

A MODIFICATION OF THE EFFROS-HANDELMAN-SHEN THEOREM WITH \mathbb{Z}_2 ACTIONS

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ABSTRACT. We show that a \mathbb{Z}_2 action on a lattice-ordered dimension group will arise as an inductive limit of \mathbb{Z}_2 actions on simplicial groups. The motivation for this study is the range of invariant problem in Elliott and Su's classification of AF type \mathbb{Z}_2 actions. We modify the proof of the Effros-Handelman-Shen theorem to include \mathbb{Z}_2 actions.

RÉSUMÉ. Nous montrons qu'une action de \mathbb{Z}_2 sur un groupe de dimension ordonné par treillis apparaît comme une limite inductive d'actions de \mathbb{Z}_2 sur des groupes simpliciaux. La motivation de cette étude est le problème de la gamme de l'invariant dans la classification d'Elliott et de Su des actions de \mathbb{Z}_2 de type AF. Nous modifions la preuve du théorème d'Effros-Handelman-Shen pour inclure les actions de \mathbb{Z}_2 .

1. Introduction The purpose of this study is to extend the Effros-Handelman-Shen theorem. We strengthen the condition on the dimension group to being a lattice-ordered group, but with \mathbb{Z}_2 actions added.

In [7], Handelman and Rossmann started to generalize the classification of AF algebras to the classification of algebras with actions on them by assuming that the algebra was a UHF algebra and the action was of product type. In [8], they generalized the study [7] to locally representable actions on an AF algebra. Blackadar showed that there are \mathbb{Z}_2 actions that are not locally representable even on UHF algebras in [1]. Elliott and Su generalized the K -theoretic classification that Handelman and Rossmann used in [8] by removing the requirement of local representability, but restricting the action to the group \mathbb{Z}_2 [5]. The range of the invariant problem for the classification of Elliott and Su has still not been completely solved. In this study, we prove a modification of the Effros-Handelman-Shen theorem which is a step toward an answer to this range of invariant problem.

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2. Main Theorem We prove the following modification of Effros, Handelman, and Shen's original theorem:

THEOREM 2.1. *If a \mathbb{Z}_2 action on a countable lattice-ordered dimension group is given, then it can be expressed as an inductive limit of \mathbb{Z}_2 actions on simplicial groups.*

Before proving Theorem 2.1, we shall need some technical results. The first is a modification of Proposition 3.15 of [6].

PROPOSITION 2.2. *Let G be a lattice-ordered dimension group, and α be an action of \mathbb{Z}_2 on G . Suppose x_1, \dots, x_n are elements of G^+ such that α acts on $\{x_1, \dots, x_n\}$ by a permutation σ . Suppose p_1, \dots, p_n are integers such that $p_1x_1 + p_2x_2 + \dots + p_nx_n = 0$. Then there exist elements y_1, \dots, y_t in G^+ such that α acts on $\{y_1, \dots, y_t\}$ by a permutation σ' and nonnegative integers q_{ij} (for $i = 1, \dots, n$, and $j = 1, \dots, t$) such that*

$$x_i = q_{i1}y_1 + \dots + q_{it}y_t \text{ and } p_1q_{1j} + \dots + p_nq_{nj} = 0$$

for all $i = 1, \dots, n$ and $j = 1, \dots, t$, and $MQ = QM'$, where M, M' are the permutation matrices giving σ, σ' respectively, and Q is the matrix of the q_{ij} s.

PROOF. The proof closely follows Goodearl's presentation [6, pp. 51–53]. From the hypotheses of the proposition, we get a positive homomorphism φ sending the simplicial basis for \mathbb{Z}^n to the elements x_i and an action α_n on \mathbb{Z}^n given by the permutation σ . The map φ sends the element $p = p_1e_1 + \dots + p_n e_n \in \mathbb{Z}^n$ to $p_1x_1 + \dots + p_nx_n$, which is 0. From the conclusion of the proposition, we construct a commuting diagram:

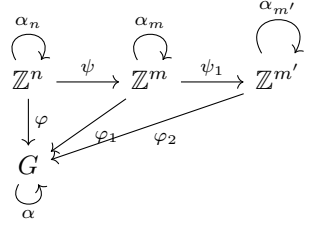
$$\left. \begin{array}{ccc} \begin{array}{c} \alpha_n \\ \curvearrowright \\ \mathbb{Z}^n \end{array} & \xrightarrow{\psi} & \begin{array}{c} \alpha_m \\ \curvearrowright \\ \mathbb{Z}^m \end{array} \\ \downarrow \varphi & \swarrow \varphi_2 & \\ \begin{array}{c} G \\ \curvearrowright \\ \alpha \end{array} & & \end{array} \right\} (*)$$

where the map ψ is given by the transpose, Q^\top , of the matrix Q in the statement of the proposition. We get two new maps ψ and φ_2 , and a new action α_m such that $\psi(p) = 0$ and $\psi \circ \alpha_n = \alpha_m \circ \psi$. The two new maps also intertwine the actions $\varphi_2 \circ \alpha_m = \alpha \circ \varphi_2$.

We will show that we may assume that for each i , either $\alpha(x_i) = x_i$ or $\alpha(x_i) \wedge x_i = 0$. Since G is a lattice-ordered group, we may introduce new variables r_i, s_i , and t_i defined by: $r_i = x_i \wedge \alpha(x_i)$, $s_i = x_i - (x_i \wedge \alpha(x_i))$, and $t_i = \alpha(x_i) - (x_i \wedge \alpha(x_i))$. These new variables satisfy $\alpha(r_i) = r_i$, $\alpha(s_i) = t_i$, $\alpha(t_i) = s_i$, $s_i \wedge t_i = 0$, $x_i = s_i + r_i$, and $\alpha(x_i) = t_i + r_i$. Notice that if $x_i = \alpha(x_i)$ then we have $r_i = x_i$ and $s_i = t_i = 0$. We replace our original list with the r_i s, and the non-zero s_i s and t_i s.

Now, we will show that the proposition for the original variables x_i follows from the statement for the new variables with the extra condition. First of all, we will show that there is a commutative diagram like (*) above with $\mathbb{Z}^n, \mathbb{Z}^m$, where m is the number of variables in our new list, and G . We get the relationship between \mathbb{Z}^n and G from the hypotheses of the proposition. For each i , we have two cases. Either e_i is fixed by α_n , or it is interchanged with another e_j . Suppose e_1, \dots, e_l are fixed and e_{l+1}, \dots, e_n are flipped in pairs by the action α_n . Then, we label the simplicial basis of \mathbb{Z}^m as $\{a_1, \dots, a_n, b_{l+1}, \dots, b_n, c_{l+1}, \dots, c_n\}$. We define a map φ_1 from \mathbb{Z}^m to G that on the generators is given by $a_i \mapsto r_i, b_i \mapsto s_i$, and $c_i \mapsto t_i$. We define an automorphism α_m of \mathbb{Z}^m given on the generators by $\alpha_m(a_i) = a_i, \alpha_m(b_i) = c_i$, and $\alpha_m(c_i) = b_i$ for each i .

