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A WEIGHTED AVERAGE OF L-FUNCTIONS OF MODULAR FORMS

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ABSTRACT. We consider a kernel function introduced by Kohnen and prove an asymptotic formula for a weighted sum of L-functions of modular forms.

RÉSUMÉ. On considère une fonction noyau introduite par Kohnen et démontre une formule asymptotique pour une somme pondérée de fonctions L de forms modulaires.

1. **Introduction** Let $k \geq 12$, $k \neq 14$ be an even positive integer. Let f be a cuspidal Hecke eigenform of weight k on $\Gamma_0(1) := SL_2(\mathbb{Z})$ with Fourier expansion

$$f(z) \;=\; \sum_{n\geq 1} a_f(n) e^{2\pi i n z}.$$

Its associated L function

$$L(f,s) := \sum_{n\geq 1} \frac{a_f(n)}{n^s}$$

has an Euler product expansion in $\Re(s) > \frac{k+1}{2}$ and can be analytically continued to the whole complex plane as an entire function such that the non-trivial zeros lie inside the critical strip $(k-1)/2 < \Re(s) < (k+1)/2$. The analogue of the Riemann Hypothesis (GRH) in this context states that they all lie on the critical line $\Re(s) = k/2$ itself.

If $k \equiv 2 \bmod 4$, then the L-function vanishes at $s = k/2$ since the functional equation has root number -1. The non-vanishing of $L(f,s)$ at the points on the interval $(k/2, (k+1)/2)$, where $12 \leq k \leq 26$ and $k \notin \{14, 24\}$ (one dimensional S_k) was proved by R. Murty (Theorem 6 in [RM83]) and in the two dimensional space S_{24}, he proved that there exists a Hecke eigenform f with $L(f,s)$ non-vanishing on $(k/2,(k+1)/2)$. For general k, however, such a result is not known.

In a different direction, W. Kohnen in [Koh97] proved a non-vanishing result for an average of L-functions of Hecke eigenforms at complex points inside the

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critical strip (outside the critical line) provided k is sufficiently large. The main tool in establishing his result is the function

$$\sum_{f\in\mathcal{B}_k} \frac{L^*(f,s)}{\langle f,f\rangle} f(z)$$

where $\mathcal{B}_k$ denotes an orthogonal basis of normalised Hecke eigenforms of S_k and $L^*(f,s)$ denotes the completed L-function attached to f. Kohnen obtained his non-vanishing result by expressing this function in terms of Poincaré series and showing that the first Fourier coefficient is non-zero for k sufficiently large.

As a corollary of his result, he deduced that given a complex number s inside the critical strip, there is a Hecke eigenform f in S_k such that $L(f,s) \neq 0$ (for large enough k). In this paper, by adapting a technique of Rankin and Swinnerton-Dyer [RSD70], we derive an asymptotic formula in terms of k for a weighted sum of L-function values, at real points σ in the critical strip and $4|k$. We also derive a lower bound for the maximum value of $|L(f,\sigma)|$ as f varies over $\mathcal{B}_k$. The method seems to have other applications which we shall pursue in our future work.

2. Kohnen's Cusp Form Let $s \in \mathbb{C}$, $1 < \Re(s) < k-1$. The kernel function $R_{k,s}(z)$, introduced by W. Kohnen [Koh97] is defined by

$$R_{k,s}(z) := \gamma_k(s) \sum{}' z^{-s}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1}$$

Here, $|_k$ denotes the standard weight k (integer) action of $GL_2^+(\mathbb{Q})$ on the functions $g : \mathbb{H} \to \mathbb{C}$, defined as

$$(g|_k\gamma)(\tau) := (ad-bc)^{k/2}(c\tau+d)^{-k} g\left(\frac{a\tau+b}{c\tau+d}\right),$$

where $\tau \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$. Moreover,

$$\gamma_k(s) := \frac{1}{2} e^{\frac{\pi i s}{2}} \Gamma(s)\Gamma(k-s)$$

and the sum $\sum'$ is taken over all the 4-tuples $(a,b,c,d) \in \mathbb{Z}^4$ such that $ad-bc = 1$.

$R_{k,s}(z)$ is a cusp form of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. If $\langle , \rangle$ denotes the usual Petersson inner product on S_k, then for any $g \in S_k$, Kohnen derived ([Koh97], Lemma 1) the relation

$$\langle g, R_{k,\bar{s}}\rangle = c_k L^*(g,s), \tag{2}$$

where,

$$c_k := (-1)^{\frac{k}{2}} \pi 2^{2-k}(k-2)!,$$

and

$$L^*(g,s) := (2\pi)^{-s}\Gamma(s)L(g,s)$$

is the completed L function of g. It is an entire function.

