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# A remark on the functoriality of the Connes-Takesaki flow of weights

*Dedicated to the memory of Uffe Haagerup*

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Abstract. The flow of weights was introduced by Connes and Takesaki as a functor on the category of von Neumann algebras with isomorphisms as maps. While it is easy to see that this functor cannot be extended to the category of all von Neumann algebra homomorphisms, it is in fact possible to extend it to a certain extent. This can also be done, fairly extensively, for the Falcone and Takesaki non-commutative flow of weights.

Résumé. Le flot des poids a été introduit par Connes et Takesaki comme foncteur sur la catégorie des algèbres de von Neumann avec isomorphismes comme flèches. On peut étendre ce foncteur jusqu'à un certain point dans les directions et covariante et contravariante. Le foncteur flot des poids non-commutatif peut aussi s'étendre, bien entendant pas aux homomorphismes arbitraires.

**1.** The main purpose of this note is to point out that, by using the theory of normal faithful semifinite (n.f.s.) operator valued weights developed by Haagerup in [8] and [9], it is possible to define a natural class of von Neumann algebra homomorphisms—namely, those (unital, normal, *-algebra homomorphisms) for which there is a n.f.s. operator valued weight from the codomain algebra to the image. Such homomorphisms form a subcategory, as, by Proposition 2.3 and Remark 2.4 of [8], the composition of two n.f.s. operator valued weights is again one. It is a proper subcategory by Proposition 5.8 of [9]. Let us refer to this subcategory as the category of tempered von Neumann algebra homomorphisms.

Takesaki's one-parameter crossed product decomposition for a von Neumann algebra, [17], was shown by Haagerup and Størmer in [11] (Proposition 12.1) to be functorial with respect to automorphisms, and a canonical construction by Falcone and Takesaki in [7] exhibited this functoriality as manifest (no calculation required). As will be shown below (Theorem 2), a refinement of the calculation of Haagerup and Størmer, using the main results of [8], shows that what Falcone and Takesaki named (in view of its canonical definition, in terms of n.f.s. weights) the non-commutative flow of weights (the semifinite von Neumann algebra with a one-parameter action scaling a n.f.s. trace obtained as the crossed product of a von Neumann algebra by the modular automorphism group of a

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n.f.s weight) is a functor on the category of von Neumann algebras with tempered homomorphisms. In fact it is only for a tempered homomorphism that the non-commutative flows of weights are compatible (Theorem 3, below). (It follows that the crossed product functor maps homomorphisms of non-commutative flows to tempered homomorphisms and is a one-sided inverse to the functor of Theorem 2; see Corollary 4, below.)

As a consequence of Theorem 2, if one restricts to the (proper) subcategory of the category of tempered homomorphisms for which the corresponding map between the non-commutative flows of weights takes the centre of the first algebra into (a subalgebra of) the centre of the second, one obtains the Connes-Takesaki flow of weights as a (covariant) functor. (We are for the present purpose taking this as a definition of the flow of weights—strictly speaking, this is valid only in the case of separable predual—see Corollary 2.5 of [5].)

Of course (regardless of how it is constructed or viewed), the Connes-Takesaki flow of weights cannot be a (covariant) functor for arbitrary homomorphisms, since a non-trivial flow (acting on a non-trivial commutative von Neumann algebra) cannot map into the trivial flow (on the algebra of complex numbers), but every von Neumann algebra maps into one with trivial flow of weights (a factor of type $\mathrm{III}_1$), simply because this is properly infinite—absorbs a factor of type I, which can be arbitrarily large. (And the embedding in the opposite direction, into a large enough factor with non-trivial flow, e.g. of type $\mathrm{II}_\infty$, shows that neither can the flow of weights be a contravariant functor in general.)

The covariant functor just described, on the deceptively easily defined subcategory of the category of tempered homomorphisms that conjures it into being (forwards embedding of centres of non-commutative flows), can in particular be seen to give rise to arbitrary homomorphisms between measure-preserving flows (i.e., flows based on commutative von Neumann algebras with separable predual preserving a n.f.s weight), as part of a routine extension of the inverse functor construction of Sutherland and Takesaki in [16] for isomorphisms (Theorem 5, below). (Thus, the construction of [16] for arbitrary isomorphisms extends to include easily also these special homomorphisms, but, it would seem, not arbitrary homomorphisms.)

A quite different subcategory of the category of tempered homomorphisms (of course, including isomorphisms as before), again with an almost vacuous definition—the corresponding map between non-commutative flows carries centres in the opposite direction!—exhibits the Connes-Takesaki flow of weights, in what seems to be a rather more limited setting, as a contravariant functor. At least, as will be shown (Theorem 6, below), besides isomorphisms, all maps between transitive flows (there aren't very many) arise in this way.

The question arises, what might be an intrinsic description of the two subcategories of the category of tempered homomorphisms introduced above, named, mnemonically, the covariant and contravariant subcategories perhaps—or right-way and wrong-way! Since both include non-isomorphisms (by Theorems 5 and 6, below), they are necessarily both proper subcategories (of the category of tempered homomorphisms). The author is indebted to Roberto Longo for suggesting that the contravariant direction might be a natural one.

The embeddings of non-commutative flows appearing in the present (right-way) extension of the Sutherland-Takesaki construction are easy to describe—and it is easy to see that they act admissibly (covariantly) on the centres—indeed, the larger algebra is generated by its centre together with the smaller one (any such inclusion acts admissibly on the centres). (Of course, both algebras are special in the construction of [16], namely, the tensor product of a commutative algebra and the amenable factor of type $\mathrm{II}_\infty$ with separable predual—and, as follows from functoriality of the non-commutative flow (with respect to isomorphisms) and the classification of amenable algebras with separable predual by the Connes-Takesaki flow ([4], [10]), the (trace-scaling) non-commutative flow (for a given commutative flow) is unique up to conjugacy.) (This was remarked in the $\mathrm{III}_1$ case in Theorem 6.18 and Problem 8.8 of [13].) It follows from [9] that the crossed product of an inclusion of non-commutative flows is tempered; see Theorem 3, below.

At least some tempered isomorphisms in the wrong-way subcategory can be described as embeddings of an algebra as one component of a tensor product—such homomorphisms are admissible (give rise to a contravariant mapping between the centres, and of course are tempered—by a Tomiyama slice map; see [19], [18]) when the flows of weights of both components, and hence (by Corollary 6.8 of [5]) that of the tensor product, are transitive (Theorem 6, below).

**2.** For the purposes of this article the non-commutative flow of weights of a von Neumann algebra [7] will just be viewed as the one-parameter crossed product by the modular automorphism group of a n.f.s. weight, together with the dual automorphism group, and the scaled n.f.s. weight associated to the dual weight. Connes showed in [3] that in a strong sense this is independent of the weight and the scaled n.f.s. trace associated to the dual weight (the modular automorphism groups differing only by a unitary cocycle), and Haagerup and Størmer in [11] (Proposition 2.1) showed that this construction is even functorial with respect to automorphisms (equivalently, isomorphisms), although it was not until [7] that this functor was shown to be canonical (i.e., obvious in view of a natural construction, independent of any choice). It is an interesting question whether the manifest functoriality of [7] for isomorphisms can be extended to the present category of tempered homomorphisms.