We construct a map $\psi : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ as follows. Consider the element $e_i \in \mathbb{Z}^n$. We have that e_i is mapped to $x_i \in G$ by φ for each i . Since $x_i = r_i$ when $i = 1, \dots, l$ and $x_i = r_i + s_i$ when $i = l + 1, \dots, n$, we set $\psi(e_i) = a_i$ when $i = 1, \dots, l$ and $\psi(e_i) = a_i + b_i$ when $i = l + 1, \dots, n$. If we can solve the problem with our new assumption, applied to our new variables and the image of p under ψ , we get maps $\psi_1 : \mathbb{Z}^m \rightarrow \mathbb{Z}^{m'}$ and $\varphi_2 : \mathbb{Z}^{m'} \rightarrow G$ giving us the commutative diagram



where $p \in \mathbb{Z}^n$ and $\varphi : e_i \rightarrow x_i$. In this diagram, the maps $\psi_1 \circ \psi$ and φ_2 solve the problem with the original variables. From now on, we may assume either $\alpha(x_i) = x_i$ or $\alpha(x_i) \wedge x_i = 0$ for each i .

First of all, we need to consider the case in which $p_i \geq 0$ for all i . If $p_i > 0$ for some i , then

$$0 \leq x_i \leq p_i x_i \leq p_1 x_1 + \dots + p_n x_n = 0,$$

and hence $x_i = 0$. In particular, if $p_i > 0$ for all i , then $x_i = 0$ for all i . In case $p_i \leq 0$ for all i , we apply the same process to the relation $(-p_1)x_1 + \dots + (-p_n)x_n = 0$.

For the general case, we assign a degree to the coefficient list p_1, \dots, p_n , and proceed by induction on the degree. The degree of a list p_1, \dots, p_n will mean the ordered pair (p, λ) where p is the maximum of the values $|p_i|$, and λ is how many times p appears in the list $|p_1|, \dots, |p_n|$.

Next, we will show that we can divide the problem into the special cases $\alpha_n(p) = -p$ and $\alpha_n(p) = p$. If the problem can be solved in these two special cases, then we can solve the problem in the general case. If $\varphi(p) = 0$, then

$\varphi(\alpha_n(p)) = 0$, and so $\varphi(p + \alpha_n(p)) = 0$ and $\varphi(p - \alpha_n(p)) = 0$. Conversely, if $\varphi(p + \alpha_n(p)) = 0$ and $\varphi(p - \alpha_n(p)) = 0$, then $\varphi(2p) = 0$. Since G is torsion free, $\varphi(p) = 0$. We look at $q_1 = p + \alpha_n(p)$ and $q_2 = p - \alpha_n(p)$. Then, $\alpha_n(q_1) = q_1$ and $\alpha_n(q_2) = -q_2$. If we can solve the first special case, then we get a new group \mathbb{Z}^{m_1} and two maps $\varphi_2 : \mathbb{Z}^{m_1} \rightarrow G$ and $\psi_1 : \mathbb{Z}^n \rightarrow \mathbb{Z}^{m_1}$, both equivariant, such that $\varphi_2 \circ \psi_1 = \varphi$ and $\psi_1(q_2) = 0$. If we can solve the second special case, then applying this to φ_2 and $\psi_1(q_1)$, we get a new group \mathbb{Z}^{m_2} and two new maps $\psi_2 : \mathbb{Z}^{m_1} \rightarrow \mathbb{Z}^{m_2}$ and $\varphi_3 : \mathbb{Z}^{m_2} \rightarrow G$ such that $\varphi_3 \circ \psi_2 = \varphi_2$ and $\psi_2(\psi_1(q_1)) = 0$. If we combine both cases, we get the commutative diagram

$$\begin{array}{ccccc}
 & \alpha_n & & \alpha_{m_1} & & \alpha_{m_2} \\
 & \curvearrowright & & \curvearrowright & & \curvearrowright \\
 \mathbb{Z}^n & \xrightarrow{\psi_1} & \mathbb{Z}^{m_1} & \xrightarrow{\psi_2} & \mathbb{Z}^{m_2} & \cdot \\
 \downarrow \varphi & \nearrow \varphi_2 & \nearrow \varphi_3 & & & \\
 G & & & & & \\
 \curvearrowright & & & & & \\
 \alpha & & & & &
 \end{array}$$

Let $\psi = \psi_2 \circ \psi_1$. Then, $\psi(2p) = \psi(q_1 + q_2) = \psi_2(\psi_1(q_1) + \psi_1(q_2)) = \psi_2(\psi_1(q_1)) = 0$. Since \mathbb{Z}^{m_2} is torsion free, $\psi(p) = 0$. Thus $\psi, \mathbb{Z}^{m_2}, \alpha_{m_2}$, and φ_3 solve the original problem. This completes the reduction to the two special cases.

Now, we look at the first special case, $\alpha_n(p) = -p$. Suppose e_1, e_2, \dots, e_j are fixed and $e_{j+1}, \alpha_n(e_{j+1}), e_{j+2}, \alpha_n(e_{j+2}), \dots, e_k, \alpha_n(e_k)$ are flipped by α_n . Then

$$p = p_1 e_1 + p_2 e_2 + \dots + p_j e_j + p_{j+1} e_{j+1} + p'_{j+1} \alpha_n(e_{j+1}) + \dots + p_k e_k + p'_k \alpha_n(e_k).$$

Since $\alpha_n(p) = -p$, we have $p_1 = p_2 = \dots = p_j = 0$ and $p_l = -p'_l$, where $l = j+1, \dots, k$. One has the relation

$$p_{j+1} x_{j+1} + \dots + p_k x_k = p_{j+1} \alpha(x_{j+1}) + \dots + p_k \alpha(x_k),$$

where $p_l \geq 0$ for all l , and we may assume p_{j+1} is the largest coefficient. We have $p_{j+1} x_{j+1} \leq p_{j+1} \alpha(x_{j+1}) + \dots + p_k \alpha(x_k)$. We may assume $x_l \wedge \alpha(x_l) = 0$ for all $l = j+1, \dots, k$. Taking wedges with $p_{j+1} x_{j+1}$ on both sides gives

$$p_{j+1} x_{j+1} \leq (p_{j+1} \alpha(x_{j+1}) + \dots + p_k \alpha(x_k)) \wedge p_{j+1} x_{j+1}.$$