For $1 < \sigma := \Re(s) < k-1$, we define

$$\begin{aligned} f_{k,s}(z) &:= 2\frac{R_{k,s}(z)}{\Gamma(s)\Gamma(k-s)} \\ &= e^{i\frac{\pi}{2}s}\sum{}' z^{-s}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \tag{3}$$

Throughout the paper, let k denote an even integer with $k \geq 12, k \neq 14$, unless further conditions are specified. Our main result in this note is the following.

THEOREM 2.1. *Let $4|k$. The cusp form $f_{k,\sigma}(z)$ in S_k satisfies the asymptotic relation*

$$f_{k,\sigma}(i) = 4 + \mathcal{O}(2^{-\frac{k}{4}})$$

at $z = i$ for all values $\sigma \in [\frac{k-1}{2}, \frac{k+1}{2})$.

One can also obtain an explicit inequality valid for all k as follows.

THEOREM 2.2. *Let k be an integer as above and divisible by 4. Then, $f_{k,\sigma}(i)$ is real valued and satisfies*

$$f_{k,\sigma}(i) \ \geq 2.745.$$

This has the following consequence on a lower bound for the values of L-function in the critical strip.

COROLLARY 2.2.1. *Given $\sigma = \frac{k}{2} + \epsilon \ \in [\frac{k-1}{2}, \frac{k+1}{2}]$ and an arbitrarily small $\delta > 0$, for sufficiently large $k \geq K_\delta$, where $4|k$, we have*

$$\max_{f \in \mathcal{B}_k} |L(f,\sigma)| \gg_\delta k^{-\epsilon-1-\delta} \ = \ k^{-(\sigma-\frac{k}{2})-1-\delta}.$$

This lower bound is meant to illustrate an immediate application of Theorem 2.1 and other methods may yield better bounds.

3. Decomposing the Sum Note that for $1 < \sigma < k-1$, it follows from (2) that

$$R_{k,\sigma} = \sum_{f \in \mathcal{B}} \langle R_{k,\sigma}, f \rangle \frac{f}{\langle f, f \rangle} = \sum_{f \in \mathcal{B}} c_k L^*(f,\sigma) \frac{f}{\langle f, f \rangle},$$

where $\mathcal{B}$ (omitting the subscript k) is the orthogonal basis consisting of the normalised Hecke eigenforms in S_k and c_k is as in (2). Dividing throughout by $\Gamma(\sigma)\Gamma(k-\sigma)$ and using the equation (3), we get

$$f_{k,\sigma}(i) = \frac{(-1)^{k/2}\pi\Gamma(k-1)}{2^{k-3}\Gamma(\sigma)\Gamma(k-\sigma)} \sum_{f\in\mathcal{B}} L^*(f,\sigma)\frac{f(i)}{\langle f,f\rangle}, \tag{4}$$

from which it follows that $f_{k,\sigma}(i)$ is real valued. Because of the functional equation, it suffices to consider $\sigma \in [\frac{k}{2}, \frac{k+1}{2}]$. Let $\sigma = \frac{k}{2}+\epsilon$. From (1), we have

$$f_{k,\sigma}(z) = e^{i\frac{\pi}{2}\sigma} \sum{}' z^{-\sigma}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \frac{k}{2} \leq \sigma \leq \frac{k+1}{2}.$$

and we note that the above series runs over a set of integral matrices with determinant 1. As we have absolute convergence for z with positive imaginary part, we can rearrange this series in terms of the sum of the squares of the entries along each row (of the matrix) to get

$$e^{i\frac{\pi}{2}\sigma} \sum_{\substack{M\geq 1\\ N\geq 1}} \sum_{a^2+b^2=\boldsymbol{M}} \sum_{\substack{c^2+d^2=\boldsymbol{N}\\ \det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=1}} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-\sigma}.$$