**Theorem.** *The non-commutative flow of weights is a functor on the category of von Neumann algebras with tempered homomorphisms.*

**Proof.** Clearly, it is enough to consider inclusions (considered as injective maps, including isomorphisms). If $N \subseteq M$ is a tempered inclusion (injective homomorphism) of von Neumann algebras, and $T$ is a n.f.s. operator valued weight from $M$ to $N$, choose a n.f.s. weight $\varphi$ on $N$. By Proposition 2.3 of [9], the composition $\varphi \cdot T$ is a n.f.s. weight on $M$. By Theorem 4.7 of [9], the modular automorphism group $\sigma^{\varphi \cdot T}$ of $\varphi \cdot T$ extends that of $\varphi$, $\sigma^{\varphi}$ (agrees with it on $N$), and so the crossed product $N \rtimes_{\sigma^\varphi} \mathbb{R}$ can be viewed naturally as contained in $M \rtimes_{\sigma^{\varphi\cdot T}} \mathbb{R}$ with the canonical one-parameter unitary group of the crossed product the same for each one.

Similarly, if now $M \subseteq P$ is a tempered inclusion of von Neumann algebras (with $M$ as above), and $S$ is a n.f.s. operator valued weight from $P$ to $M$, then the composition $(\varphi \cdot T) \cdot S$ is a n.f.s. weight on $P$, its modular automorphism group extends that of $\varphi \cdot T$, and the crossed product $M \rtimes_{\sigma^{\varphi\cdot T}} \mathbb{R}$ may be viewed as contained in $P \rtimes_{\sigma^{\varphi\cdot T\cdot S}} \mathbb{R}$, again with the same canonical one-parameter unitary group.

Similarly, again, one has $N \rtimes_{\sigma^\varphi} \mathbb{R} \subseteq P \rtimes_{\sigma^{\varphi\cdot(T\cdot S)}} \mathbb{R}$.

In this way, with respect to the particular n.f.s. weights $\varphi$ and $(\varphi \cdot T) \cdot S = \varphi \cdot (T \cdot S)$ we see that the crossed product $N \rtimes_{\sigma^\varphi} \mathbb{R}$ sits in the same way inside the crossed product $P \rtimes_{\sigma^{\varphi\cdot(T\cdot S)}} \mathbb{R}$ whether one considers the two steps $N \subseteq M$ with $T$ and $M \subseteq P$ with $S$, or just the single step $N \subseteq P$ with $T \cdot S$.

But (as in [11]) this is only the first step towards proving functoriality of the crossed product (the non-commutative flow of weights), for the category of tempered homomorphisms. While the dual one-parameter automorphism group and dual weight, and the canonical scaled trace, will not depend on the n.f.s. weights the (three) constructions are carried out with respect to, the inclusions will of course to some extent depend on these, although (as in [11]—but the situation here is somewhat more complicated) they may be reconciled through the use of Connes's cocycle Radon-Nikodym derivative ([3], Théorème 1.2; see also [15], [6]). (Our model for this calculation is the functorial construction of [11].)

We must show that the three inclusions between the crossed products are still compatible in the natural way if each one is constituted arbitrarily just like that for the given inclusion $N \subseteq M$, i.e., with an arbitrary n.f.s. weight on the subalgebra and an arbitrary n.f.s. operator valued weight from $M$ to $N$—not the composites $\varphi \cdot T$ for $M \subseteq P$ and $T \cdot S$ for $N \subseteq P$. Thus, let $\psi$ be an arbitrary n.f.s. weight on $M$, let $U$ be an arbitrary n.f.s. operator valued weight from $P$ to $N$, and let $\rho$ be an arbitrary n.f.s weight on $P$. (Note that $T$ from $M$ to $N$ and $S$ from $P$ to $M$ are already arbitrary, as is the n.f.s. weight $\varphi$ on $N$.) We must show that the natural inclusions between the crossed products $N \rtimes_{\sigma^\varphi} \mathbb{R}$, $M \rtimes_{\sigma^{\varphi\cdot T}} \mathbb{R}$, $M \rtimes_{\sigma^\psi} \mathbb{R}$, $P \rtimes_{\sigma^{\varphi\cdot(T\cdot S)}} \mathbb{R}$, $P \rtimes_{\sigma^{\psi\cdot S}} \mathbb{R}$, and $P \rtimes_{\sigma^\rho} \mathbb{R}$, most of them obtained by using Connes's Radon-Nikodym derivative, are compatible.

More specifically, if for the chosen n.f.s weight $\varphi$ on $N$ the canonical one-parameter unitary group of the crossed product is denoted by $u_\varphi$, and similarly for other n.f.s weights on other von Neumann algebras, then, by Theorem 4.7(1) of [8] (the first of the two main results of [8]), the modular automorphism groups $\sigma^\varphi$, $\sigma^{\varphi\cdot T}$, and $\sigma^{\varphi\cdot T\cdot S}$ are compatible, and so we have simple inclusions

$$N \rtimes_{\sigma^\varphi} \mathbb{R} \subseteq M \rtimes_{\sigma^{\varphi\cdot T}} \mathbb{R} \subseteq P \rtimes_{\sigma^{\varphi\cdot T\cdot S}} \mathbb{R},$$

extending the given inclusions $N \subseteq M \subseteq P$, in which the canonical one-parameter unitary groups of the crossed products are identified:

$$u_\varphi = u_{\varphi \cdot T} = u_{\varphi \cdot T \cdot S}.$$

By Connes's unitary cocycle Radon-Nikodym theorem ([3]), we in fact have, altogether, inclusions (injective homomorphisms) of $N \rtimes_{\sigma^\varphi} \mathbb{R}$ in $M \rtimes_{\sigma^\psi} \mathbb{R}$ and of each of $M \rtimes_{\varphi \cdot T} \mathbb{R}$ and $M \rtimes_{\sigma^\psi} \mathbb{R}$ in each of $P \rtimes_{\sigma^{\varphi \cdot T \cdot S}} \mathbb{R}$, $P \rtimes_{\sigma^{\psi \cdot S}} \mathbb{R}$, and $P \rtimes_{\sigma^\rho} \mathbb{R}$, defined by the identifications

$$\begin{aligned}
u_\varphi &= u_{\varphi \cdot T} = (D\varphi \cdot T : D\psi) u_\psi, \\
u_{\varphi \cdot T} &= u_{\varphi \cdot T \cdot S}, \\
u_{\varphi \cdot T} &= u_{\varphi \cdot T \cdot S} = (D\varphi \cdot T \cdot S : D\psi \cdot S) u_{\psi \cdot S}, \\
u_{\varphi \cdot T} &= u_{\varphi \cdot T \cdot S} = (D\varphi \cdot T \cdot S : D\rho) u_\rho, \\
u_\psi &= u_{\psi \cdot S} = (D\psi \cdot S : D\varphi \cdot T \cdot S) u_{\varphi \cdot T \cdot S}, \\
u_\psi &= u_{\psi \cdot S}, \text{ and} \\
u_\psi &= (D\psi \cdot S : D\rho) u_\rho.
\end{aligned}$$