Since $x_{j+1} \wedge \alpha(x_{j+1}) = 0$, we have

$$\begin{aligned}
 p_{j+1} x_{j+1} &\leq p_{j+2} \alpha(x_{j+2}) + \dots + p_k \alpha(x_k) \\
 &\leq p_{j+1} \alpha(x_{j+2}) + \dots + p_{j+1} \alpha(x_k) \\
 &= p_{j+1} (\alpha(x_{j+2}) + \dots + \alpha(x_k)),
 \end{aligned}$$

and so

$$x_{j+1} \leq \alpha(x_{j+2}) + \cdots + \alpha(x_k).$$

By Riesz decomposition, which by [2] holds in a lattice-ordered group, $x_{j+1} = z_{j+2} + \cdots + z_k$ for some elements $z_i \in G^+$ such that $z_i \leq \alpha(x_i)$ for each $i = j+2, \dots, k$. Also, $\alpha(x_{j+1}) = \alpha(z_{j+2}) + \cdots + \alpha(z_k)$, $\alpha(z_i) \leq x_i$, and $\alpha(z_i) \wedge z_i = 0$, for each $i = j+2, \dots, k$.

Observe that

$$p_{j+1}x_{j+1} + \cdots + p_k x_k = p_{j+1}\alpha(x_{j+1}) + \cdots + p_k\alpha(x_k).$$

Since $x_{j+1} = z_{j+2} + \cdots + z_k$ and $\alpha(x_{j+1}) = \alpha(z_{j+2}) + \cdots + \alpha(z_k)$,

$$\begin{aligned} p_{j+1}(z_{j+2} + \cdots + z_k) + p_{j+2}x_{j+2} + \cdots + p_k x_k \\ = p_{j+1}(\alpha(z_{j+2}) + \cdots + \alpha(z_k)) + p_{j+2}\alpha(x_{j+2}) + \cdots + p_k\alpha(x_k). \end{aligned}$$

This implies

$$\begin{aligned} p_{j+1}(z_{j+2} + \cdots + z_k) + p_{j+2}x_{j+2} + p_{j+2}z_{j+2} - p_{j+2}z_{j+2} \\ + \cdots + p_k x_k + p_k z_k - p_k z_k \\ = p_{j+1}(\alpha(z_{j+2}) + \cdots + \alpha(z_k)) + p_{j+2}\alpha(x_{j+2}) + p_{j+2}\alpha(z_{j+2}) - p_{j+2}\alpha(z_{j+2}) \\ + \cdots + p_k\alpha(x_k) + p_k\alpha(z_k) - p_k\alpha(z_k). \end{aligned}$$

So,

$$\left. \begin{aligned} \sum_{i=j+2}^k (p_{j+1} - p_i)z_i + \sum_{i=j+2}^k p_i(x_i - \alpha(z_i)) \\ = \sum_{i=j+2}^k (p_{j+1} - p_i)\alpha(z_i) + \sum_{i=j+2}^k p_i(\alpha(x_i) - z_i). \end{aligned} \right\} \quad (\dagger)$$

We label the collection of the new variables as follows:

$$\begin{aligned} & \underbrace{z_{j+2}, \dots, z_k}_{\vec{z}}, \\ & \underbrace{x_{j+2} - \alpha(z_{j+2}), \dots, x_k - \alpha(z_k)}_{\vec{x} - \alpha(\vec{z})}, \\ & \underbrace{\alpha(z_{j+2}), \dots, \alpha(z_k)}_{\alpha(\vec{z})}, \end{aligned}$$

and

$$\underbrace{\alpha(x_{j+2}) - z_{j+2}, \dots, \alpha(x_k) - z_k}_{\alpha(\vec{x}) - \vec{z}}.$$

The relation (\dagger) has lower degree and satisfies condition 1. If p_1 still occurs, it occurs one time less. By the induction hypothesis, there exist elements y_1, \dots, y_t in G^+ such that α permutes these with a permutation σ' and nonnegative integers r_{il} , s_{il} , r'_{il} , and s'_{il} for $i = j + 2, \dots, k$ and $l = 1, \dots, t$ such that

$$z_i = r_{i1}y_1 + \dots + r_{it}y_t,$$

$$x_i - \alpha(z_i) = s_{(i)1}y_1 + \dots + s_{it}y_t,$$

$$\alpha(z_i) = \alpha(r_{i1}y_1 + \dots + r_{it}y_t) = r'_{i1}y_1 + \dots + r'_{it}y_t,$$

and

$$\alpha(x_i) - z_i = \alpha(x_i - \alpha(z_i)) = s'_{i1}y_1 + \dots + s'_{it}y_t,$$

for $i = j + 2, \dots, k$.

We get a matrix $R^\top = \begin{pmatrix} r & s & r' & s' \end{pmatrix}$ where r, s, r' , and s' are as in the equations above. Then, there exists a matrix M_σ which gives the permutation of the generators $\vec{z}, \vec{x} - \alpha(\vec{z}), \alpha(\vec{z}),$ and $\alpha(\vec{x}) - \vec{z}$ such that

$$M_\sigma \begin{pmatrix} \vec{z} & \vec{x} - \alpha(\vec{z}) & \alpha(\vec{z}) & \alpha(\vec{x}) - \vec{z} \end{pmatrix}^\top = \begin{pmatrix} \alpha(\vec{z}) & \alpha(\vec{x}) - \vec{z} & \vec{z} & \vec{x} - \alpha(\vec{z}) \end{pmatrix}^\top.$$

Also, the action of α on the y_t s is given by a permutation σ' , expressed by a matrix $M_{\sigma'}$. That $M_\sigma R = R M_{\sigma'}$ follows from the induction hypothesis.