We now proceed to split this sum into a main term and an error term. We use the rearrangement of the series as done in [RSD70]. For $M, N, M_0, N_0 \in \mathbb{N}$, let us set

$$T_{\boldsymbol{M_0},\boldsymbol{N_0},\sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{a^2+b^2=\boldsymbol{M_0}} \sum_{\substack{c^2+d^2=\boldsymbol{N_0}\\ det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=1}} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

$$T_{\boldsymbol{M_0},\geq\boldsymbol{N_0},\sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{a^2+b^2=\boldsymbol{M_0}} \sum_{N\geq\boldsymbol{N_0}} \sum_{\substack{c^2+d^2=N\\ det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=1}} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

$$T_{\geq\boldsymbol{M_0},\boldsymbol{N_0},\sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{M\geq\boldsymbol{M_0}} \sum_{a^2+b^2=M} \sum_{\substack{c^2+d^2=\boldsymbol{N_0}\\ det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=1}} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

$$T_{\geq\boldsymbol{M_0},\geq\boldsymbol{N_0},\sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{M\geq\boldsymbol{M_0}} \sum_{a^2+b^2=M} \sum_{N\geq\boldsymbol{N_0}} \sum_{\substack{c^2+d^2=N\\ det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=1}} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

where $T_{-,-,\sigma}(z)$ denotes the unique complex valued holomorphic function associated to the respective series in right hand side.

4. **Proof of Theorem 2.1** $T_{1,1}(z)$ is formed by summing $z^{-\sigma}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over the collection $\left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. We get

$$T_{1,1}(i) = \; 2e^{i\frac{\pi}{2}\sigma}\left\{(i)^{-\sigma} + (-i)^{-k}\left(\tfrac{1}{-i}\right)^{-\sigma}\right\} = \; 4.$$

Next, we have

$$|T_{\geq 2,\geq 1,\sigma}(i)| \; \leq \; \sum_{\substack{M\geq 2\\ N\geq 1}} \; \sum_{\substack{a^2+b^2=M\\ c^2+d^2=N}} \frac{1}{(c^2+d^2)^{(k-\sigma)/2}}\frac{1}{(a^2+b^2)^{\sigma/2}}.$$

Given M, there are $\leq 2\sqrt{M}+1$ possible values of a and at most 2 values of b. Given a and b, since $ad-bc=1$, the restriction $c^2+d^2=N$ implies that c satisifes a quadratic equation, and so given N there are ≤ 2 possible values of c and at most a single value of d. Hence, the right hand side above is

$$\ll \left(\sum_{M\geq 2}\frac{\sqrt{M}}{M^{\sigma/2}}\right)\left(\sum_{N\geq 1}\frac{1}{N^{(k-\sigma)/2}}\right). \tag{5}$$

The following result is convenient for estimating the above sums.

LEMMA 4.1. *We have for $x>1$ and $r\geq 1$,*

$$\sum_{\substack{a\in\mathbb{Z}\\ a\geq r}}\frac{1}{a^x} \; \leq \; \zeta(x)/r^{x-1}.$$

PROOF. We have

$$\sum_{a\geq r}\frac{1}{a^x} \; = \; \sum_{m\geq 1}\sum_{a=mr}^{(m+1)r-1}\frac{1}{a^x} \; \leq \sum_{m\geq 1}\frac{r}{(mr)^x} \; = \; r^{1-x}\zeta(x).$$

□

Using this and the fact that $\sigma \geq \; k/2$, we see that (5) is

$$\ll \zeta(\frac{\sigma-1}{2})\zeta(\frac{k-\sigma}{2})/2^{(\sigma-3)/2} \; \ll \; 2^{-k/4}.$$

Note that $f_{k,\sigma}(i) = T_{1,1,\sigma}(i) + T_{\geq 2,\geq 1,\sigma}(i) + T_{1,\geq 2,\sigma}(i)$. An argument similar to that used for deducing eqn (5) gives

$$|T_{1,\geq 2,\sigma}(i)| \; \ll \; \sum_{N\geq 2}\frac{1}{N^{(k-\sigma)/2}} \ll \zeta(\frac{k-\sigma}{2})/2^{(k-\sigma-2)/2} \; \ll \; 2^{-k/4}.$$

5. Proof of Theorem 2.2 We define

$$T_{main,\sigma}(z) := T_{1,1,\sigma}(z) + T_{1,2,\sigma}(z) + T_{2,1,\sigma}(z),$$
$$T_{error,\sigma}(z) := f_{k,\sigma}(z) - T_{main,\sigma}(z). \tag{6}$$

For the rest of section 5, we drop the parameter σ whenever σ is clear from the context.