These are single step inclusions; we also have the following double step, or leap-frog, inclusions (or injective homomorphisms), besides that mentioned above of $N \rtimes_{\sigma^\varphi} \mathbb{R}$ in $P \rtimes_{\sigma^{\varphi \cdot T \cdot S}} \mathbb{R}$ directly through $M \rtimes_{\sigma^{\varphi \cdot T}} \mathbb{R}$: namely, that of $N \rtimes_{\sigma^\varphi} \mathbb{R}$ in $P \rtimes_{\sigma^{\varphi \cdot T \cdot S}} \mathbb{R}$ through $M \rtimes_{\sigma^\psi} \mathbb{R}$, that of $N \rtimes_{\sigma^\varphi} \mathbb{R}$ in each of $P \rtimes_{\sigma^{\psi \cdot S}} \mathbb{R}$ and $P \rtimes_{\sigma^\rho} \mathbb{R}$ through each of $M \rtimes_{\sigma^{\varphi \cdot T}} \mathbb{R}$ and $M \rtimes_{\sigma^\psi} \mathbb{R}$, and, finally, the direct inclusion of $N \rtimes_{\sigma^\varphi} \mathbb{R}$ in $P \rtimes_{\sigma^{\varphi \cdot U}}$, defined by the identification

$$u_\varphi = u_{\varphi \cdot U},$$

and the two direct inclusions of $N \rtimes_{\sigma^\varphi} \mathbb{R}$ in $P \rtimes_{\sigma^\rho} \mathbb{R}$ defined by the identifications

$$\begin{aligned}
u_\varphi &= u_{\varphi \cdot T \cdot S} = (D\varphi \cdot T \cdot S : D\rho) u_\rho \text{ and} \\
u_\varphi &= u_{\varphi \cdot U} = (D\varphi \cdot U : D\rho) u_\rho.
\end{aligned}$$

We must show that all of these two-step (or leap-frog) maps, either direct or composed of two separate (single-step) homomorphisms, are the same, in the natural sense, taking into account the properties of Connes's Radon-Nikodym derivative.

In the simplest case, the two inclusions defined by $u_\varphi = u_{\varphi \cdot T}$ and $u_{\varphi \cdot T} = u_{\varphi \cdot T \cdot S}$ combine, clearly, to yield the single inclusion defined by $u_\varphi = u_{\varphi \cdot (T \cdot S)}$.

A much more interesting case is that of the two inclusions

$$N \rtimes_{\sigma^\varphi} \mathbb{R} \hookrightarrow M \rtimes_{\sigma^\psi} \mathbb{R} \hookrightarrow P \rtimes_{\sigma^\rho} \mathbb{R},$$

defined by the identifications (or assignments) $u_\varphi = (D\varphi \cdot T : D\psi) u_\psi$ and $u_\psi = (D\psi \cdot S : D\rho) u_\rho$, the composition of which is to be shown to be equal to the single (leap-frog) inclusion $N \rtimes_{\sigma^\varphi} \mathbb{R} \hookrightarrow P \rtimes_{\sigma^\rho} \mathbb{R}$ defined by the identification $u_\varphi = (D\varphi \cdot (T \cdot S) : D\rho) u_\rho$. The composed inclusion is defined by the composed identification (or assignment)

$$u_\varphi = (D\varphi \cdot T : D\psi)(D\psi \cdot S : D\rho) u_\rho.$$

But by Theorem 4.7(2) of [8] (the second of the two main results of [8]—the first of which, 4.7(1), was used above),

$$(D\varphi \cdot T : D\psi) = (D(\varphi \cdot T) \cdot S : D\psi \cdot S).$$

The composed identification above thus becomes

$$u_\varphi = (D\varphi \cdot T \cdot S : D\psi \cdot S)(D\psi \cdot S : D\rho) u_\rho$$

which by Connes's chain rule for cocycle Radon-Nikodym derivatives ([3]) contracts to

$$u_\varphi = (D\varphi \cdot T \cdot S : D\rho) u_\rho,$$

the defining identification for the leap-frog inclusion $N \rtimes_{\sigma^\varphi} \mathbb{R} \hookrightarrow P \rtimes_{\sigma^\rho} \mathbb{R}$, as desired.

A similarly interesting case is that of the two inclusions

$$N \rtimes_{\sigma^\varphi} \mathbb{R} \hookrightarrow M \rtimes_{\sigma^\psi} \mathbb{R} \hookrightarrow P \rtimes_{\sigma^{\varphi \cdot T \cdot S}} \mathbb{R},$$

defined by the identifications

$$u_\varphi = (D\varphi \cdot T : D\psi) u_\psi \text{ and } u_\psi (D\psi \cdot S : D\varphi \cdot T \cdot S) u_{\varphi \cdot T \cdot S},$$

the composition of which is defined by the concatenated identification (or assignment)

$$u_\varphi = (D\varphi \cdot T : D\psi)(D\psi \cdot S : D\varphi \cdot T \cdot S) u_{\varphi \cdot T \cdot S}.$$

Application of Theorem 4.7(2) of [8] together with the chain rule of [3], as above, now yields the identification

$$u_\varphi = (D\varphi \cdot T : D\psi)(D\psi : D\varphi \cdot T) u_{\varphi \cdot T \cdot S} = u_{\varphi \cdot T \cdot S},$$

as expected.

The other combined pairs of inclusions mentioned above need no calculation, as they involve only a single non-trivial Radon-Nikodym derivative, and the composition of the canonical identifications (or assignments) of one-parameter unitary groups is seen immediately to coincide with the expected single (leap-frog) identification.

The overall functorial nature of the present situation, short of using the completely canonical approach of Falcone and Takesaki in [7], can best be systematized using the approach of Sutherland and Takesaki in the proof of Theorem

1.1 of [16] (the reverse construction from flows to von Neumann algebras, by way of non-commutative flows). Namely, instead of the crossed product of a given von Neumann algebra by the modular automorphism group of a single n.f.s. weight, consider simultaneously the crossed products corresponding to all such weights, and focus on the subalgebra of the Cartesian product consisting of all families the coordinates of which are related by the canonical isomorphisms between the different crossed products—assigning the canonical one-parameter unitary group $u_\varphi$ of the crossed product corresponding to the weight $\varphi$ to the Radon-Nikodym multiple $(D\varphi : D\psi)u_\psi$ of the canonical unitary group $u_\psi$ corresponding to the weight $\psi$. This relation between coordinates of a family in the Cartesian product is an equivalence relation because of the chain rule ([3]). (In [16], the analogous relation was an equivalence relation because of the usual chain rule—this will of course be used in the proof of the extension of Theorem 1.1 of [16] in Theorem 4, below.)

