

## KAM-RENORMALIZATION AND HERMAN RINGS FOR 2D MAPS

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**ABSTRACT.** In this note, we extend the renormalization horseshoe we have recently constructed with N. Goncharuk for analytic diffeomorphisms of the circle to their small two-dimensional perturbations. As one consequence, Herman rings with rotation numbers of bounded type survive on a codimension one set of parameters under small two-dimensional perturbations.

**RÉSUMÉ.** On étend le fer à cheval de renormalisation récemment construit avec N. Goncharuk pour les difféomorphismes analytiques du cercle à leurs petites perturbations à deux dimensions. Il suit que les anneaux de Herman à nombre de rotation de type borné survivent sur un ensemble de paramètres à codimension un sous petites perturbations à deux dimensions.

**1. Foreword** In a recent paper [5], N. Goncharuk and the author constructed a renormalization operator for complex-analytic maps of the annulus, which leaves rigid rotations invariant. This operator is complex analytic and real-symmetric with respect to the natural Banach space structure given by the sup-norm, and its differential is compact. Furthermore, the set of Brjuno rotations is a hyperbolic invariant set of this operator, with one-dimensional unstable bundle. This set has a stable foliation of codimension one, whose leaves consist of maps which are analytically conjugate to Brjuno rotations.

The construction of the renormalization operator in [5] parallels the cylinder renormalization operator introduced by the author in [6]. The hyperbolic horseshoe for cylinder renormalization consists of analytic critical circle maps. In another recent work [3], D. Gaidashev and the author extended a subset of this horseshoe, corresponding to the rotation numbers of bounded type, to two-dimensional complex-analytic maps. In the real slice of the corresponding Banach manifold, the leaves of the stable foliation have codimension one, and consist of real-analytic dissipative maps of the annulus with a critical circle attractor, the dynamics on which is homeomorphically but not smoothly conjugate to the rigid rotation.

Finally, in [1], a similar set of ideas was used by D. Gaidashev, R. Radu, and the author to prove that Siegel disk boundaries of highly dissipative Hénon

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maps with golden-mean semi-Siegel fixed points are topological circles; the corresponding renormalization picture was constructed in [4].

This note synthesizes the above ideas, and adds some new ones, to extend the renormalization picture of [5] restricted to rotation numbers of bounded type to two-dimensional complex-analytic dissipative maps (Theorem 3.5). The maps populating the stable leaves possess analytic invariant circles, the dynamics on which is analytically conjugate to the rotation (Theorem 4.1). In particular, this implies that Herman rings with rotation numbers of bounded type survive on a codimension one subset of parameters under small two-dimensional perturbations.

This note is not supposed to be a self-contained introduction to the subject. Rather, we assume that the reader is either familiar with the above quoted works, or will look up the relevant statements there – what we do here is connect the dots.

## 2. Function Spaces and 1D Renormalization

*2.1. Commuting pairs* Denote  $U_r(z_0) = \{|z - z_0| < r\}$ ; for a set  $S \subset \mathbb{C}$  set

$$U_r(S) = \bigcup_{z \in S} U_r(z).$$

Let  $\mathbf{B}$  denote Brjuno rotation numbers. For  $M \in \mathbb{N}$  let  $\mathbf{T}_M$  denote the irrational rotation numbers of type bounded by  $M$  (that is,  $M$  is an upper bound for the terms in the continued fraction expansion of the rotation number).

Let us set

$$T_\theta(z) = z + \theta, \text{ and } \alpha(z) = T_1(z) = z + 1.$$

Let us fix a real-symmetric topological disk  $W \ni [0, 1]$ . We define  $\mathcal{C}_W$  to be the space of maps  $\beta$  such that:

- $\beta$  is a bounded analytic map in  $W$ , continuous up to the boundary;
- $\alpha \circ \beta = \beta \circ \alpha$  where defined.

Equipped with the uniform norm,  $\mathcal{C}_W$  is a Banach manifold (since  $\beta(z) - z$  is a 1-periodic function).

We will refer to pairs  $(\alpha, \beta)$  where  $\beta \in \mathcal{C}_W$  as *normalized commuting pairs*.