5.1. The main term $T_{main}(i)$

The matrices which form the term $T_{1,2}(i)$ are

$$\left\{\pm\begin{pmatrix}1&0\\1&1\end{pmatrix}, \pm\begin{pmatrix}0&-1\\1&1\end{pmatrix}, \pm\begin{pmatrix}1&0\\-1&1\end{pmatrix}, \pm\begin{pmatrix}0&1\\-1&1\end{pmatrix}\right\}.$$

$$T_{1,2,(i)} = 2e^{i\frac{\pi}{2}\sigma}\left\{ (i+1)^{-k}\left(\frac{i}{i+1}\right)^{-\sigma} + (i+1)^{-k}\left(\frac{-1}{1+i}\right)^{-\sigma} + (-i+1)^{-k}\left(\frac{i}{-i+1}\right)^{-\sigma} + (-i+1)^{-k}\left(\frac{1}{-i+1}\right)^{-\sigma}\right\}.$$

After few simplifications, $T_{1,2}(i)$ is obtained as

$$T_{1,2}(i) = \begin{cases} \frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\cos\frac{\pi\epsilon}{4} & k \equiv 0 \bmod 16, \\ \frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\sin\frac{\pi\epsilon}{4} & k \equiv 4 \bmod 16, \\ -\frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\cos\frac{\pi\epsilon}{4} & k \equiv 8 \bmod 16, \\ -\frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\sin\frac{\pi\epsilon}{4} & k \equiv 12 \bmod 16. \end{cases}$$

The term $T_{2,1}(i)$ is formed by the matrices in

$$\left\{\pm\begin{pmatrix}1&-1\\1&0\end{pmatrix}, \pm\begin{pmatrix}1&1\\-1&0\end{pmatrix}, \pm\begin{pmatrix}1&1\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&-1\\0&1\end{pmatrix}\right\}.$$

So,

$$T_{2,1}(i) = 2e^{i\frac{\pi}{2}\sigma}\left\{ (i)^{-k}\left(\frac{i-1}{i}\right)^{-\sigma} + (-i)^{-k}\left(\frac{i+1}{-i}\right)^{-\sigma} + (i+1)^{-\sigma} + (i-1)^{-\sigma}\right\}.$$

Hence, we obtain

$$T_{2,1}(i) = \begin{cases} \frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 0 \bmod 16, \\ -\frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 4 \bmod 16, \\ -\frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 8 \bmod 16, \\ \frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 12 \bmod 16. \end{cases}$$

Adding $T_{1,1}, T_{1,2}$ and $T_{2,1}$ at $z = i$, we get $T_{main}(i)$ as

$$T_{main}(i) = \begin{cases} 4 + \frac{\left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 0 \bmod 16, \\ 4 + \frac{\left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 4 \bmod 16, \\ 4 - \frac{\left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 8 \bmod 16, \\ 4 - \frac{\left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 12 \bmod 16. \end{cases}$$

We note that the explicit determination of $T_{main}(i)$ above gives an inequality valid for all k. Indeed, $T_{main}(i) \geq 4$ in the first two cases as $\epsilon \in [0, \frac{1}{2}]$. In the third case, the term $\left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right) \cos \frac{\pi\epsilon}{4}$ is a decreasing function on $[0, \frac{1}{2}]$ (with respect to ϵ) and

$$\left| \frac{1}{2^{\frac{k}{4}-3}} \left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right) \cos \frac{\pi\epsilon}{4} \right| \leq 0.25$$

if $k \geq 24$. Thus, $T_{main}(i) \geq 3.75$. Similarly, the term $\left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right) \sin \frac{\pi\epsilon}{4}$ attains its maximum at $\epsilon = 0.5$ as it is an increasing function on $[0, \frac{1}{2}]$. So, in the fourth case where $k \equiv 12 \bmod 16$, we have

$$\left| \frac{1}{2^{\frac{k}{4}-3}} \left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right) \sin \frac{\pi\epsilon}{4} \right| \leq 0.134$$

and hence, $T_{main}(i) \geq 3.866$. Thus,

$$T_{main}(i) \geq \begin{cases} 4 & k \equiv 0 \bmod 16, \\ 4 & k \equiv 4 \bmod 16, \\ 3.75 & k \equiv 8 \bmod 16, \\ 3.866 & k \equiv 12 \bmod 16. \end{cases} \tag{7}$$

5.2. The error term $T_{error}(i)$ Recall from eqn (6) that

$$T_{error}(i) = T_{1,\geq 5}(i) + T_{2,\geq 5}(i) + T_{\geq 5,1}(i) + +T_{\geq 5,2}(i) + T_{2,2}(i) + T_{\geq 5,\geq 5}(i).$$