By using the matrix $\begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix}$, where E_k is a $k \times k$ identity matrix, we get $\begin{pmatrix} \vec{x} \\ \alpha(\vec{x}) \end{pmatrix}$ from $\begin{pmatrix} \vec{z} & \vec{x} - \alpha(\vec{z}) & \alpha(\vec{z}) & \alpha(\vec{x}) - \vec{z} \end{pmatrix}^\top$. The matrix $M_{\sigma''} = \begin{pmatrix} 0 & E_k \\ E_k & 0 \end{pmatrix}$ gives a permutation such that $M_{\sigma''} \begin{pmatrix} \vec{x} & \alpha(\vec{x}) \end{pmatrix}^\top = \begin{pmatrix} \alpha(\vec{x}) & \vec{x} \end{pmatrix}^\top$. We have

$$M_{\sigma''} \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} = \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} M_\sigma.$$

Now we can check that $M_{\sigma''} Q = Q M_{\sigma'}$, where $Q = \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} R$, and

$$Q \vec{y} = \begin{pmatrix} 0 & E_k & E_k & 0 \\ E_k & 0 & 0 & E_k \end{pmatrix} \begin{pmatrix} \vec{z} \\ \vec{x} - \alpha(\vec{z}) \\ \alpha(\vec{z}) \\ \alpha(\vec{x}) - \vec{z} \end{pmatrix} = \begin{pmatrix} \vec{x} \\ \alpha(\vec{x}) \end{pmatrix}.$$

Next, we look at the second special case $\alpha_n(p) = p$. In this case, the coefficient of e_i is equal to the coefficient of $\alpha_n(e_i)$. Suppose e_1, \dots, e_h , and also e_{m+1}, \dots, e_s , are fixed, and $e_{h+1}, \alpha_n(e_{h+1}), \dots, e_m, \alpha_n(e_m)$ and $e_{s+1}, \alpha_n(e_{s+1}), \dots, e_l, \alpha_n(e_l)$ are flipped by α_n . Then,

$$p = p_1 e_1 + \dots + p_h e_h + p_{h+1} e_{h+1} + p_{h+1} \alpha_n(p_{h+1}) + \dots + p_m e_m + p_m \alpha_n(e_m)$$

$$-p_{m+1}e_{m+1} - \cdots - p_s e_s - p_{s+1}e_{s+1} - p_{s+1}\alpha_n(e_{s+1}) - \cdots - p_l e_l - p_l \alpha_n(e_l)$$

with all $p_i \geq 0$. Then our relation becomes

$$\begin{aligned} & p_1 x_1 + \cdots + p_h x_h + p_{h+1} x_{h+1} + p_{h+1} \alpha(x_{h+1}) + \cdots + p_m x_m + p_m \alpha(x_m) \\ & = p_{m+1} x_{m+1} + \cdots + p_s x_s + p_{s+1} x_{s+1} + p_{s+1} \alpha(x_{s+1}) + \cdots + p_l x_l + p_l \alpha(x_l) \end{aligned}$$

in G . We label the collection of variables in the original relation as follows:

$$\begin{aligned} & \underbrace{x_1, \cdots, x_h}_{\mathcal{X}_1} \\ & \underbrace{x_{h+1}, \cdots, x_m}_{\mathcal{X}_2} \\ & \underbrace{\alpha(x_{h+1}), \cdots, \alpha(x_m)}_{\alpha(\mathcal{X}_2)} \\ & \underbrace{x_{m+1}, \cdots, x_s}_{\mathcal{X}_3} \\ & \underbrace{x_{s+1}, \cdots, x_l}_{\mathcal{X}_4} \end{aligned}$$

and

$$\underbrace{\alpha(x_{s+1}), \cdots, \alpha(x_l)}_{\alpha(\mathcal{X}_4)}.$$

We put these together in one vector,

$$\vec{x} = (\mathcal{X}_1 \quad \mathcal{X}_2 \quad \alpha(\mathcal{X}_2) \quad \mathcal{X}_3 \quad \mathcal{X}_4 \quad \alpha(\mathcal{X}_4))^\top,$$

and we get a permutation matrix $M_{\sigma''}$ such that

$$M_{\sigma''} \vec{x} = (\mathcal{X}_1 \quad \alpha(\mathcal{X}_2) \quad \mathcal{X}_2 \quad \mathcal{X}_3 \quad \alpha(\mathcal{X}_4) \quad \mathcal{X}_4)^\top.$$

In the case $\alpha_n(p) = p$, we have two situations to consider: whether the biggest coefficient is one of the fixed ones, or one of the flipped ones.

Suppose it is one of the fixed ones, p_1 . We have $p_1 x_1 \leq q$ with $q = p_{m+1} x_{m+1} + \cdots + p_s x_s + p_{s+1} (x_{s+1} + \alpha(x_{s+1})) + \cdots + p_l (x_l + \alpha(x_l))$. Suppose $x_i + \alpha(x_i) = w_i$. We have $x_1, x_{m+1}, \cdots, x_s, w_{s+1}, \cdots, w_l$ are all in G^α , the fixed point subgroup. Since G^α is a lattice-ordered dimension group, we can write $x_1 = z_{m+1} + \cdots + z_s + y_{s+1} + \cdots + y_l$ with $0 \leq z_i \leq x_i$ and $0 \leq y_j \leq w_j$, where $z_i, y_j \in G^\alpha$. We have $w_j = x_j + \alpha(x_j)$ with $x_j \wedge \alpha(x_j) = 0, y_j \leq w_j$, and write $y_j = z_j + z'_j$ with $z_j \leq x_j$ and $z'_j \leq \alpha(x_j)$. Since $x_i \wedge \alpha(x_i) = 0$, we get $z_j = y_j \wedge x_j, z'_j = y_j \wedge \alpha(x_j)$. Also, since $\alpha(y_j) = y_j$, we get $\alpha(z_j) = z'_j$. Then, we get $x_1 = z_{m+1} + \cdots + z_s + z_{s+1} + \alpha(z_{s+1}) + \cdots + z_l + \alpha(z_l)$.

Observe that

$$\begin{aligned} & p_1x_1 + \cdots + p_hx_h + p_{h+1}x_{h+1} + p_{h+1}\alpha(x_{h+1}) + \cdots + p_mx_m + p_m\alpha(x_m) \\ &= p_{m+1}x_{m+1} + \cdots + p_sx_s + p_{s+1}x_{s+1} + p_{s+1}\alpha(x_{s+1}) + \cdots + p_lx_l + p_l\alpha(x_l). \end{aligned}$$

Since $x_1 = z_{m+1} + \cdots + z_s + z_{s+1} + \alpha(z_{s+1}) + \cdots + z_l + \alpha(z_l)$, we get

$$\begin{aligned} & p_1(z_{m+1} + \cdots + z_s + z_{s+1} + \alpha(z_{s+1}) + \cdots + z_l + \alpha(z_l)) + p_2x_2 + \cdots \\ & \quad + p_hx_h + p_{h+1}x_{h+1} + p_{h+1}\alpha(x_{h+1}) + \cdots + p_mx_m + p_m\alpha(x_m) \\ &= p_{m+1}x_{m+1} + \cdots + p_sx_s + p_{s+1}x_{s+1} + p_{s+1}\alpha(x_{s+1}) + \cdots + p_lx_l + p_l\alpha(x_l). \end{aligned}$$