The connection with the usual definition of commuting pairs (which we will refer to as *non-normalized* commuting pairs, to avoid confusion) [3] is as follows. Let  $\zeta = (\eta, \xi)$  be a commuting pair of analytic diffeomorphisms. The model case to consider is a map  $F$  of an annulus around  $\mathbb{T}$  which is close to an irrational rotation  $T_\theta$ , and

$$\eta = F^{q_n}, \text{ and } \xi = F^{q_{n+1}}.$$

Now, let  $\Psi$  be a locally conformal solution (cf. [5]) of the functional equation

$$\Psi^{-1} \circ \eta \circ \Psi = \alpha$$

and set  $\beta = \Psi^{-1} \circ \xi \circ \Psi$ . Such a solution exists and can furthermore be chosen to analytically depend on  $\eta$  (see [5] for the construction, and also [2] for a constructive version), associating a normalized commuting pair to a non-normalized one.

Renormalization  $\mathcal{R}$  on normalized commuting pairs is simply the cylinder renormalization defined in [5], since  $B$  naturally projects to a map of the quotient  $W/z \sim z+1$  and vice versa. We thus have the following Renormalization Hyperbolicity Theorem [5]:

**THEOREM 2.1.** *There exists a domain  $W$  such that the following holds. The operator  $\mathcal{R}$  maps an open subset  $\mathcal{U}$  of  $\mathcal{C}_W$  to a pre-compact subset of  $\mathcal{C}_W$ . The invariant set  $\{T_\theta \mid \theta \in \mathbf{B}\} \subset \mathcal{C}_W$  is hyperbolic, with one-dimensional unstable direction. It has a codimension one strong stable foliation by analytic submanifolds, and every  $\beta \in \mathcal{C}_W \cap W^s(T_\theta)$  with  $\theta \in \mathbf{B}$  is analytically conjugate to  $T_\theta$  in a neighborhood of  $[0, 1]$ . Furthermore, for every  $M \in \mathbb{N}$ , hyperbolicity is uniform on the set  $\{T_\theta \mid \theta \in \mathbf{T}_M\}$ .*

**2.2. Almost commuting pairs** We say that  $(\alpha, \beta)$  is a normalized almost commuting pair if

- $\beta$  is a bounded analytic map in  $W$ , continuous up to the boundary;
- $[\alpha, \beta](z) \equiv \alpha \circ \beta(z) - \beta \circ \alpha(z) = o(z^2)$ .

We denote the space of such maps  $\beta$  by  $\mathcal{AC}_W$ ; as seen in [3] it is a Banach submanifold of the space of bounded analytic functions in  $W$ , continuous up to the boundary. The definition of  $\mathcal{R}$  is naturally extended to these pairs; the image of an almost commuting pair is again an almost commuting pair.

Just as before, these pairs correspond to conformal rescalings of *non-normalized* almost commuting pairs [1, 3]  $\zeta = (\eta, \xi)$  with

$$[\zeta](z) = \eta \circ \xi(z) - \xi \circ \eta(z) = o(z^2).$$

It turns out that almost commutativity improves under renormalization:

**THEOREM 2.2.** *Fix  $\tau \in (0, 1)$ , and let  $M \in \mathbb{N}$ . There exist  $\delta > 0$ , and  $\ell \in \mathbb{N}$  such that for every  $\theta \in \mathbf{B}$ , there exists  $\epsilon = \epsilon(\theta)$  such that the following statement is true. Let  $\beta \in \mathcal{AC}_W$  be  $\epsilon$ -close to  $T_\theta$ , and set  $\nu = (\alpha, \beta)$ . Then*

$$(2.1) \quad \|[\mathcal{R}^\ell \nu]\|_{\tilde{U}_\delta(0)}^\infty < \tau \|[\nu]\|_{\tilde{U}_\delta(0)}^\infty.$$

*Moreover, suppose  $\theta \in \mathbf{T}_M$ . Then  $\epsilon$  can be chosen depending only on  $M$ .*

It is easier to prove the estimate for non-rescaled pairs  $\zeta = (\eta, \xi)$ . We will need the following lemma:

**LEMMA 2.3.** *Let  $\zeta = (\eta, \xi)$ , and denote  $p\mathcal{R}^\ell \zeta = (\eta_\ell, \xi_\ell)$  the  $\ell$ -th pre-renormalization of  $\zeta$ . Then,*

$$(2.2) \quad \eta_\ell \circ \xi_\ell = f_\ell \circ \eta \circ \xi \text{ and } \xi_\ell \circ \eta_\ell = f_\ell \circ \xi \circ \eta,$$

where  $f_\ell$  is the same composition of iterates of  $\eta$  and  $\xi$ . In particular, the commutator  $[p\mathcal{R}^\ell\zeta]$  has the form

$$(2.3) \quad [p\mathcal{R}^\ell\zeta] = f_\ell \circ \eta \circ \xi - f_\ell \circ \xi \circ \eta.$$