For fixed $N \in \mathbb{N}$, $T_{N,1}(i)$ is formed precisely from the matrices

$$\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

where a runs over all $\mathbb{Z}$ such that $a^2 + 1 = N$. Using this,

$$\begin{aligned} T_{\geq 5,1}(i) &= \sum_{N \geq 5} T_{N,1}(i) \\ &= e^{i\frac{\pi}{2}\sigma} \sum_{|a| \geq 2} \left\{ i^{-k} \left(\frac{ai-1}{i}\right)^{-\sigma} + (-i)^{-k} \left(\frac{ai+1}{-i}\right)^{-\sigma} \right. \\ &\qquad \left. + (i+a)^{-\sigma} + (-1)^{-k} \left(\frac{-i+a}{-1}\right)^{-\sigma} \right\} \\ &= 4e^{i\frac{\pi}{2}\sigma} \sum_{|a| \geq 2} (a+i)^{-\sigma}. \end{aligned}$$

Thus,

$$|T_{\geq 5,1}(i)| \leq 4 \sum_{|a| \geq 2} \left| \frac{1}{1+a^2} \right|^{\frac{k}{4}+\frac{\epsilon}{2}} \leq 8 \sum_{a=2}^{\infty} \frac{1}{a^{\frac{k}{2}}} \leq \frac{\zeta(\frac{k}{2})}{2^{\frac{k}{2}-4}}.$$

In the final estimation above, we used Lemma 4.1. For fixed $M \in \mathbb{N}$, $T_{1,M}(\theta)$ is formed precisely by the matrices in

$$\left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & c \end{pmatrix} \middle| c \in \mathbb{Z}, c^2 + 1 = M \right\}.$$

Using this,

$$\begin{aligned} T_{1,\geq 5}(i) &= \sum_{M\geq 5} T_{1,M}(i) \\ &= e^{i\frac{\pi}{2}\sigma} \sum_{|c|\geq 2} \Bigg\{ (ci+1)^{-k}\left(\frac{i}{ci+1}\right)^{-\sigma} + (ci-1)^{-k}\left(\frac{-i}{ci-1}\right)^{-\sigma} \\ &\qquad + (-i+c)^{-k}\left(\frac{1}{-i+c}\right)^{-\sigma} + (i+c)^{-k}\left(\frac{-1}{i+c}\right)^{-\sigma} \Bigg\}. \end{aligned}$$

Thus,

$$\left|\sum_{M\geq 5} T_{1,M}(i)\right| \leq 4\sum_{|c|\geq 2}\left|\frac{1}{1+c^2}\right|^{\frac{k}{4}-\frac{\epsilon}{2}} \leq 8\sum_{c=2}^{\infty}\frac{1}{c^{\frac{k}{2}-\epsilon}} \leq \frac{\zeta\left(\frac{k}{2}-\frac{1}{2}\right)}{2^{\frac{k}{2}-\frac{9}{2}}},$$

where we have again used Lemma 4.1 in the final step.

Next, we consider terms of the form $T_{N,2}$ where $N \geq 5$. Note that these are formed by matrices in $SL_2(\mathbb{Z})$ which are one among

$$\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} a & b \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ -1 & -1 \end{pmatrix}, \text{ where } |a| \geq 1, |b| \geq 1 \text{ and } a^2+b^2 = N.$$

We claim that terms of the form $T_{N,2}$ where N is NOT a sum of squares of consecutive integers do not survive; (for example, $T_{2,2}, T_{10,2}, T_{17,2}, T_{26,2}$). Indeed, for any $\gamma = \begin{pmatrix} a & b \\ * & * \end{pmatrix}$ from the above collection, by the determinant condition, it follows that a and b must satisfy either

$$|a+b| = 1 \quad \text{OR} \quad |a-b| = 1. \tag{8}$$

Clearly then, a and b cannot be both simultaneously odd nor simultaneously even. Further, we claim that in both these cases, their absolute values differ by 1 exactly. Suppose $|a|$ and $|b|$ differ by an integer ≥ 2, i.e.,

$$\Big||a|-|b|\Big| \geq 2.$$

Then,

$$|a-b| \;\geq ||a|-|b|| \geq 2,$$

and the same holds for $|a+b|$, and this is a contradiction to eqn (8). Hence, we conclude that $T_{N,2}$ is formed only by the following matrices:

$$\pm\begin{pmatrix} n+1 & n \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} n & -(n+1) \\ 1 & -1 \end{pmatrix}, \pm\begin{pmatrix} n+1 & -n \\ -1 & 1 \end{pmatrix}, \pm\begin{pmatrix} n & n+1 \\ -1 & -1 \end{pmatrix},$$

where $n \in \mathbb{N}$ such that $n^2 + (n+1)^2 = N$. Thus, we see that

$$\begin{aligned}
T_{\geq 5,2}(i) &= \sum_{n\geq 1} T_{n^2+(n+1)^2,2}(i) \\
&= 2e^{i\frac{\pi}{2}\sigma} \sum_{n\geq 1} \Bigg\{ (i+1)^{-k} \left(\frac{(n+1)i+n}{i+1}\right)^{-\sigma} + (i-1)^{-k} \left(\frac{ni-(n+1)}{i-1}\right)^{-\sigma} \\
&\quad + (-i+1)^{-k} \left(\frac{(n+1)i-n}{-i+1}\right)^{-\sigma} + (-i-1)^{-k} \left(\frac{ni+n+1}{-i-1}\right)^{-\sigma} \Bigg\}.
\end{aligned}$$