This gives

$$\left. \begin{aligned} & \sum_{i=m+1}^s (p_1 - p_i)z_i + \sum_{i=s+1}^l (p_1 - p_i)z_i \\ & + \sum_{i=s+1}^l (p_1 - p_i)\alpha(z_i) + \sum_{i=2}^h p_ix_i + \sum_{i=h+1}^m p_ix_i \\ & + \sum_{i=h+1}^m p_i\alpha(x_i) \\ &= \sum_{i=m+1}^s p_i(x_i - z_i) + \sum_{i=s+1}^l p_i(x_i - z_i) \\ & + \sum_{i=s+1}^l p_i(\alpha(x_i) - \alpha(z_i)). \end{aligned} \right\} \quad (\dagger\dagger)$$

We label the new variables as follows:

$$\begin{aligned} & \underbrace{z_{m+1}, \dots, z_s}_{z_1} \\ & \underbrace{z_{s+1}, \dots, z_l}_{z_2} \\ & \underbrace{\alpha(z_{s+1}), \dots, \alpha(z_l)}_{\alpha(z_2)} \\ & \underbrace{x_2, \dots, x_h}_{x_1'} \\ & \underbrace{x_{h+1}, \dots, x_m}_{x_2} \\ & \underbrace{\alpha(x_{h+1}), \dots, \alpha(x_m)}_{\alpha(x_2)} \\ & \underbrace{x_{m+1} - z_{m+1}, \dots, x_s - z_s}_{x_3 - z_1} \\ & \underbrace{x_{s+1} - z_{s+1}, \dots, x_l - z_l}_{x_4 - z_2} \end{aligned}$$

and

$$\underbrace{\alpha(x_{s+1} - z_{s+1}), \dots, \alpha(x_l - z_l)}_{\alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2)}.$$

We have that α acts on these new variables by a permutation, which we call σ . Let M_σ denote the matrix implementing this permutation.

This relation $(\dagger\dagger)$ has lower degree and satisfies condition 2, i.e., the relation is invariant under the automorphism. If p_1 still occurs, it occurs one time less. By the induction hypothesis, there exists $\vec{y} = (y_1 \cdots y_t)$ for $y_i \in G^+$ such that α permutes these with a permutation σ' , implemented by a matrix $M_{\sigma'}$, and matrices $r, s, t, r', s', t', r'', s''$ and t'' whose entries are nonnegative integers such that

$$\begin{aligned} \mathcal{Z}_1 &= a\vec{y}^\top, \\ \mathcal{Z}_2 &= a'\vec{y}^\top, \\ \alpha(\mathcal{Z}_2) &= \alpha(a'\vec{y}^\top) = a''\vec{y}^\top, \\ \mathcal{X}'_1 &= b\vec{y}^\top, \\ \mathcal{X}_2 &= b'\vec{y}^\top, \\ \alpha(\mathcal{X}_2) &= \alpha(b'\vec{y}^\top) = b''\vec{y}^\top, \\ \mathcal{X}_3 - \mathcal{Z}_1 &= c\vec{y}^\top, \\ \mathcal{X}_4 - \mathcal{Z}_2 &= c'\vec{y}^\top, \end{aligned}$$

and

$$\alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) = \alpha(c'\vec{y}^\top) = c''\vec{y}^\top,$$

and such that $M_\sigma R = R M_{\sigma'}$, where $R^\top = (a \ a' \ a'' \ b \ b' \ b'' \ c \ c' \ c'')$. So, we get a commutative diagram,

$$\begin{array}{ccc} \begin{array}{c} \textcirclearrowleft \\ \mathbb{Z}^\gamma \end{array} & \xrightarrow{R^\top} & \begin{array}{c} \textcirclearrowleft \\ \mathbb{Z}^t \end{array} \\ \downarrow & \swarrow & \\ \begin{array}{c} \textcirclearrowleft \\ G \end{array} & & \end{array},$$

where γ is the number of variables in $(\dagger\dagger)$ above.

Write

$$\mathcal{X}_1 = \begin{pmatrix} x_1 \\ \mathcal{X}'_1 \end{pmatrix} = \left(\begin{array}{c|c} (1, \dots, 1) & 0 \\ \hline 0 & E_{k'_1} \end{array} \right) \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \alpha(\mathcal{Z}_2) \\ \mathcal{X}'_1 \end{pmatrix},$$

where $E_{k'_1}$ is the $(h-1) \times (h-1)$ identity matrix. From the block matrix above, we can define the new matrices

$$P_{k_i} = \begin{pmatrix} 1, \dots, 1 \\ 0 \end{pmatrix}_{h \times (h-1)} \quad \text{and} \quad \widetilde{E}_{k'_1} = \begin{pmatrix} 0, \dots, 0 \\ E_{k'_1} \end{pmatrix}_{h \times (h-1)}$$

for $i = 3, 4$. By using the matrix

$$R' = \begin{pmatrix} P_{k_3} & P_{k_4} & P_{k_4} & \widetilde{E}_{k'_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{k_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{k_2} & 0 & 0 & 0 \\ E_{k_3} & 0 & 0 & 0 & 0 & 0 & E_{k_3} & 0 & 0 \\ 0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 \\ 0 & 0 & E_{k_4} & 0 & 0 & 0 & 0 & 0 & E_{k_4} \end{pmatrix},$$

we get the old variables $(\mathcal{X}_1 \ \mathcal{X}_2 \ \alpha(\mathcal{X}_2) \ \mathcal{X}_3 \ \mathcal{X}_4 \ \alpha(\mathcal{X}_4))^\top$ from the new ones

$$(\mathcal{Z}_1 \ \mathcal{Z}_2 \ \alpha(\mathcal{Z}_2) \ \mathcal{X}'_1 \ \mathcal{X}_2 \ \alpha(\mathcal{X}_2) \ \mathcal{X}_3 - \mathcal{Z}_1 \ \mathcal{X}_4 - \mathcal{Z}_2 \ \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2))^\top.$$

Let σ'' denote the permutation of the original variables given by α , and let $M_{\sigma''}$ denote the corresponding permutation matrix. One can check that $R' M_\sigma = M_{\sigma''} R'$. So, we get a commuting diagram:

$$\begin{array}{ccc} & M_{\sigma''} & M_\sigma \\ & \curvearrowright & \curvearrowright \\ \mathbb{Z}^n & \xrightarrow{R'^\top} & \mathbb{Z}^\gamma \\ & \downarrow & \swarrow \\ & G & \\ & \curvearrowright & \\ & \alpha & \end{array}.$$

If we put together the two diagrams above, then we get the result that $M_{\sigma''} Q = Q M_{\sigma'}$, where $Q = R' R$.