To describe the functorial embedding of what we might call the family construction of the non-commutative flow of weights of one von Neumann algebra, $N$, into that of a larger one, $M$, with respect to a tempered embedding of $N$ in $M$, let us first note the role of the existence of a n.f.s. operator valued weight from $M$ onto $N$. As mentioned earlier, by Theorem 4.7 of [8], the existence of a n.f.s. operator valued weight implies the existence of a pair of n.f.s. weights $\varphi$ and $\psi$ on $N$ and $M$ such that $\sigma^\psi$ is compatible with $\sigma^\varphi$ (agrees with it on $N$). (In fact, by Theorem 5.1 of [9] the converse of this is also true, and we shall need this in the proof of Theorem 4, below.) Thus, whenever the inclusion of $N$ in $M$ is tempered, there is in the first instance an embedding of at least one coordinate algebra in the family construction of the non-commutative flow of $N$ into at least one coordinate algebra for $M$, namely, the one for $N$ for an arbitrary n.f.s weight $\varphi$ on $N$ and the one for $M$ corresponding to some $\psi$ on $M$ with $\sigma^\psi$ compatible with $\sigma^\psi$. Adjoining the natural isomorphisms, one then has a commutative diagram of inclusions from every coordinate algebra for $N$ into every such algebra for $M$, and (owing to the Radon-Nikodym chain rule, [3]) this is independent of the choice of $\varphi$—and, for any choice of $\varphi$, independent of the corresponding choice of $\psi$ (such that $\psi \cdot T = \varphi$ for some n.f.s. operator valued weight $T$ from $M$ to $N$—not in general unique). The calculations above (again using the chain rule, together with Theorem 4.7(2) of [8]) show that, if $M \subseteq P$ is another tempered inclusion, then the concatenation of the two commutative diagrams arising from the two tempered inclusions separately contracts under composition of maps to the single commutative diagram (of families of coordinate algebras) arising from the inclusion $N \subseteq P$. In this way, much as in [16] (and in Theorem 5, below) for the reverse construction, we see the non-commutative flow of weights exhibited as a functor, on the category of von Neumann algebras with tempered homomorphisms.

It almost goes without saying that, in the inclusion of non-commutative flows for a tempered inclusion $N \subseteq M$, as represented by the commutative diagram of inclusions between isomorphic families, if the centres are mapped either covariantly or contravariantly by what might be called the seed inclusion, from

$N \rtimes_{\sigma^\varphi} \mathbb{R}$ to $M \rtimes_{\sigma^\psi} \mathbb{R}$ (as above, with $\sigma^\psi$ extending $\sigma^\varphi$), then as all the other inclusions between algebras in the respective families are just equal to this one composed with an isomorphism on either side, then all the other inclusions of coordinate algebras map the centres in the same way, and so this is purely a property of the inclusion $N \subseteq M$. (It is not clear how to identify this property intrinsically, in either case, but interesting examples of both behaviours are described in Theorem 5 (covariant) and Theorem 6 (contravariant), below.)

**3.** The proof of Theorem 2, above, depended on the results of Haagerup on operator valued weights in [8]. The following converse to Theorem 2, stating that tempered homomorphisms are exactly those for which there exists a map between the non-commutative flows of weights, depends on the further results of Haagerup in [9].

**Theorem.** *Let $N \subseteq M$ be von Neumann algebras, and suppose that this inclusion extends to an inclusion between the non-commutative flows of weights of $N$ and $M$ (defined in [7]). It follows that the inclusion $N \subseteq M$ is tempered.*

**Proof.** The hypothesis may be sated more explicitly as the existence of a von Neumann algebra inclusion of the crossed product $N \rtimes_{\sigma^\varphi} \mathbb{R}$ in the crossed product $M \rtimes_{\sigma^\psi} \mathbb{R}$, where $\varphi$ and $\psi$ are n.f.s. weights on $N$ and $M$, which is compatible with both the given inclusion and the dual automorphism groups $(\sigma^\varphi)\hat{}$ and $(\sigma^\psi)\hat{}$.

Note that, by Theorem 5.1 of [9] and Theorem 4.7 of [8], a given von Neumann algebra inclusion is tempered if, and only if, there exist n.f.s. weights on the two given algebras having compatible modular automorphism groups (one extending the other). (The "if" is Theorem 5.1 of [9] and the "only if" Theorem 4.7 of [8].)

The asserted conclusion follows immediately. (Of course, it follows just from the "if", after passing to the crossed product of the given inclusion with respect to the common flow—the two compatible dual one-parameter automorphism groups each scaling a n.f.s. trace—incidentally, from the "only if" one sees that the given inclusion of non-commutative flow algebras is tempered (the two traces have trivial modular automorphism groups) but we don't need this. By Takesaki duality (specifically, Theorem 6.7 of [17]), the (compatible) double dual actions are just the original modular automorphism groups of the two n.f.s. weights $\varphi$ and $\psi$ tensored with a single automorphism group of a type $\mathrm{I}_\infty$ factor. The original modular automorphism groups must therefore be compatible, and so the "if" above yields the existence of a n.f.s. operator valued weight from $M$ to $N$—expressing $\psi$ in terms of $\varphi$.)

(The question arises whether any crossed product of a tempered inclusion by a group action compatible with the inclusion is again a tempered inclusion. One might also ask if it is sufficient for some modular automorphism group of the larger algebra to leave the subalgebra invariant: is the restriction automatically itself a modular automorphism group so that the criterion of Theorem 5.1 of [9] applies?)

**4**. **Theorem.** *The (manifest) functor (the crossed product) from the category of non-commutative flows (continuous, infinite, von Neumann algebras with a (specified) one-parameter automorphism group scaling a n.f.s. trace) to the category of von Neumann algebras is a one-sided inverse to the functor of Theorem 2 (from tempered von Neumann algebra homomorphisms to non-commutative flows).*

**Proof.** By Theorem 3, above (which depends on Theorem 5.1 of [9]) together with Theorem 8.3 of [17], the crossed product functor maps into the category of tempered homomorphisms (as the dual one-parameter groups are compatable). Strictly speaking, it is a one-sided inverse to the functor of Theorem 2, by Takasaki duality—as pointed out in the proof of Theorem 3—, only in the sense of the combined functor's being the double crossed product construction, i.e., by Theorem 6.7 of [17], tensoring with a factor of type $\mathrm{I}_\infty$.

**5.** In the setting of amenable von Neumann algebras with separable predual, Sutherland and Takesaki in [16] showed the existence of an inverse functor, for automorphisms (or isomorphisms), from the category of flows (one-parameter automorphism groups of commutative von Neumann algebras) to the category of von Neumann algebras. Since inner automorphisms of a von Neumann algebra act trivially on the flow of weights, this was just a one-sided inverse. It is not difficult to extend this inverse functor to a fairly general class of homomorphisms between flows (i.e., von Neumann algebra homomorphisms compatible with the one-parameter groups), into (by Theorem 3, above) the category of von Neumann algebra with tempered homomorphisms.