PROOF. The proof of (2.2) is easily supplied by induction. Indeed, if we assume that it holds for some  $\ell$ , then

$$\xi_{\ell+1} \circ \eta_{\ell+1} = \xi_\ell^r \circ \eta_\ell \circ \xi_\ell = \xi_\ell^r \circ f_\ell \circ \eta \circ \xi, \text{ and } \eta_{\ell+1} \circ \xi_{\ell+1} = \xi_\ell \circ \xi_\ell^r \circ \eta_\ell = \xi_\ell^r \circ f_\ell \circ \xi \circ \eta,$$

and so we get the desired claim by setting

$$f_{\ell+1} = -\xi_\ell^r \circ f_\ell.$$

□

Set  $\lambda_\ell = |\xi_\ell(0)|$ . In view of Lemma 2.3, we have the following first-order estimate:

$$[p\mathcal{R}^\ell\zeta] = \eta_\ell \circ \xi_\ell - \xi_\ell \circ \eta_\ell \sim f'_\ell(\eta \circ \xi(0))[\zeta] \sim f'_\ell(\eta \circ \xi(0)) \cdot cz^2.$$

Assuming the pair is sufficiently close to linear, the derivative  $f'_\ell$  is uniformly bounded. Thus,

$$|[p\mathcal{R}^\ell\zeta](z)| \sim \lambda_\ell |f'_\ell(\eta \circ \xi(0))| \cdot |c||z|^2 \lesssim \tau|z|^2.$$

for  $\ell$  large enough and  $\zeta$  sufficiently close to linear. The statement for normalized almost commuting pairs clearly follows.

As a corollary of Theorem 2.2, the operator  $\mathcal{R}$  in  $\mathcal{AC}_W$  does not have any new unstable directions:

**THEOREM 2.4.** *There exists a domain  $W$  such that the following holds. The operator  $\mathcal{R}$  maps an open subset  $\tilde{U}$  of  $\mathcal{AC}_W$  to a compact subset of  $\mathcal{AC}_W$ . Its differential at every point is a compact linear operator. Fix  $M \in \mathbb{N}$ . The invariant set  $\{T_\theta \mid \theta \in \mathbf{T}_M\} \subset \mathcal{AC}_W$  is uniformly hyperbolic, with one-dimensional unstable direction. It has a codimension one strong stable foliation by analytic submanifolds.*

PROOF. The map  $\beta \in \mathcal{AC}_W$  projects from the interval  $[0, 1]$  to a smooth map of the unit circle which is analytic at every point except one. When the commutator  $[\alpha, \beta]$  is small, this map can be extended to a quasiregular map of a neighborhood with a small dilatation. Applying the Measurable Riemann Mapping Theorem produces a conformal map of an annulus around the circle, whose lift  $\tilde{\beta}$  is close to  $\beta$ : the norm of the distance is comparable to the norm of  $[\alpha, \beta]$ . Since  $\tilde{\beta}$  commutes with  $\alpha$ , Theorem 2.2 implies that the operator  $\mathcal{R}$  contracts the distance to  $\mathcal{C}_W$ . The statement follows by Theorem 2.1. □

### 3. Definition of 2D Renormalization

3.1. *Preliminaries: multi-indices of renormalization* We follow the notation of [1]. Namely, let us consider the space  $\mathcal{J}$  of multi-indices  $\bar{s} = (a_1, b_1, a_2, b_2, \dots, a_m, b_m)$  where  $a_j \in \mathbb{N}$  for  $2 \leq m$ ,  $a_1 \in \mathbb{N} \cup \{0\}$ ,  $b_j \in \mathbb{N}$  for  $1 \leq j \leq m-1$ , and  $b_m \in \mathbb{N} \cup \{0\}$ .