Thus, we have

$$|T_{\geq 5,2}(i)| \leq 2 \times 4 \times \sum_{n\geq 1} \left(\frac{1}{n^2+(n+1)^2}\right)^{\frac{k}{4}+\frac{\epsilon}{2}} \left(\frac{1}{2}\right)^{\frac{k}{4}-\frac{\epsilon}{2}} \leq \frac{1}{2^{\frac{k}{2}-3}} \zeta\left(\frac{k}{2}\right).$$

Our next term in the $T_{error}(i)$ is of the form $\sum_{M\geq 5} T_{2,M}(i)$. As above, the terms of the form $T_{2,M}(i)$ are formed by matrices which are one among

$$\pm \begin{pmatrix} 1 & 1 \\ n & n+1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ -n & n+1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ -(n+1) & n \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ n+1 & n \end{pmatrix},$$

where $n \in \mathbb{N}$ such that $M = n^2 + (n+1)^2$. Moreover,

$$\begin{aligned}
&T_{2,n^2+(n+1)^2}(i) \\
&= 2e^{i\frac{\pi}{2}\sigma} \Bigg\{ (ni+n+1)^{-k} \left(\frac{i+1}{ni+n+1}\right)^{-\sigma} + (-ni+n+1)^{-k} \left(\frac{i-1}{-ni+n+1}\right)^{-\sigma} \\
&+ (-(n+1)i+n)^{-k} \left(\frac{-i+1}{-(n+1)i+n}\right)^{-\sigma} + ((n+1)i+n)^{-k} \left(\frac{-i-1}{(n+1)i+n}\right)^{-\sigma} \Bigg\}.
\end{aligned}$$

We deduce that

$$|T_{2,\geq 5}(i)| \leq 2 \times 4 \times \sum_{n\geq 1} \left(\frac{1}{n^2+(n+1)^2}\right)^{\frac{k}{4}-\frac{\epsilon}{2}} \left(\frac{1}{2}\right)^{\frac{k}{4}+\frac{\epsilon}{2}} \leq \frac{1}{2^{\frac{k}{2}-3}} \zeta\left(\frac{k}{2}-\epsilon\right).$$

Finally, we obtain an upper bound for $T_{\geq 5,\geq 5}(i)$. We have

$$\begin{aligned}
|T_{\geq 5,\geq 5}(i)| &\leq \sum_{M\geq 5} \sum_{c^2+d^2=M} \sum_{N\geq 5} \sum_{\substack{a^2+b^2=N \\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} |ci+d|^{-(k-\sigma)} |ai+b|^{-\sigma} \\
&\leq \left(\sum_{M\geq 5} \sum_{c^2+d^2=M} |ci+d|^{-(k-\sigma)}\right) \left(\sum_{N\geq 5} \sum_{a^2+b^2=N} |ai+b|^{-\sigma}\right).
\end{aligned}$$

Thus, we have

$$|T_{\geq 5, \geq 5}(i)| \leq \left(\sum_{M\geq 5} \frac{4\sqrt{M}}{M^{\frac{k}{4}-\frac{\epsilon}{2}}}\right)\left(\sum_{N\geq 5} \frac{4\sqrt{N}}{N^{\frac{k}{4}+\frac{\epsilon}{2}}}\right)$$
$$\leq \frac{16}{5^{\frac{k}{2}-3}}\zeta\left(\frac{k}{4}-\frac{1}{2}+\frac{\epsilon}{2}\right)\zeta\left(\frac{k}{4}-\frac{1}{2}-\frac{\epsilon}{2}\right),$$

where in the final step we have again used Lemma 4.1. Thus, we have the following numerical estimates:

$$(9) \qquad |T_{error}(i)| \leq \begin{cases} 0.22148 & k \equiv 0 \bmod 16, \\ 0.05402 & k \equiv 4 \bmod 16, \\ 0.01346 & k \equiv 8 \bmod 16, \\ 1.12052 & k \equiv 12 \bmod 16. \end{cases}$$

The result follows from

$$f_{k,\sigma}(i) \geq \; T_{main}(i) \; - \; |T_{error}(i)|.$$

From (7) and (9), when $4|k$, clearly, $f_{k,\sigma}(i) > 0$ and in fact,

$$f_{k,\sigma}(i) \geq 3.866 \; - \; 1.12052 \; = \; 2.74548$$

completing the proof.