**Theorem.** (Cf. [16].) *There exists a covariant functor from the subcategory described below of the category of (separable) flows (i.e., weak* continuous one-parameter automorphism groups of commutative von Neumann algebras with separable predual, with compatible homomorphisms as maps) to the category of von Neumann algebras with tempered homomorphisms, which, on its domain, is a one-sided inverse to the (covariant) flow of weights functor of Section 1—consisting of the non-commutative flow of weights for tempered homomorphisms (Theorem 2, above), restricted to those tempered homomorphisms for which the centres of the non-commutative flows are mapped in the forwards (covariant) direction (see Section 1), followed by this map between the centres.*

*The subcategory referred to consists of all objects (flows), but only certain homomorphisms, namely, those for which the domain and codomain algebras have n.f.s. weights which evolve in the same way along corresponding paths: in other words, if $F \hookrightarrow G$ is the homomorphism of flows, then there are n.f.s. weights $\varphi$ and $\psi$ on $F$ and $G$ such that, for each $t \in \mathbb{R}$, the Radon-Nikodym derivative $d\psi \cdot G_t/d\psi$ is affiliated with (the image of) $F$ (i.e., is invariant under the equivalence relation and is equal to the image of $d\varphi \cdot F_t/d\varphi$). (This occurs for instance if there exist invariant n.f.s. weights $\varphi$ on $F$ and $\psi$ on $G$. Note that if $F$ is just $\mathbb{C}$, then $\psi$ must be invariant. The restriction is vacuous if $F \hookrightarrow G$ is an isomorphism.)*

**Proof.** First, we must check that the homomorphisms between flows described form a subcategory. Suppose that $E \hookrightarrow F$ ad $F \hookrightarrow G$ are homomorphisms as stipulated, and let $\rho,\ \varphi,\ \varphi'$, and $\psi'$ be n.f.s. weights on $E,\ F$, and $G$—both $\varphi$ and $\varphi'$ on $F$—such that

$$d\varphi \cdot F_t/d\varphi = d\rho \cdot E_t/d\rho \text{ and } d\psi' \cdot G_t/d\psi' = d\varphi' \cdot F_t/d\varphi'.$$

We wish to find a n.f.s. weight $\psi$ on $G$ such that

$$d\psi \cdot G_t/d\psi = d\varphi \cdot F_t/d\varphi.$$

Set $\psi = \psi'(d\varphi/d\varphi')$. Then $d\psi'/d\psi = d\varphi'/d\varphi$, and hence also

$$d\psi \cdot G_t/d\psi' \cdot G_t = (d\psi/d\psi') \cdot G_t = d\varphi \cdot F_t/d\varphi' \cdot F_t.$$

Then, by the (ordinary) chain rule,

$$\begin{aligned} d\varphi \cdot F_t/d\varphi &= (d\varphi \cdot F_t/d\varphi' \cdot F_t)(d\varphi' \cdot F_t/d\varphi')(d\varphi'/d\varphi) \\ &= d\psi \cdot G_t/d\psi' \cdot G_t)(d\psi' \cdot G_t/d\psi')(d\psi'/d\psi) \\ &= d\psi \cdot G_t/d\psi, \end{aligned}$$

and so $d\psi \cdot G_t/d\psi = d\rho \cdot E_t/d\rho$, as desired.

If $F \hookrightarrow G$ is an inclusion (injective homomorphism) of flows, belonging to the specified subcategory, it is enough to extend this to an inclusion of non-commutative flows $F \otimes R_{0,1} \hookrightarrow G \otimes R_{0,1}$, where $R_{0,1}$ is the amenable factor of type $\mathrm{II}_\infty$ with separable predual ([14], [4]), and by a non-commutative flow we mean a one-parameter automorphism group of a von Neumann algebra scaling a n.f.s. trace. We must carry this out in such a way as to obtain a functor. Combining this factor with the functor of Corollary 4, above, i.e., passing to the one-parameter crossed product of the resulting inclusion of non-commutative flows, then yields the desired inclusion of von Neumann algebras.

As far as the one-sided inverse property is concerned, by Theorem 4 it is sufficient to show that the (covariant) functor from the subcategory of flows considered is (on its domain) a one-sided inverse to the (covariant) inclusion map between the centres of those inclusions of non-commutative flows for which (see Section 1) the centres are mapped one into the other covariantly. But this is automatic from the nature of the functor now to be constructed.

By Lemma 8.2 of [17] (applied in the amenable type $\mathrm{III}_1$ case), there exists a one-parameter automorphism group $\alpha$ of $R_{0,1}$ scaling the trace (composing with $\alpha_t$ multiplies the trace by $e^{-t}$). As pointed out in [13] (Theorem 6.18 and Problem 8.8), and more abstractly in Corollary 6.4 of [17], $\alpha$ is unique up to conjugacy, as a consequence of the uniqueness of the amenable factor of type $\mathrm{III}_1$ with separable predual ([10]). (A similar uniqueness statement holds for any non-commutative flow on an amenable von Neumann algebra of type $\mathrm{II}_\infty$ with separable predual—not necessarily a factor—as a consequence of the uniqueness of the amenable continuous von Neumann algebra with separable predual with a given flow of weights ([4], [10]).) (But we shall not use this. For that matter we shall not use the $\mathrm{II}_\infty$ factor case either.)

Let us consider first the special case that each of $F$ and $G$ has an invariant n.f.s. weight (trace). In this case, for each of $F$ and $G$, the tensor product of the given flow with the non-commutative flow $\alpha$ on $R_{0,1}$ constitutes a non-commutative flow on the tensor product, scaling the (n.f.s) tensor product trace, as desired. This construction is manifestly functorial with respect to flows with an invariant n.f.s. weight (trace). (Recall that a homomorphism in the category of non-commutative flows is required to be compatible only with the flows, not the scaled traces.)

Let us now consider the general case. Given an inclusion of flows $F \hookrightarrow G$ (with separable preduals), and n.f.s. weights $\varphi$ and $\psi$ on $F$ and $G$ such that the Radon-Nikodym derivative $d\psi \cdot G_t/d\psi$ belongs to $F$—or, if it is unbounded, is affiliated with $F$ (the supremum of an increasing sequence of positive elements of $F$)—, and is equal to $d\varphi \cdot F_t/d\varphi$, consider the non-commutative flows on $R \otimes R_{0,1}$ and $G \otimes R_{0,1}$ of [16], scaling the n.f.s. traces $\varphi \otimes \mathrm{tr}$ and $\psi \otimes \mathrm{tr}$, where tr denotes a choice of n.f.s. trace on $R_{0,1}$. The assumption on $\varphi$ and $\psi$ is equivalent to saying that the inclusion $F \otimes R_{0,1} \subseteq G \otimes R_{0,1}$ is compatible with the flows.

To check the functorial property, let $E \hookrightarrow F$ and $F \hookrightarrow G$ be inclusions of flows in the specified subcategory. As in the construction of the non-commutative flow corresponding to a flow in [16], the construction of a functorial homomorphism between non-commutative flows must not depend on the choices of n.f.s. weights on the different flow algebras. As done in [16] to exhibit functoriality for isomorphisms, let us consider simultaneously all n.f.s. weights on a flow, and inside the Cartesian product of all the corresponding non-commutative flows, indexed by these weights, consider the families of elements compatible with the natural isomorphisms between all these non-commutative flows, which as shown in [16], form a commutative diagram. These constitute a choice-independent construction of the non-commutative flow corresponding to a given (commutative) flow.