For a pair of maps  $\zeta = (\eta, \xi)$  and  $\bar{s}$  as above we will write

$$\zeta^{\bar{s}} \equiv \xi^{b_m} \circ \eta^{a_m} \circ \dots \circ \xi^{b_2} \circ \eta^{a_2} \circ \xi^{b_1} \circ \eta^{a_1}.$$

Similarly,

$$\zeta^{-\bar{s}} \equiv (\zeta^{\bar{s}})^{-1} = (\eta^{a_1})^{-1} \circ (\xi^{b_1})^{-1} \circ \dots \circ (\eta^{a_m})^{-1} \circ (\xi^{b_m})^{-1}.$$

Consider the  $n$ -th pre-renormalization of  $\zeta$ :

$$p\mathcal{R}^n \zeta = \zeta_n = (\eta_n|_{Z_n}, \xi_n|_{W_n}),$$

where  $Z_n = \mathbf{1}_n(Z)$ ,  $W_n = \mathbf{1}_n(W)$ , and

$$(3.1) \quad \mathbf{1}_n(z) = \eta_n(0)z.$$

We define  $\bar{s}_n, \bar{t}_n \in \mathcal{J}$  to be such that

$$\eta_n = \zeta^{\bar{s}_n}, \text{ and } \xi_n = \zeta^{\bar{t}_n}.$$

A straightforward induction shows:

LEMMA 3.1. *For  $n \geq 1$ , let  $\bar{r} = \bar{s}_n$  or  $\bar{t}_n$ . Write  $\bar{r} = (a_1, b_1, a_2, b_2, \dots, a_{m_n}, b_{m_n})$ . Then  $b_{m_n} = 0$ , and either*

$$a_{m_n} \geq 2$$

or

$$a_{m_n} = b_{m_n-1} = 1.$$

Furthermore, if  $\bar{s}_n$  ends in  $\dots, 1, 1, 0$  then so also does  $\bar{t}_n$ .

Let  $\bar{s}_n$  be given by  $(a_1, b_1, a_2, b_2, \dots, a_{m_n}, 0)$ . Write

$$\hat{s}_n = \begin{cases} (a_1, b_1, a_2, b_2, \dots, a_{m_n} - 2, 0), & a_{m_n} \geq 2 \\ (a_1, b_1, a_2, b_2, \dots, 0, 0, 0), & a_{m_n} = 1 \end{cases},$$

$$\phi_0(x) = \begin{cases} \eta^2, & a_{m_n} \geq 2 \\ \eta \circ \xi, & a_{m_n} = 1 \end{cases}.$$

Define  $\hat{t}_n$  in an identical way to  $\hat{s}_n$ . Then  $p\mathcal{R}^n \zeta$  can be written as

$$p\mathcal{R}^n \zeta = \phi_0 \circ (\zeta^{\hat{s}_n}, \zeta^{\hat{t}_n}).$$

3.2. *Renormalization of two-dimensional maps* Let us fix  $W$  as in Theorem 2.4, and let  $\Omega = W \times W$ . We also let  $\Gamma = U_R([0, 1]) \times U_R([0, 1])$  for  $R > 0$ .

We denote by  $\mathcal{O}_{\Gamma, \Omega}$  the Banach space of pairs of bounded analytic functions continuous up to the boundary,

$$F = (F_1 : \Gamma \rightarrow \mathbb{C}^2, F_2 : \Omega \rightarrow \mathbb{C}^2),$$

equipped with the norm

$$(3.2) \quad \|F\| = \frac{1}{2} \left( \sup_{(x,y) \in \Gamma} |F_1(x, y)| + \sup_{(x,y) \in \Omega} |F_2(x, y)| \right).$$

We define a transformation  $\iota$  which sends the pair  $\nu = (\alpha, \beta)$  to the pair of functions  $\iota(\nu)$ :

$$(3.3) \quad \left( \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha(x) \\ \alpha(x) \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \beta(x) \\ \beta(x) \end{pmatrix} \right).$$

Next, we follow [1, 3] to define renormalization of small (of size  $O(\delta)$  for some small  $\delta > 0$ ) two-dimensional perturbations of one-dimensional almost commuting pairs. Namely, we consider pairs of maps of the form

$$(3.4) \quad A(x, y) = (a(x, y), h(x, y)) = (a_x(x), h_y(x)),$$

$$(3.5) \quad B(x, y) = (b(x, y), g(x, y)) = (b_x(x), g_y(x)),$$

such that

- (1) the pair  $(A(x, y), B(x, y))$  is in a  $\delta$ -neighborhood of  $\iota(\mathcal{U})$ ,
- (2)  $(h, g)$  are such that  $|\partial_x h(x, 0)| > 0$  and  $|\partial_x g(x, 0)| > 0$ , and

$$(h(x, y) - h(x, 0), g(x, y) - g(x, 0)) \text{ have norms bounded by } \delta.$$

Suppose that  $\Sigma = (A, B)$  as above is a small perturbation of a pair  $\iota(\nu)$  where  $\nu = (\alpha, \beta)$ .

