ is defined as usually by

$$\partial[I] := \sum_{j>0} (-1)^j [i_0, \dots, \widehat{i_j}, \dots, i_l], \quad I = [i_0, \dots, i_l].$$

REMARK 6.3. It is a well-known fact that if \mathcal{U} is a weakly good cover of X then the homology of the nerve complex $C_\bullet(\mathcal{U})$ coincides with the singular homology of X (see e.g. [H, Corollary 4G.3]).

Every triangulable set X has a weakly good cover. Indeed, if T is a triangulation of X with $V := \{1, \dots, |T|\}$ being the set of vertices of T , then let $\mathcal{U} := \{U_i\}_{i \in V}$ be a cover of X , where U_i is the star of vertex i . We claim that $\{U_i\}_{i \in V}$ is a weakly good cover. Let $I := (i_0, \dots, i_l)$ and assume that $U_I := U_{i_0} \cap \dots \cap U_{i_l} \neq \emptyset$. Every simplex of dimension $\dim X$ in the closure of U_I contains the vertex i_0 . Therefore, it is possible to deformation retract the closure of U_I to i_0 . Consequently, every finite intersection U_I is contractible.

LEMMA 6.4. *Assume that X is a compact contractible semialgebraic set. There exists $p \gg 1$ such that for every closed k -form ω in $\Lambda_{L^p}^k(X_{\text{reg}})$, $k \geq 1$ there exists a form $\xi \in \Omega_{L^p}^{k-1}(X)$ such that*

$$(6.1) \quad \begin{cases} \omega = d\xi \text{ on } X_{\text{reg}}, \\ \|\xi\|_{L^p(X)} \leq C\|\omega\|_{L^p(X)}. \end{cases}$$

Let \mathcal{U} be a weakly good cover of X , $\omega \in \Omega_{L^p}^k(X)$ a closed form, where p is from Lemma 6.4 and c is a singular cycle in X . According to Remark 6.3, singular homology is isomorphic to the homology of $C_\bullet(\mathcal{U})$ and therefore, by means of this isomorphism, every singular cycle can be identified with a cycle $\sum_I a_I [I]$ for some real numbers a_I in the nerve complex of \mathcal{U} . To define the integral of ω over c we construct a collection of forms $\xi_I^s \in \Omega_{L^p}^{k-s-1}(U_I)$ as follows:

$$d\xi_I^{s+1} = (\delta\xi^s)_I \text{ on } U_I, \quad \text{for } s \geq 0,$$

where

$$(\delta\xi^s)_{i_0, \dots, i_{s+1}} := \sum_l (-1)^l \xi_{i_0, \dots, \widehat{i}_l, \dots, i_{s+1}}^s$$

and $\xi_{i_0}^0$ is a solution to $d\xi_{i_0}^0 = \omega|_{U_{i_0}}$ satisfying $\|\xi_{i_0}^0\|_{L^p(U_{i_0})} \lesssim \|\omega\|_{L^p(U_{i_0})}$ given by Lemma 6.4. Note that the forms ξ_I^{s+1} exist according to Lemma 6.4 since the forms $(\delta\xi^s)_I$ are closed on U_I . Indeed, $d(\delta\xi^s)_I = \delta(d\xi^s)_I = (\delta(\delta\xi^{s-1}))_I = 0$.

Define

$$(\text{Int } \omega)c = (-1)^{\lfloor \frac{k}{2} \rfloor} \sum a_I \delta\xi_I^{k-1}.$$

REMARK 6.5. It is proved in [S1] that if X is a compact manifold and ω is a closed smooth k -form on M then $(\text{Int } \omega)c = \int_c \omega$ for every cycle c in X .

Moreover, one can show (see *e.g.* [BT], [S1]) that the homomorphism Int is an isomorphism of $H_{L^p}^k(X)$ with $H^k(X)$, where $H^k(X)$ is the cohomology of the dual complex to $C_\bullet(\mathcal{U})$.

As a consequence of Lemma 6.4 and Remark 6.5, one may extend the definition of an integral over the cycles in X to all closed L^p bounded forms as follows:

$$\int_c \omega := (\text{Int } \omega)c = (-1)^{\lfloor \frac{k}{2} \rfloor} \sum a_I \delta\xi_I^{k-1},$$

where $c = \sum_I a_I [I]$ is a cycle in X .

6.2. *Global L^p inequality.* A global analog of problem (4.1) can be formulated as follows. Say that X satisfies the global L^p inequality for forms if there exists a constant $C > 0$ such that for every closed form $\omega \in \Lambda_{L^p}^k(X)$, $k > 0$ with zero integrals over every cycle in X , there is a form $\xi \in \Lambda_{L^p}^{k-1}(X)$ such that

$$(6.2) \quad \begin{cases} \omega = d\xi \text{ on } X_{\text{reg}}, \\ \|\xi\|_{L^p(X)} \leq C \|\omega\|_{L^p(X)}. \end{cases}$$

PROPOSITION 6.6. *For sufficiently large $p \gg 1$ if ω is a closed k -form in $\Lambda_{L^p}^k(X)$ and $\int_c \omega = 0$ for every $c \in H_k(X)$ then ω is exact and (6.2) holds for ω .*

The proof of this proposition follows the same argument as the proof of Theorem 3.1 in [S1]. The form ξ is constructed by following faithfully the structure of the construction in [S1, Sections 3.1 and 3.2] for a finite weakly good cover \mathcal{U} of X , but replacing the ‘ (p, q) Poincaré inequality’ for forms by Lemma 6.4.

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