(For the convenience of the reader, let us recapitulate the construction of [16] both of the non-commutative flow corresponding to a single n.f.s. weight on a given flow algebra, $F$, and of the natural isomorphism between the non-commutative flows corresponding to two given n.f.s. weights on $F$. If $\varphi$ is a n.f.s. weight on $F$, channel the flow $F$ into the von Neumann algebra $F \otimes R_{0,1}$ as the non-commutative flow

$$\mathbb{R} \ni t \mapsto (F_t \otimes \alpha_t) \cdot \alpha(\log(d\varphi \cdot F_t/d\varphi)),$$

where $\alpha$ is as above, and the second factor acts on $F \otimes R_{0,1}$ by fixing $F \otimes 1$ and acting by $\alpha(\log(d\varphi \cdot F_t/d\varphi)(x))$ on the fibre $R_{0,1}$ at the point $x$ in the direct integral decomposition of $F \otimes R_{0,1}$ with respect to (the commutative von Neumann algebra) $F$ (all on a separable Hilbert space). If $\varphi'$ is another n.f.s. weight on $F$, then the natural automorphism of $F \otimes R_{0,1}$, again fixing $F \otimes 1$, and acting as $\alpha(\log(d\varphi'/d\varphi)(x))$ on the fibre $R_{0,1}$ at the point $x$, transforms the scaled tensor product trace $\varphi \otimes \mathrm{tr}$ into $\varphi' \otimes \mathrm{tr}$, and the non-commutative flow corresponding to $\varphi$ into that corresponding to $\varphi'$. The functorial non-commutative flow associated to the flow $F$ is then the family if non-commutative flows indexed by n.f.s. weights simultaneously identified in this way. An arbitrary automorphism of the flow $F$ is then manifestly transported functorially into an automorphism of this global non-commutative flow.)

Now, with the given inclusions of flows $E \hookrightarrow F$ and $F \hookrightarrow G$, in the specified subcategory, so that, as shown, the combined inclusion $E \hookrightarrow G$ also belongs to this subcategory, let us show that there are manifest inclusions (homomorphisms) between the (global) non-commutative flows on $E \otimes R_{0,1}$, $F \otimes R_{0,1}$, and $G \otimes R_{0,1}$—between the first two, the last two, and the first and the last—such that the inclusion constructed directly between the first and the third, without mentioning the other two, is just equal to the composition of the other two. (Note that homomorphisms should include isomorphisms, and in particular, automorphisms, and so the present case will be modelled on—indeed, be dealt with in exactly the same way as—the case of automorphisms recalled above from [16].)

Let, then, n.f.s. weights $\rho$ and $\varphi$ be given on $E$ and $F$, and $\varphi'$ and $\psi'$ on $F$ and $G$, both pairs of weights compatible with the inclusions as specified for the subcategory, and consider the corresponding homomorphisms between the non-commutative flows on $E \otimes R_{0,1}$ and $F \otimes R_{0,1}$, and on $F \otimes R_{0,1}$ and $G \otimes R_{0,1}$, specific to these pairs of weights, constructed as above. Consider, more pertinently, also the homomorphism between the non-commutative flows on $F \otimes R_{0,1}$ and $G \otimes R_{0,1}$ specific to the pair of weights $\varphi$ and $\psi = \psi'(d\varphi/d\varphi')$; as shown above this pair is also compatible with the inclusion (homomorphism) $F \hookrightarrow G$, and the pair $\rho$, $\psi$ is compatible with the combined inclusion $E \hookrightarrow G = E \hookrightarrow F \hookrightarrow G$. It is immediate that the three inclusions (homomorphisms) between the specific non-commutative flows on $E \otimes R_{0,1}$, $F \otimes R_{0,1}$, and $G \otimes R_{0,1}$ corresponding to $\rho$, $\varphi$, and $\psi$ are compatible. Furthermore, owing to the ordinary Radon-Nikodym theorem, the diagram of all homomorphisms and isomorphisms arising in this way, among specific non-commutative flows corresponding to $E$, $F$, and $G$, is commutative, as asserted.

**6.** In the same setting as Section 4, of amenable von Neumann algebras with separable predual, it is also not difficult, using the machinery of the Connes-Takesaki flow of weights, specifically, Corollary 6.8 of [5] (and again Theorem 3, above, i.e., Theorem 5.1 of [9]), to construct a contravariant functor from the category of transitive flows to the category of tempered homomorphisms between von Neumann algebras—a one-sided inverse to the flow of weights restricted to the proper subcategory introduced in Sections 1 and 2 on which it is a contravariant functor.

**Theorem.** *There exists a contravariant functor from the category of transitive flows (i.e., translation on $\mathbb{R}$ or the canonical action of $\mathbb{R}$ on $\mathbb{R}/(\log\lambda)\mathbb{Z}$,*

*$0 < \lambda < 1$, or on a single point—considered as actions on* $\mathrm{L}^\infty(\mathbb{R})$, $\mathrm{L}^\infty(\mathbb{T})$, *or* $\mathbb{C}$, *with compatible homomorphisms as maps) to the category of amenable von Neumann algebras with separable predual, with tempered homomorphisms as maps, which is a one-sided inverse to the (contravariant) flow of weights functor of Section 1 (see also Section 2)—consisting of the non-commutative flow of weights functor for tempered homomorphisms (Theorem 2), restricted to those tempered homomorphisms for which the centres of the non-commutative flow algebras are mapped in the backwards (contravariant) direction (see Sections 1 and 2), followed by this map between the centres.*

**Proof.** The flows in question are (necessarily) ergodic, and so the corresponding (continuous, infinite) von Neumann algebras are factors, in the amenable case unique ([4], [10]), namely,

$$R_{0,1},\ R_\lambda\ (0 < \lambda < 1),\ \text{and}\ R_1\ (\text{type III}_1).$$

(In the present setting, this will be taken just as notation—for the factors with the specified flows of weights.) (Although we shall see that $R_{0,1}$ is of type $\mathrm{II}_\infty$.)

The maps between the flows are also relatively few, at most one for each pair (and in most cases none at all). So it is just a question of constructing a map between factors in the (rare) case that a map exists between the flows, verifying its action on the flows, and checking that composition of maps behaves well.

The maps between factors that we will consider are obtained by decomposing the second (codomain) factor as a tensor product of the first (domain) factor and another one, according to the well-known rule (according to Corollary 6.8 of [5])

$$\begin{aligned} R_{0,1} \otimes R_\lambda &\cong R_\lambda, && 0 < \lambda \le 1, \\ R_{\lambda^n} \otimes R_\lambda &\cong R_\lambda, && 0 < \lambda < 1,\ n = 1, 2, \ldots,\ \text{and} \\ R_\lambda \otimes R_1 &\cong R_1, && 0 < \lambda < 1. \end{aligned}$$

(These do not include quite all such relations, but they are enough for the present purpose.)

The flows pertinent to those inclusions (the left tensor product component as a subalgebra of the tensor product), expressed in terms of the underlying measurable spaces $\mathbb{R}$, $\mathbb{R}/(\log\lambda)\mathbb{Z}$ for $0 < \lambda < 1$, and a single point, and the maps between them compatible with the $\mathbb{R}$-action, are just the obvious ones, $\mathbb{R}$ mapping onto everything and $\mathbb{R}/(\log\lambda)\mathbb{Z}$ mapping onto $\mathbb{R}/n(\log\lambda)\mathbb{Z} = \mathbb{R}/(\log\lambda^n)\mathbb{Z}$, for any $n = 1, 2, \ldots$, or onto a point.

It remains to verify that the flows of weights for the smaller and larger algebras in the inclusions indicated are in fact related in the contravariant way—and that the maps between them are the ones listed. This description would be in fact, automatic, given a contravariant inclusion of the flows (necessarily unique for transitive flows), but it seems necessary to compute the specific relation between the flows to prove it is an inclusion in the first place.