Set

$$\hat{p}\mathcal{R}^n \Sigma = (F \circ \Sigma^{\hat{s}^n} \circ A, F \circ \Sigma^{\hat{t}^n} \circ A),$$

where  $F = A$  if  $a_n \geq 2$  and  $F = B$  if  $a_n = 1$ .

We will write

$$\pi_1(x, y) = x \text{ and } \pi_2(x, y) = y.$$

Set

$$\phi_y(x) = \phi(x, y) := \begin{cases} \pi_1 A^2(x, y), & a_n \geq 2 \\ \pi_1 A \circ B(x, y), & a_n = 1 \end{cases}.$$

Furthermore, set

$$q_z(x) \equiv q(x, z) = \pi_2 F(x, z) = \begin{cases} h_z(x), & a_n \geq 2 \\ g_z(x), & a_n = 1 \end{cases}.$$

Also, set

$$w_z(x) \equiv w(x, z) := q_z(\phi_z^{-1}(x)).$$

Notice, that  $\partial_z w_z(x)$  and  $\partial_z w_z^{-1}(x)$  are functions whose uniform norms are  $O(\delta)$ . Define the following transformation:

$$(3.6) \quad H_\Sigma(x, y) = (a_y(x), w_{q_0^{-1}(y)}^{-1}(y)).$$

We have

$$(3.7) \quad A \circ H_\Sigma^{-1}(x, y) = (x, h(\alpha^{-1}(x), y)) + O(\delta).$$

We use  $H_\Sigma(x, y)$  to pull back  $\hat{p}\mathcal{R}^n\Sigma$  to a neighborhood of definition of the  $n$ -th pre-renormalization of a pair  $(\alpha, \beta)$ :

$$p\mathcal{R}^n\Sigma = (\bar{A}, \bar{B}) = H_\Sigma \circ F \circ (\Sigma^{\hat{s}_n}, \Sigma^{\hat{t}_n}) \circ A \circ H_\Sigma^{-1}(x, y).$$

The proof of the following statement is straightforward (compare with [3]):

**PROPOSITION 3.2.** *For every  $n$  there exists  $\delta$  small enough such that the following statement holds. There exists a unique triple  $(d_0, d_1, d_2) \in \mathbb{C}^3$  which analytically depends on the pair  $\Sigma$  such that the almost commutation condition*

$$(3.8) \quad \pi_1(\tilde{A} \circ \tilde{B})(x, 0) - \tilde{B} \circ \tilde{A}(x, 0) = o(x^2)$$

holds for the pair

$$(3.9) \quad (\tilde{A}, \tilde{B}) \equiv (\bar{A}, \bar{B}) + \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} d_0 + d_1x + d_2x^2 \\ d_0 + d_1x + d_2x^2 \end{pmatrix} \right).$$

Note that:

**PROPOSITION 3.3.** *If  $A$  and  $B$  commute then*

$$(\tilde{A}, \tilde{B}) = (\bar{A}, \bar{B}).$$

Set

$$\tilde{\alpha} \equiv \pi_1 \tilde{A}(x, 0),$$

and let  $\psi(x)$  be a solution of the functional equation

$$\psi^{-1} \circ \tilde{\alpha} \circ \psi(x) = T(x)$$

chosen to analytically depend on  $\tilde{\alpha}$ , satisfying  $\psi(0) = 0$ . Set

$$\Psi(x, y) = (\psi(x), y)$$

and

$$\hat{A} \equiv \Psi^{-1} \tilde{A}(\Psi(x, y)), \quad \hat{B} \equiv \Psi^{-1} \tilde{B}(\Psi(x, y)).$$

In the same way as in [3], we have

**THEOREM 3.4.** *For every  $n, M \in \mathbb{N}$ , there exist constants  $\delta_0 > 0$ ,  $C > 0$ , a topological disk  $\hat{W} \supset W$ , and a neighborhood  $\hat{\mathcal{U}} \in \mathcal{O}_{\Gamma, \Omega}$  of  $\iota\{(T, T_\theta) \mid \theta \in \mathbf{B}\}$  such that the following statement holds. Set  $\hat{\Omega} = \hat{W} \times \hat{W}$ . Assume  $\delta < \delta_0$ , and  $\Sigma = (A, B) \in \hat{\mathcal{U}}$ . Then  $(\hat{A}, \hat{B})$  lies in  $\mathcal{O}_{\Gamma, \hat{\Omega}}$ , and*

$$\text{dist}((\hat{A}, \hat{B}), \iota(\mathcal{AC}_W)) < C\delta^2.$$

Let us fix the quantifiers as in Theorem 3.4 and define the *renormalization operator of level  $n$*  as

$$\mathcal{R}_n : (A, B) \mapsto (\hat{A}, \hat{B}).$$

Theorems 2.4 and 3.4 imply the following:

**THEOREM 3.5.** *The operator  $\mathcal{R}_n$  is a compact analytic operator of an open neighborhood  $\hat{\mathcal{U}} \in \mathcal{O}_{\Gamma, \Omega}$  of  $\iota\{(T, T_\theta) \mid \theta \in \mathbf{T}_M\}$  to  $\mathcal{O}_{\Gamma, \Omega}$ . Its differential is a compact linear operator. The invariant set  $\iota\{(T, T_\theta) \mid \theta \in \mathbf{T}_M\}$  is uniformly hyperbolic, with one dimensional unstable direction. It has a codimension one strong stable foliation by analytic submanifolds.*

**4. Rotation Domains** For  $r > 0$  set

$$\mathbb{A}_r \equiv \{|\text{Im } z| < r\}/\mathbb{Z}.$$

We say that a dissipative Hénon-like map  $H$  has a *rotation domain* if there exists an  $H$ -invariant domain  $C$  and an analytic isomorphism

$$\Phi : C \rightarrow \mathbb{A}_r \times \mathbb{D} \text{ for some } r > 0,$$

which conjugates  $H$  to the linear map

$$L(x, y) = (e^{2\pi i \theta} x, sy) \text{ with } \theta \in \mathbb{R} \setminus \mathbb{Q}.$$

Furthermore, let us say that a pair  $(A, B) \in \mathcal{O}_\Omega$  is a renormalization of a map  $H$  if there exists  $m \in \mathbb{N}$  and a linear map  $\Lambda$  such that

$$(A, B) = \Lambda \circ (H^{q_m}, H^{q_{m+1}}) \circ \Lambda^{-1}.$$

Finally, for a Brjuno number  $\theta \in (0, 1)$  let us write

$$Y_m(\theta) = \sum_{j=0}^m \theta_{-1} \theta_0 \theta_1 \cdots \theta_{j-1} \log \frac{1}{\theta_j}, \text{ where } \theta_{-1} = 1, \theta_0 = \theta, \text{ and } \theta_{j+1} = \{1/\theta_j\}.$$

Note that  $Y_\infty(\theta) < \infty$  is the Brjuno-Yoccoz function of  $\theta \in \mathbf{B}$ .

THEOREM 4.1. *Suppose a renormalization of  $H$  lies in the strong stable foliation of*

$$\iota\{(T, T_\theta) \mid \theta \in \mathbf{T}_M\}$$

*constructed in Theorem 3.5. Then  $H$  possesses a rotation domain.*

PROOF. Denote by  $(A, B)$  the renormalization of  $H$  in the strong stable manifold of  $\iota(T, T_\theta)$  with  $\theta \in \mathbf{B}$ , and let  $(A_k, B_k) = (p\mathcal{R}^n)^k(A, B)$ . Consider a shadowing  $p\mathcal{R}^n$ -orbit of a normalized almost-commuting pair  $(T, g)$  (note that this is a pre-renormalization orbit, which thus consists of non-normalized pairs),

$$p\mathcal{R}^n : (T, g_k) \mapsto (T, g_{k+1}), \quad g_0 = g$$

so that

$$\text{dist}(G_k, B_k) \lesssim \delta^{2^k} \quad \text{where } G_k = \iota(g_k).$$

Let  $h_k$  be the corresponding linearizing map

$$h_k \circ g_k \circ h_k^{-1}(z) = 1.$$

Let  $H_k$  be the lift of  $(h_k(x), h_k(y))$  to the domain of definition of  $(A, B)$  via the renormalization microscope (see [1] for the details of the construction). Replacing  $(A, B)$  with  $\mathcal{R}_n^l(A, B)$  for a sufficiently large  $l$  if needed, we can guarantee that  $H_k$  is defined in a polydisk, whose  $x$ -projection is a rectangle of height

$$|\text{Im } z| > \Delta - C - Y_{kn}(\theta),$$

where  $C$  is a constant independent of  $k$  and  $\Delta$  is large compared to  $C + Y_\infty(\theta)$ . Passing to a limit  $H_\infty$  completes the proof.  $\square$

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