Now suppose the largest coefficient is one of the flipped ones, p_{h+1} . We have $p_{h+1}(x_{h+1} + \alpha(x_{h+1})) \leq q$, where $q = p_{m+1}x_{m+1} + \dots + p_s x_s + p_{s+1}x_{s+1} + p_{s+1}\alpha(x_{s+1}) + \dots + p_l x_l + p_l \alpha(x_l)$. Suppose $x_j + \alpha(x_j) = v_j$. Then $v_j \in G^\alpha$ for each j . Since G^α is a lattice-ordered dimension group, we can write $x_{h+1} + \alpha(x_{h+1}) = z_{m+1} + \dots + z_s + r_{s+1} + \dots + r_l$ with $0 \leq z_i \leq x_i$ and $0 \leq r_j \leq v_j$, where $z_i, r_j \in G^\alpha$. We have $v_j = x_j + \alpha(x_j)$ with $x_j \wedge \alpha(x_j) = 0$, $r_j \leq v_j$, and write $r_j = z_j + z'_j$ with $z_j \leq x_j$ and $z'_j \leq \alpha(x_j)$. Since $x_i \wedge \alpha(x_i) = 0$, we get $z_j = v_j \wedge x_j$ and $z'_j = v_j \wedge \alpha(x_j)$. Also, since $\alpha(v_j) = v_j$, we get $\alpha(z_j) = z'_j$. Then we get

$$x_{h+1} + \alpha(x_{h+1}) = z_{m+1} + \dots + z_s + (z_{s+1} + \alpha(z_{s+1})) + \dots + (z_l + \alpha(z_l)).$$

Observe that

$$\begin{aligned} & p_1 x_1 + \dots + p_h x_h + p_{h+1} x_{h+1} + p_{h+1} \alpha(x_{h+1}) + \dots + p_m x_m + p_m \alpha(x_m) \\ &= p_{m+1} x_{m+1} + \dots + p_s x_s + p_{s+1} x_{s+1} + p_{s+1} \alpha(x_{s+1}) + \dots + p_l x_l + p_l \alpha(x_l). \end{aligned}$$

Since $x_{h+1} + \alpha(x_{h+1}) = z_{m+1} + \cdots + z_s + (z_{s+1} + \alpha(z_{s+1})) + \cdots + (z_l + \alpha(z_l))$, we get

$$\begin{aligned} & p_1 x_1 + \cdots + p_h x_h + p_{h+1}(z_{m+1} + \cdots + z_s + (z_{s+1} + \alpha(z_{s+1}))) \\ & + \cdots + (z_l + \alpha(z_l)) + p_{h+2}(x_{h+2} + \alpha(x_{h+2})) + \cdots + p_m(x_m + \alpha(x_m)) \\ & = p_{m+1}x_{m+1} + \cdots + p_s x_s + p_{s+1}x_{s+1} + p_{s+1}\alpha(x_{s+1}) + \cdots + p_l x_l + p_l \alpha(x_l). \end{aligned}$$

Thus, we have

$$\left. \begin{aligned} & \sum_{i=1}^h p_i x_i + \sum_{i=m+1}^s (p_{h+1} - p_i) z_i + \sum_{i=s+1}^l (p_{h+1} - p_i) z_i \\ & + \sum_{i=s+1}^l (p_{h+1} - p_i) \alpha(z_i) + \sum_{i=h+2}^m p_i (x_i + \alpha(x_i)) \\ = & \sum_{i=m+1}^s p_i (x_i - z_i) + \sum_{i=s+1}^l p_i (x_i - z_i) \\ & + \sum_{i=s+1}^l p_i (\alpha(x_i) - \alpha(z_i)). \end{aligned} \right\} (\dagger \dagger \dagger)$$

We need to split the sum $x_{h+1} + \alpha(x_{h+1})$ into the two terms x_{h+1} and $\alpha(x_{h+1})$. Define new variables as follows:

$$t_{m+1}, \dots, t_s \text{ where } t_i = z_i \wedge x_{h+1},$$

$$r_{m+1}, \dots, r_s \text{ where } r_i = z_i \wedge \alpha(x_{h+1}),$$

$$v_{s+1}, \dots, v_l \text{ where } v_j = z_j \wedge x_{h+1},$$

and

$$w_{s+1}, \dots, w_l \text{ where } w_j = z_j \wedge \alpha(x_{h+1}).$$

To make our new collection of variables invariant under α , we also include $\alpha(v_{s+1}), \dots, \alpha(v_l)$, and $\alpha(w_{s+1}), \dots, \alpha(w_l)$. Notice that $\alpha(t_i) = r_i$.

Then,

$$x_{h+1} = t_{m+1} + \cdots + t_s + v_{s+1} + \cdots + v_l + \alpha(w_{s+1}) + \cdots + \alpha(w_l),$$

$$\alpha(x_{h+1}) = r_{m+1} + \cdots + r_s + \alpha(v_{s+1}) + \cdots + \alpha(v_l) + w_{s+1} + \cdots + w_l,$$

$$z_i = t_i + r_i \text{ for } i = m+1, \dots, s,$$

and

$$z_j = v_j + w_j \text{ for } j = s+1, \dots, l.$$

We label the collection of the new variables as follows:

$$\underbrace{x_1, \dots, x_h}_{\mathcal{X}_1},$$

$$\underbrace{t_{m+1}, \dots, t_s}_{\mathcal{J}}$$

$$\begin{array}{c}
\underbrace{r_{m+1}, \dots, r_s}_{\mathcal{R}} \\
\underbrace{v_{s+1}, \dots, v_l}_{\mathcal{V}} \\
\underbrace{\alpha(v_{s+1}), \dots, \alpha(v_l)}_{\alpha(\mathcal{V})} \\
\underbrace{w_{s+1}, \dots, w_l}_{\mathcal{W}} \\
\underbrace{\alpha(w_{s+1}), \dots, \alpha(w_l)}_{\alpha(\mathcal{W})} \\
\underbrace{x_{h+2}, \dots, x_m}_{\mathcal{X}'_2} \\
\underbrace{\alpha(x_{h+2}), \dots, \alpha(x_m)}_{\alpha(\mathcal{X}'_2)} \\
\underbrace{x_{m+1} - z_{m+1}, \dots, x_s - z_s}_{\mathcal{X}_3 - \mathcal{Z}_1} \\
\underbrace{x_{s+1} - z_{s+1}, \dots, x_l - z_l}_{\mathcal{X}_4 - \mathcal{Z}_2}
\end{array}$$

and

$$\underbrace{\alpha(x_{s+1}) - \alpha(x_l), \dots, \alpha(x_l) - \alpha(z_l)}_{\alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2)}.$$