Fix $0 < \lambda \leq 1$. Consider first the inclusion of $R_{0,1}$ as $R_{0,1} \otimes 1$ in $R_{0,1} \otimes R_\lambda \cong R_\lambda$. Denote by $\tau$ the n.f.s. trace on the type $\mathrm{II}_\infty$ factor $R_{0,1}$ (unique up to a positive multiple). Since $\sigma^\tau = \mathrm{id}$, the crossed product $R_{0,1} \rtimes_{\sigma^\tau} \mathbb{R}$ is isomorphic to $R_{0,1} \otimes \mathrm{L}^\infty(\mathbb{R})$, and the dual automorphism group is translation on $\mathbb{R}$; in particular, the flow of weights (the action of $(\sigma^\tau)\widehat{\ }$ on $\mathrm{Centre}(R_{0,1} \rtimes_{\sigma^\tau} \mathbb{R})$—as we are presuming) on $R_{0,1}$ is translation on $\mathbb{R}$, i.e., on $\mathrm{L}^\infty(\mathbb{R}) = \mathrm{Centre}(R_{0,1} \otimes \mathrm{L}^\infty(\mathbb{R}))$, and since $\sigma^\tau$ is trivial, this centre is generated by the canonical one-parameter group $u_\tau$ of the crossed product. (In particular, we see that $R_{0,1}$ must in fact be semifinite, as by [4] only one amenable factor can have this flow of weights.)

The centre of $R_\lambda \rtimes_{\sigma^\varphi} \mathbb{R}$, on the other hand, with the restriction of $(\sigma^\varphi)\widehat{\ }$, is by definition $\mathrm{L}^\infty(\mathbb{R}/(\log\lambda)\mathbb{Z})$, with translation. By Theorems 1.3.2(d) and 3.4.1 of [3], the automorphism $\sigma^\varphi(2\pi/\log\lambda)$ is inner, and one may choose the n.f.s. weight $\varphi$ so that it is in fact the identity. If $u_\varphi$ denotes the canonical one-parameter unitary group in the crossed product $R_\lambda \rtimes_{\sigma^\varphi} \mathbb{R}$, the unitary $u_\varphi(2\pi/\log\lambda)$ is then in the centre of $R_\lambda \rtimes_{\sigma^\varphi} \mathbb{R}$, and since it is an eigenunitary for the dual action $(\sigma^\varphi)\widehat{\ }$, with

$$(\sigma^\varphi)\widehat{\ }_s(u_\varphi(2\pi/\log\lambda)) = \mathrm{e}^{is2\pi/\log\lambda} u_\varphi(2\pi/\log\lambda), \qquad s \in \mathbb{R},$$

and since, furthermore, the canonical unitary generator of $\mathrm{L}^\infty(\mathbb{R}/(\log\lambda)\mathbb{Z})$ satisfies the same equation, and the flow is ergodic, these two unitaries must differ by a scalar multiple. It follows immediately that $u_\varphi(2\pi/\log\lambda)$ generates the centre of the crossed product.

The unitary group $u_\varphi$ may also be identified as the canonical one-parameter unitary group $u_{\tau\otimes\varphi}$ in the crossed product $(R_{0,1} \otimes R_\lambda) \rtimes_{\sigma^{\tau\otimes\varphi}} \mathbb{R}$, since $\sigma^\tau$ is trivial and the argument above shows that $u_{\tau\otimes\varphi}(2\pi/\log\lambda)$ also generates the centre of this crossed product (which is the same as the centre of $R_\lambda \rtimes_{\sigma^\varphi} \mathbb{R}$). But the same one-parameter unitary group $u_{\tau\otimes\varphi}$ may also be viewed as the canonical one-parameter unitary group in the crossed product $R_{0,1} \rtimes_{\sigma^\tau} \mathbb{R} \subseteq (R_{0,1} \otimes R_\lambda) \rtimes_{\sigma^{\tau\otimes\varphi}} \mathbb{R}$. This reveals the centre of the larger crossed product as a subalgebra, in a way compatible with the dual action (which determines the flow of weights), of the centre $\mathrm{L}^\infty(\mathbb{R})$ of $R_{0,1} \rtimes_{\sigma^\tau} \mathbb{R}$, as desired.

Now fix $n = 2, 3, \ldots$ and consider the inclusion of $R_{\lambda^n}$ as $R_{\lambda^n} \otimes 1$ in $R_{\lambda^n} \otimes R_\lambda \cong R_\lambda$. As above, choose n.f.s. weights $\varphi$ on $R_{\lambda^n}$ and $\psi$ on $R_\lambda$ such that $\sigma^\varphi_{2\pi/\log\lambda^n}$ and $\sigma^\psi_{2\pi/\log\lambda}$ are both the identity. As above, the centre of $R_{\lambda^n} \rtimes_{\lambda^\varphi} \mathbb{R}$ is generated by $u_\varphi(2\pi/\log\lambda^n)$, and (as $R_{\lambda^n} \otimes R_\lambda \cong R_\lambda$), the centre of $(R_{\lambda^n} \otimes R_\lambda) \rtimes_{\sigma^{\varphi\otimes\psi}} \mathbb{R}$ is generated by $u_{\varphi\otimes\psi}(2\pi/\log\lambda)$. Somewhat as in the previous case (but just a little more subtly), let us consider the natural embedding of $R_{\lambda^n} \rtimes_{\sigma^\varphi} \mathbb{R}$ in $(R_{\lambda^n} \otimes R_\lambda) \rtimes_{\sigma^{\varphi\otimes\psi}} \mathbb{R}$ obtained by identifying $u_\varphi$ with $u_{\varphi\otimes\psi}$ (or just making this assignment). Since $u_{\sigma^{\varphi\otimes\psi}}(2\pi/\log\lambda)$ generates the centre of $(R_{\lambda^n} \otimes R_\lambda) \rtimes_{\sigma^{\varphi\otimes\psi}} \mathbb{R}$, and $u_\varphi(2\pi/\log\lambda)$ is contained in (in fact, generates) the centre of $R_{\lambda^n} \rtimes_{\sigma^\varphi} \mathbb{R} = (R_{\lambda^n} \otimes 1) \rtimes_{\sigma^{\varphi\otimes\psi}} \mathbb{R}$, and $u_\varphi$ is identified with $u_{\varphi\otimes\psi}$, so that

$$u_{\varphi\otimes\psi}(2\pi/\log\lambda) = u_\varphi(2\pi/\log\lambda) = (u_\varphi(2\pi/\log\lambda^n))^n,$$

we see that $u_{\varphi\otimes\psi}(2\pi/\log\lambda)$ is contained in the centre of $R_{\lambda^n} \rtimes_{\sigma^\varphi} \mathbb{R}$, whence the whole centre of $(R_{\lambda^n} \otimes R_\lambda) \rtimes_{\sigma^{\varphi\otimes\psi}} \mathbb{R}$ is contained—contravariantly—in the centre of $R_\lambda$, as desired.

The last case, $R_\lambda$ mapping into $R_\lambda \otimes R_1 \cong R_1$, is trivial, since the flow of weights on $R_1$ is trivial, and so is included (as an algebra, $\mathbb{C}$) in any flow.