6. **Application of the Theorem: Corollary 2.2.1** First, we observe (recall eqn (4)) that since

$$f_{k,\sigma}(i) = \frac{\pi\Gamma(k-1)}{2^{k-3}\Gamma(\sigma)\Gamma(k-\sigma)} \sum_{f\in\mathcal{B}} \frac{L^*(f,\sigma)}{\langle f,f\rangle} f(i),$$

we remark that for $\sigma \in [\frac{k-1}{2}, \frac{k+1}{2}]$, there exists at least one Hecke eigenform $f \in \; S_k$ (dependent on σ) whose L-value is real and non-zero.

It is well-known that

$$\langle f,f\rangle = \frac{2\Gamma(k)}{\pi(4\pi)^k} L(\mathrm{Sym}^2(f), k)$$

for any normalised Hecke eigenform f for S_k. Moreover,

$$k^{-\delta_1} \ll L(\mathrm{Sym}^2(f), k) \ll k^{\delta_1}$$

holds for arbitrarily small $\delta_1 > 0$ ([Luo17], p.4), so we get that

$$\langle f, f\rangle \gg \frac{\Gamma(k)}{(4\pi)^k k^{\delta_1}}.$$

From Theorem (2.2), it follows that when $4|k$,

$$2.745 < f_{k,\sigma}(i) = \frac{\pi\Gamma(k-1)}{2^{k-3}\Gamma(\sigma)\Gamma(k-\sigma)} \sum_{f\in\mathcal{B}} \frac{L^*(f,\sigma)}{\langle f, f\rangle} f(i).$$

$$\ll \frac{(2\pi)^{k-\sigma}}{\Gamma(k-\sigma)} k^{\delta_1} \max_{f\in\mathcal{B}} |L(f,\sigma)| \left|\sum_{n\geq 1} a_f(n)e^{-2\pi n}\right|.$$

Using the Ramanujan-Petersson estimate $|a_f(n)| \ll_{\delta_2} n^{\frac{k-1}{2}+\delta_2}$ for arbitrary $\delta_2 > 0$, we have

$$\left|\sum_{n\geq 1} a_f(n)e^{-2\pi n}\right| \ll_{\delta_2} \sum_{n\geq 1} n^{\frac{k-1}{2}+\delta_2} e^{-2\pi n}. \tag{10}$$

The sum on the right can be estimated using the following lemma.

LEMMA 6.1. *(i) For any fixed $a > 0$,*

$$\sum_{m\geq 1} m^a e^{-mz} = \mathcal{O}_a\left(\frac{1}{z}\right)^{a+1} \quad \textit{for each } z > 0.$$

(ii)

$$\max_{z>0} z^{a+1} \sum_{m\geq 1} m^a e^{-mz} \ll \left(\frac{a+1}{e}\right)^{a+1} \qquad (a\to\infty).$$

PROOF. For the first part, we refer to Lemma (9.3.13) of [Coh-Str17] for the proof. As for the second part, for fixed $z > 0$, consider the function $f_z(x) = x^a e^{-xz}$ on the domain $\{x \in \mathbb{R} \mid x \geq 0\}$. This function is increasing on the interval $[0, a/z]$ and decreasing on $[a/z, \infty)$. Let $N := \left\lfloor \frac{a}{z} \right\rfloor$. Hence, the lower Riemann sum upto $\frac{a}{z}$ satisfies

$$\sum_{n=0}^{N-1} f_z(n) + f_z(N)(a/z - N) < \int_0^{a/z} f_z(x)dx. \text{ Similarly,}$$

$$\sum_{n=N+2}^{\infty} f_z(n) + f_z(N+1)(N+1-a/z) < \int_{a/z}^{\infty} f_z(x)dx.$$

From the above two inequalities, it follows that

$$\sum_{m\geq 0} f_z(m) < \int_0^\infty x^a e^{-xz} dx + f_z(N)(N+1-a/z) + f_z(N+1)(a/z-N)$$
$$\leq \frac{\Gamma(a+1)}{z^{a+1}} + f_z(a/z).$$