This relation that we get from $(\dagger \dagger \dagger)$ has lower degree and satisfies condition 2. If p_{h+1} still occurs, it occurs one time less. By the induction hypothesis, there exists $\vec{y} = (y_1 \cdots y_t)$ with $y_i \in G^+$ such that α permutes these with a permutation σ' , implemented by a matrix $M_{\sigma'}$, and there exist matrices $a, a', a'', b, c, d, d', e, e', f, f'$, and f'' whose entries are nonnegative integers such that

$$\begin{aligned}
\mathcal{X}_1 &= a\vec{y}^\top, \\
\mathcal{X}'_2 &= a'\vec{y}^\top, \\
\alpha(\mathcal{X}'_2) &= a''\vec{y}^\top, \\
\mathcal{T} &= b\vec{y}^\top, \\
\mathcal{R} &= c\vec{y}^\top, \\
\mathcal{V} &= d\vec{y}^\top, \\
\alpha(\mathcal{V}) &= d'\vec{y}^\top, \\
\mathcal{W} &= e\vec{y}^\top, \alpha(\mathcal{W}) = e'\vec{y}^\top,
\end{aligned}$$

$$\begin{aligned}\mathcal{X}_3 - \mathcal{Z}_1 &= f\bar{y}^\top, \\ \mathcal{X}_4 - \mathcal{Z}_2 &= f'\bar{y}^\top,\end{aligned}$$

and

$$\alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2) = f''\bar{y}^\top,$$

and such that $M_\sigma R = RM_{\sigma'}$, where

$$R^\top = (a \ a' \ a'' \ b \ c \ d \ d' \ e \ e' \ f \ f' \ f'').$$

So, one can check that we get the following commutative diagram:

$$\begin{array}{ccc} & & M_{\sigma'} \\ & & \curvearrowright \\ M_\sigma & & \\ \curvearrowright & & \\ \mathbb{Z}^m & \xrightarrow{R^\top} & \mathbb{Z}^{m'} \\ \downarrow & \swarrow & \\ G & & \\ \downarrow & & \\ \alpha & & \end{array}.$$

We write

$$\mathcal{X}_2 = \begin{pmatrix} x_{h+1} \\ \mathcal{X}'_2 \end{pmatrix} = \left(\begin{array}{c|c} (1, \dots, 1) & 0 \\ \hline & E_{k'_2} \end{array} \right) \begin{pmatrix} \mathcal{T} \\ \mathcal{V} \\ \hline \alpha(\mathcal{W}) \\ \mathcal{X}'_2 \end{pmatrix}$$

and

$$\alpha(\mathcal{X}_2) = \begin{pmatrix} \alpha(x_{h+1}) \\ \alpha(\mathcal{X}'_2) \end{pmatrix} = \left(\begin{array}{c|c} (1, \dots, 1) & 0 \\ \hline & E_{k'_2} \end{array} \right) \begin{pmatrix} \mathcal{R} \\ \alpha(\mathcal{V}) \\ \mathcal{W} \\ \hline \alpha(\mathcal{X}'_2) \end{pmatrix},$$

where $E_{k'_2}$ is the $(m-h-2) \times (m-h-2)$ identity matrix. From the block matrix above, we can define the new matrices

$$P_{k_i} = \left(\begin{array}{c|c} 1, \dots, 1 & \\ \hline & 0 \end{array} \right)_{(m-h-1) \times (m-h-2)}$$

and

$$\widetilde{E}_{k'_2} = \left(\begin{array}{c|c} 0, \dots, 0 & \\ \hline & E_{k'_2} \end{array} \right)_{(m-h-1) \times (m-h-2)}$$

for $i = 3, 4$. By using the matrix

$$R' = \begin{pmatrix} E_{k_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{E}_{k'_2} & 0 & P_{k_3} & 0 & P_{k_4} & 0 & 0 & P_{k_4} & 0 & 0 & 0 \\ 0 & 0 & \widetilde{E}_{k'_2} & 0 & P_{k_3} & 0 & P_{k_4} & P_{k_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{k_3} & E_{k_3} & 0 & 0 & 0 & 0 & E_{k_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & E_{k_4} & 0 & 0 & E_{k_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_{k_4} & 0 & E_{k_4} & 0 & 0 & E_{k_4} \end{pmatrix},$$

we get the old variables

$$\mathcal{X}_1, \mathcal{X}_2, \alpha(\mathcal{X}_2), \mathcal{X}_3, \mathcal{X}_4, \text{ and } \alpha(\mathcal{X}_4)$$

from the new ones

$$\begin{aligned} &\mathcal{X}_1, \mathcal{X}'_2, \alpha(\mathcal{X}'_2), \mathcal{J}, \mathcal{R}, \mathcal{V}, \alpha(\mathcal{V}), \mathcal{W}, \alpha(\mathcal{W}), \\ &\mathcal{X}_3 - \mathcal{Z}_1, \mathcal{X}_4 - \mathcal{Z}_2, \text{ and } \alpha(\mathcal{X}_4) - \alpha(\mathcal{Z}_2). \end{aligned}$$

Let $M_{\sigma''}$ denote the matrix that gives the permutation of the original variables. So, we get a commuting diagram:

$$\begin{array}{ccc} \begin{array}{c} \textcirclearrowleft \\ \mathbb{Z}^n \end{array} & \xrightarrow{R'^T} & \begin{array}{c} \textcirclearrowleft \\ \mathbb{Z}^\gamma \end{array} \\ \downarrow & \swarrow & \\ \begin{array}{c} \textcirclearrowleft \\ G \end{array} & & \end{array}$$

If we put together the two diagrams above, then we get the result that $M_{\sigma''}Q = QM_{\sigma'}$, where $Q = R'R$. \square

PROPOSITION 2.3. *Let G_1 be a simplicial group with simplicial basis $\{e_1, \dots, e_n\}$ and \mathbb{Z}_2 action α_1 given by the permutation σ . Let G be a lattice-ordered dimension group with \mathbb{Z}_2 action α , and let $g_1 : G_1 \rightarrow G$ be a positive equivariant homomorphism. Then there exist a simplicial group G_2 with \mathbb{Z}_2 action α_2 , and positive equivariant homomorphisms $h : G_1 \rightarrow G_2$ and $g_2 : G_2 \rightarrow G$ such that $g_1 = g_2 \circ h$ and $\ker(g_1) = \ker(h)$. In other words, we have a commuting diagram:*

$$\begin{array}{ccc} \begin{array}{c} \textcirclearrowleft \\ G_1 \end{array} & \xrightarrow{h} & \begin{array}{c} \textcirclearrowleft \\ G_2 \end{array} \\ \downarrow g_1 & \swarrow g_2 & \\ \begin{array}{c} \textcirclearrowleft \\ G \end{array} & & \end{array}$$