The inclusions constructed, exhausting maps between transitive flows (contravariantly) are tempered because in fact of the existence of a faithful normal projection of norm one onto the subalgebra (e.g. a Tomiyama slice map—the tensor product of the identity map on the first component in a tensor product of two von Neumann algebras and a faithful normal state on the second component—see [19], [18], and Corollary 2.10(2) of [8].)

The construction may be viewed as natural, if not strictly speaking functorial, as, in the progression from $R_{\lambda^{m+n}}$ to $R_{\lambda^{m+n}} \otimes R_{\lambda^n} \cong R_{\lambda^n}$ to $(R_{\lambda^{m+n}} \otimes R_{\lambda^n}) \otimes R_\lambda \cong R_\lambda$ the composed embedding $R_{\lambda^{m+n}} \to (R_{\lambda^{m+n}} \otimes R_{\lambda^n}) \otimes R_\lambda = R_{\lambda^{m+n}} \otimes (R_{\lambda^n} \otimes R_\lambda)$ is (since $R_{\lambda^n} \otimes R_\lambda \cong R_\lambda$) essentially the single embedding $R_{\lambda^{m+n}} \to R_{\lambda^{m+n}} \otimes R_\lambda$ of our construction. (The same for the composed embedding $R_{0,1} \to (R_{0,1} \otimes R_{\lambda^n}) \otimes R_\lambda = R_{0,1} \otimes (R_{\lambda^n} \otimes R_\lambda)$.)

**7.** It would be desirable to extend the result of [5] and [12] that an automorphism of an amenable von Neumann algebra with separable predual acts trivially on the flow of weights if ([12]) and only if ([5]) it is approximately inner. A related question is the continuity of the functors considered above—the non-commutative flow of weights for tempered homomorphisms (Theorem 2), both covariant and contravariant flows of weights on the appropriate proper subcategories of tempered homomorphisms (Section 1 and (proof of) Theorem 2), and the one-sided inverses to these (Theorem 3, Theorem 4, Theorem 5, and Theorem 6).

In the context of the extension of the construction of [16] described in Theorem 5, above, it would seem to be an interesting question whether, in the inclusion $N \subseteq M$ constructed, the relative commutant of $N$ must be the centre of $M$. More generally, this question can be formulated for any tempered inclusion $N \subseteq M$ for which the associated inclusion of non-commutative flows maps the centre of the smaller one (covariantly—see Section 1 and proof of Theorem 2) into the centre of the larger—is this covariance equivalent to the condition $N' \cap M =$ Centre $M$?

Note that this condition on the tempered inclusion $N \subseteq M$ (i.e., $N' \cap M =$ Centre $M$) is equivalent to uniqueness of the n.f.s. operator valued weight from $M$ to $N$ (up to a central multiple—a scalar multiple in the case that $M$ is a factor).

Indeed, the relative Radon-Nikodym derivative of Definition 6.2(2) of [9] between two such operator valued weights, which belongs to $N' \cap M$, and so in the case $N' \cap M =$ Centre $M$ belongs to the centre of $M$, must be a one-parameter unitary group—as the modular automorphism group of an operator valued weight of Definition 6.2(1) of [9] acts trivially on the center of $M$, considered as a subalgebra of the relative commutant of $N$ in $M$ ($N' \cap M$) (it is the restriction to $N' \cap M$ of the modular automorphism group of a n.f.s. weight on $M$) and the Radon-Nikodym derivative of a second such weight with respect to the first is a unitary cycle with respect to this group. This is what it means for the second operator valued weight to be a central multiple of the first.

Conversely, if $T$ is a n.f.s. operator valued weight from $M$ to $N$ and $k$ is a self-adjoint element of $N' \cap M$ which is not in the centre of $M$, consider the perturbation $\psi^k$ constructed by Araki in [2] of the n.f.s. weight $\psi = \varphi \cdot T$ on $M$ where $\varphi$ is a fixed n.f.s. weight on $N$. The Radon-Nikodym derivative $(D\psi^k : D\psi)$ of $\psi^k$ with respect to $\psi$ is equal to what Araki referred to in [1] as the expansional of $k$ with respect to $\sigma^\psi$. Since $\sigma^\psi = \sigma^{\varphi \cdot T}$ leaves $N$ invariant (as by Theorem 4.7 of [8], $\sigma^{\varphi \cdot T}$ agrees with $\sigma^\varphi$ on $N$) it also leaves $N' \cap M$ invariant, and since $k \in N' \cap M$, also the expansional of $k$, which (see Section 2 of [1] and (1.8) of [2]) is defined for a strongly continuous one-parameter automorphism group of a von Neumann algebra, belongs to $N' \cap M$. Thus, the automorphism group $\sigma^{\psi^k} = \mathrm{Ad}(D\psi^k : D\varphi \cdot T) \cdot \sigma^{\varphi \cdot T}$ agrees with $\sigma^{\varphi \cdot T}$ on $N$, and so by Theorem 5.1 of [9] there exists a (unique) n.f.s. operator valued weight $T^k$ from $M$ to $N$ such that $\psi = \varphi \cdot T^k$.

The Radon-Nikodym derivative $(DT^k : DT) = (D\varphi \cdot T^k : D\varphi \cdot T)$, belonging to $N' \cap M$, is a unitary cocycle with respect to $\sigma^{\varphi \cdot T} | N' \cap M =: \sigma^T$, and what is meant by saying that $T^k$ is not a central multiple of $T$ is that the unitary cocycle $(DT^k : DT)$ is not just a one-parameter unitary group in the centre of $M$. But if it is not fixed by $\sigma^T$, then it is not a one-parameter group at all, and if fixed, again inspection of the construction of $(D\psi^k : D\varphi \cdot T)$ in [2] (the equation (1.8) of [2]) shows that it is the group $\mathrm{e}^{itk}$, which is not in the centre of $M$ since $k$ is not.

With regard to Theorem 6, it would be desirable to locate more general contravariant (reverse) inclusions of the flows of weights for tempered inclusions of von Neumann algebras—other than just for the transitive flows. For instance, it might be interesting to consider the case of a Kronecker foliation mapped onto another one by an endomorphism of the torus, $\mathbb{T}^2$ (for instance, squaring both generators).

For that matter, although Theorem 5 includes many homomorphisms between flows, including arbitrary homomorphisms between flows which have invariant n.f.s. weights (i.e., traces, or measures), it would be desirable to identify more fully the exact subcategory of flows and homomorphisms that arise covariantly from the non-commutative flow of weights functor for tempered inclusions, as established in Theorem 2.

Concerning flows with an invariant n.f.s. weight (measure), it is perhaps worth pointing out that it is not necessary to assume that, in an inclusion of such flows, there is any relation between given such weights on the codomain and domain flow algebras—the restriction of the former need not be assumed to be semifinite. An example of this is the flow by translation on $\mathrm{L}^\infty(\mathbb{R})$, which includes any periodic flow, but the restriction of Lebesgue measure on $\mathrm{L}^\infty(\mathbb{R})$ to $\mathrm{L}^\infty(\mathbb{R}/(\log\lambda)\mathbb{Z})$ is not semifinite. Note that this inclusion arises both covariantly (Theorem 5) and contravariantly (Theorem 6).

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