In other words,

$$z^{a+1} \sum_{m\geq 1} m^a e^{-mz} < \Gamma(a+1) + z\left(\frac{a}{e}\right)^a. \tag{11}$$

Similarly, by considering the upper sums, one can find that

$$\Gamma(a+1) - z\left(\frac{a}{e}\right)^a < z^{a+1} \sum_{m\geq 1} m^a e^{-mz}.$$

Thus, it follows that $\lim_{z\to 0} z\sum_{m\geq 1} (mz)^a e^{-mz} = \Gamma(a+1)$. Also, $\lim_{z\to\infty} z\sum_{m\geq 1}(mz)^a e^{-mz} = 0$. And, on any compact interval $[\delta, T]$ with $\delta > 0$, due to Weierstrass M-test, the partial sums $-\sum_{1\leq m\leq n} m^{a+1}e^{-mz}$ converge uniformly and absolutely and hence, derivative of $\sum_{m\geq 1} m^a e^{-mz}$ is the sum of its termwise derivatives.
Now, applying the product rule on $g_a(z) := z^{a+1}\sum_{m\geq 1} m^a e^{-mz}$, we see that

$$\frac{d}{dz} g_a(z) = z^a \left((a+1)\sum_{m\geq 1} m^a e^{-mz} - z\sum_{m\geq 1} m^{a+1} e^{-mz} \right) < 0$$

if $z \geq a+1$. Hence, we can see that the maximum of $g_a(z)$ in $[\delta, T]$ occurs at some point $z_0 < a+1$ (we choose $T \gg a+1$ sufficiently large and $\delta > 0$ sufficiently small). Thus, by eqn (11),

$$\max_{\delta\leq z\leq T} z^{a+1} \sum_{m\geq 1} m^a e^{-mz} < \Gamma(a+1) + (a+1)\left(\frac{a}{e}\right)^a \ll \left(\frac{a+1}{e}\right)^{a+1}.$$

We remark that $\Gamma(a+1) = o\left((\frac{a+1}{e})^{a+1} \right)$. This concludes the proof. □

Using the above lemma, the right hand side of (10) is estimated to be

$$\ll_{\delta_2} \left(\frac{\frac{k+1}{2} + \delta_2}{2\pi e} \right)^{\frac{k+1}{2}+\delta_2}.$$

By Stirling's estimate, $\Gamma(k-\sigma) \gg (\frac{k-\sigma}{e})^{k-\sigma}\sqrt{\frac{1}{k-\sigma}}$. Hence,

$$1 \ll_{\delta_2} \left(\frac{2\pi e}{k-\sigma}\right)^{k-\sigma} (k-\sigma)^{1/2} k^{\delta_1} \max_{f\in\mathcal{B}} |L(f,\sigma)| \left(\frac{\frac{k+1}{2}+\delta_2}{2\pi e}\right)^{\frac{k+1}{2}+\delta_2}$$

for large enough $k \geq K(\delta_2)$. Writing $\sigma = \frac{k}{2}+\epsilon$, this simplifies to

$$1 \ll_{\delta_2} k^{1+\delta_1+\delta_2+\epsilon} \max_{f\in\mathcal{B}} |L(f,\sigma)|.$$

Thus, for $k \gg_{\delta_2} 1$ and $\sigma \geq \frac{k-1}{2}$, we have

$$\max_{f\in\mathcal{B}} |L(f,\sigma)| \gg_{\delta} k^{-(\sigma-\frac{k}{2})-1-\delta},$$

where $\delta = \delta_1+\delta_2$ is any arbitrarily small positive real number.

7. Further Work We remark that in this context, it is interesting to ask the following question: quantify the number of modular forms of given weight whose L-values are non-vanishing at a point. More precisely, given s, a complex point inside the critical strip $\frac{k-1}{2} \leq \Re(s) \leq \frac{k+1}{2}$, consider the number

$$N_k(s) := \#\{f \in \mathcal{B}_k \mid L(f,s) \neq 0\}$$

Luo showed that for the central critical point $s = k/2$ ([Luo15], (4)), we have for $4|k$

$$N_k(k/2) \gg k,$$

and so there is a positive proportion of Hecke eigenforms in $\mathcal{B}_k$ with non-vanishing central values as $k \to \infty$ through multiples of 4. Assuming the truth of the Riemann Hypothesis, at points s other than those on the critical line, we expect the size of $N_k(s)$ to be roughly $k/12$. In a subsequent work, we quantify $N_k(s)$ for any s inside the critical strip with an arbitrarily fixed imaginary part (except those on the critical line $\Re(s) = k/2$) by providing an explicit lower bound in terms of the weight k.

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