PROOF. The proof proceeds as in [6, Proposition 3.16] with our Proposition 2.2 in place of [6, Proposition 3.15]. We want to show that a finite set of generators a_1, \dots, a_k for $\ker(g_1)$ also lie in $\ker(h)$. The proof is by induction. In this proof, we add \mathbb{Z}_2 actions α_1 and α for G_1 and G , respectively. Since g_1 is an equivariant homomorphism, it follows that $\alpha(x_i) = \alpha(g_1(e_i)) = g_1(\alpha_1(e_i))$ for each $i = 1, \dots, n$. First, we need to check when $k = 1$. If G_1 is the zero group, we may take G_2 to also be the zero group and h, g_2 the zero maps. According to Proposition 2.2 above, there exist elements y_1, \dots, y_t in G^+ such that α acts on

$\{y_1, \dots, y_t\}$ by a permutation σ' and nonnegative integers q_{ij} (for $i = 1, \dots, n$, and $j = 1, \dots, t$) such that $x_i = q_{i1}y_1 + \dots + q_{it}y_t$ and $p_1q_{1j} + \dots + p_nq_{nj} = 0$ for all i and j , and $MQ = QM'$, where M, M' are the permutation matrices giving σ, σ' respectively, and Q is the matrix of the q_{ij} 's. Define a \mathbb{Z}_2 action α_2 on G_2 by the permutation matrix M' . Then, it follows that $h(\alpha_1(e_i)) = q_{i1}\alpha_2(f_1) + \dots + q_{it}\alpha_2(f_t) = \alpha_2(h(e_i))$ for $i = 1, \dots, n$ and $\alpha(g_2(f_j)) = \alpha(y_j) = g_2(\alpha_2(f_j))$ for $j = 1, \dots, t$. So, the maps intertwine the actions.

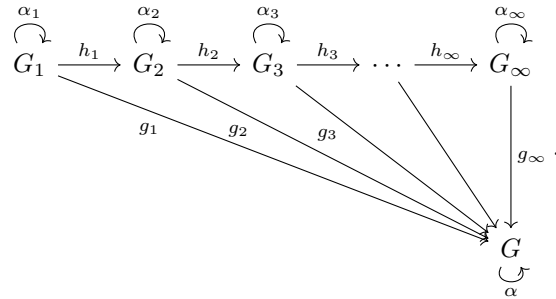
We apply these to the induction hypothesis in the original proof [6, pp. 53–54].

The rest of the proof is similar to that of [6, Proposition 3.16] with our Proposition 2.2 in place of [6, Proposition 3.15]. □

Now, we shall prove Theorem 2.1, which is the modified Effros-Handelman-Shen theorem, with the aid of Propositions 2.2 and 2.3.

PROOF OF THEOREM 2.1. Let G be a countable lattice-ordered dimension group with an action α of \mathbb{Z}_2 . Write $G^+ = \{x_1, x_2, \dots\}$. We shall construct a sequence of simplicial groups G_1, G_2, \dots with the actions $\alpha_1, \alpha_2, \dots$ and positive equivariant homomorphisms $g_n : G_n \rightarrow G$ and $h_n : G_n \rightarrow G_{n+1}$ for all $n \in \mathbb{N}$ such that $x_n \in g_n(G_n^+)$, $g_{n+1} \circ h_n = g_n$, and $\ker(g_n) = \ker(h_n)$ for all $n \in \mathbb{N}$. Also, we shall consider the limit of the sequence that we construct, G_∞ , with a positive equivariant homomorphism $g_\infty : G_\infty \rightarrow G$, and a \mathbb{Z}_2 action on G_∞ , α_∞ .

We set $G_1 = \mathbb{Z}^2$ with the \mathbb{Z}_2 action α_1 that flips the elements, i.e., $\alpha_1(e_1) = e_2$ and $\alpha_1(e_2) = e_1$. We define a positive homomorphism $g_1 : G_1 \rightarrow G$ by the rule $g_1(e_1) = x_1$ and $g_1(e_2) = g_1(\alpha_1(e_1)) = \alpha(x_1)$. Suppose that we have constructed $g_1, G_1, \alpha_1, \dots, g_n, G_n, \alpha_n$ which meet the requirements. We would like to construct the next one: g_{n+1}, G_{n+1} , and α_{n+1} . The direct product $H = G_n \oplus \mathbb{Z}^2$ is a simplicial group and we define a positive homomorphism $g : H \rightarrow G$ by the rule $g(a, k, l) = g_n(a) + kx_{n+1} + l\alpha(x_{n+1})$, and a \mathbb{Z}_2 action α' on H such that $\alpha'(a, (0, 0)) = (\alpha_n(a), (0, 0))$ for $a \in G_n$ and $\alpha'(0, (k, l)) = (0, (l, k))$. Then, $\alpha(g(a, k, l)) = g(\alpha'(a, k, l))$. We get a commutative diagram:



The rest of the proof is similar to that of [6, Proposition 3.17] with our Proposition 2.3 in place of [6, Proposition 3.16]. □

REFERENCES

1. B. Blackadar, *Symmetries of the CAR algebra*, Ann. of Math. (2) **131**(3) (1990), 589–623.
2. G. Birkhoff, *Lattice-ordered groups*, Ann. of Math. (2) **43**(2) (1942), 298–331.
3. B. Choi, *A modification of the Effros-Handelman-Shen theorem*, Master's Thesis, Lakehead University, 2020.
4. E. G. Effros, D. E. Handelman, and C.-L. Shen, *Dimension groups and their affine representations*, American Journal of Mathematics **102**(2) (1980), 385–407.
5. G. A. Elliott and H. Su, *K-theoretic classification for inductive limit Z_2 actions on AF algebras*, Canadian Journal of Mathematics **48**(5) (1996), 946–958.
6. K. R. Goodearl, *Partially Ordered Abelian Groups with Interpolation*, American Mathematical Society, Providence, 1986.
7. D. E. Handelman and W. Rossmann, *Product type actions of finite and compact groups*, Indiana University Mathematics Journal **33**(4) (1984), 479–509.
8. D. E. Handelman and W. Rossmann, *Actions of compact groups on AF C^* -algebras*, Illinois Journal of Mathematics **29**(1) (1985), 51–95.

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