

FREE PROBABILITY AND COMPLEX COBORDISM

ROLAND FRIEDRICH AND JOHN MCKAY, FRSC

ABSTRACT. We show how Voiculescu’s S -transform [15] in free probability relates to the space of complex genera in complex cobordism, and endows it with the structure of an infinite Lie group, with the underlying manifold being isomorphic to the group of automorphisms of the formal disc. In addition, this connection gives a link with C^* -algebras. The group operation introduced can be understood in terms of non-crossing partitions [13], [14]. We then connect both via the Landweber–Novikov algebra with the Faà di Bruno Hopf algebra and the combinatorial structure of rooted trees. Finally, we provide a bridge between free probability, conformal field theory and the KP-hierarchy.

RÉSUMÉ. Nous montrons comment la transformée S de Voiculescu [15] de la théorie des probabilités libres rapporte à l’espace de genres complexes dans la théorie de cobordisme complexe, et lui confère une structure de groupe de Lie infini, avec la variété sous-jacente isomorphe au groupe des automorphismes du disque formel; plus loin, cette connexion permet une liaison avec les C^* -algèbres. L’opération de groupe introduite est comprise en termes de partitions non-franchissant [13], [14]. L’algèbre de Landweber–Novikov relie ces deux choses à l’algèbre de Hopf de Faà di Bruno et la combinatoire des arbres enracinés. Enfin, nous fournissons une passerelle entre la théorie des probabilités libres, la théorie conforme des champs et l’hierarchie KP.

1. Introduction. A profound connection between conformal field theory and complex cobordism has been previously established in [11], [12]; a relation between the positive part of the Witt algebra and the Landweber–Novikov algebra was found in [2].

In this note, we extend this picture by showing that Voiculescu’s S -transform from non-commutative probability theory [15] is intrinsically related to complex genera [9], [10]. This bridge yields a rich correspondence between previously seemingly disconnected mathematical areas.

Further, we add yet another direction of recent mathematical development to this picture, namely the combinatorial Hopf algebra structure of rooted trees [6], [3], as underlying renormalisation [3]. As an example, we show how the Faà di Bruno Hopf algebra [5], [6], and in particular the Connes–Moscovici algebra [4], are related to the Landweber–Novikov algebra [1], [2].

Received by the editors on August 13, 2011.

AMS Subject Classification: Primary: 46L54; secondaries: 55N22, 57R77, 81T40.

Keywords: free probability and free operator algebras, bordism and cobordism theories, formal group laws, complex cobordism, conformal field theory.

© Royal Society of Canada 2011.

We begin by introducing some spaces of formal power series. Let $\mathcal{O} := \mathbb{C}[[z]] = \sum_{n=0}^{\infty} a_n z^n$ denote the ring of formal complex power series in an indeterminate z . It is a local ring with unique maximal ideal $\mathfrak{m} = z\mathbb{C}[[z]]$, which also induces the \mathfrak{m} -adic topology. Let $\text{Aut}(\mathcal{O}) := \{a_1 z + a_2 z^2 + \dots \mid a_1 \neq 0\}$ be the infinite Lie group of automorphisms of $\mathbb{C}[[z]]$, considered as a \mathbb{C} -algebra with the group operation ‘ \circ ’ given by composition of power series. These automorphisms can be thought of as the automorphisms of the formal complex line that fix the origin. $\text{Aut}_+(\mathcal{O})$ is then the Lie subgroup of all automorphisms of the form $z + \sum_{n=2}^{\infty} a_n z^n$.

Let $\Lambda(\mathbb{C}) := 1 + z\mathbb{C}[[z]]$ be the sub-ring of $\mathbb{C}[[z]]$ with respect to multiplication of power series, ‘ \cdot ’, *i.e.*, the set of all power series with constant term equal to 1.

Finally, $\mathbb{C}^* \rtimes \Lambda(\mathbb{C})$ is isomorphic to $\mathbb{C}[[z]]^\times = \{a_0 + \sum_{n=1}^{\infty} a_n z^n \mid a_0 \neq 0\}$, *i.e.*, the multiplicative subgroup of invertible elements of $\mathbb{C}[[z]]$.

2. Free probability. Let us recall some definitions and facts from free probability [15], [13], [14].

A pair (A, ϕ) , is called a *non-commutative probability space*, if it consists of a unital \mathbb{C} -algebra A and a linear functional $\phi: A \rightarrow \mathbb{C}$ such that $\phi(1_A) = 1$.

The elements $a \in A$ are viewed as the *random variables* and every $a \in A$ defines a *distribution* or *law*, μ_a . Namely, let $\mu_a: \mathbb{C}[X] \rightarrow \mathbb{C}$ be the linear functional which on the infinite \mathbb{C} -basis $\{X^n\}_{n \in \mathbb{N}}$ of $\mathbb{C}[X]$ is given by $\mu_a(1) = 1$ and $\mu_a(X^n) := \phi(a^n)$.

Consider the following sets of linear functionals: $\Sigma := \{\mu: \mathbb{C}[X] \rightarrow \mathbb{C} \mid \mu \text{ is linear, } \mu(1) = 1\}$, $\Sigma^\times := \{\mu: \mathbb{C}[X] \rightarrow \mathbb{C} \mid \mu \text{ is linear, } \mu(1) = 1, \mu(X) \neq 0\}$ and $\Sigma_1 := \{\mu: \mathbb{C}[X] \rightarrow \mathbb{C} \mid \mu \text{ is linear, } \mu(1) = 1, \mu(X) = 1\}$. They satisfy the strict inclusions: $\Sigma_1 \subset \Sigma^\times \subset \Sigma$.

Let $\mu, \nu \in \Sigma$ and let a, b be *free random variables* in a non-commutative probability space, *e.g.*, a C^* -algebra, having μ and ν as distributions, respectively. Then the *multiplicatively free convolution* of μ and ν is defined as the distribution of ab and it is denoted by $\mu \boxtimes \nu$. It is important to note that \boxtimes is a commutative operation, *cf.* [15].

The *S-transform* defines a bijection $S: \Sigma^\times \rightarrow \mathbb{C}[[z]]^\times$, $\mu \mapsto S_\mu$, which implies that $(\Sigma^\times, \boxtimes)$ is an abelian group, in fact an infinite-dimensional Lie group [15].

If a and b are free random variables in a non-commutative probability space (A, ϕ) , with non-vanishing first moments, *i.e.*, $\phi(a), \phi(b) \neq 0$, and with corresponding distributions μ_{ab} , μ_a and μ_b respectively, then the following holds [15]:

$$S_{\mu_{ab}}(z) = S_{\mu_a}(z) \cdot S_{\mu_b}(z).$$

According to [13], [14], let $f(z) \in \text{Aut}(\mathcal{O})$, $n \geq 1$, and define the operator $[\text{coef}(n)](f) := \frac{1}{n!} \frac{d^n f(z)}{dz^n} \Big|_{z=0}$, which returns the coefficient of z^n of f . Furthermore, let $\pi = \{B_1, \dots, B_r\}$ denote a partition of $\{1, \dots, n\}$, where the sets B_1, \dots, B_r are called the blocks of π , and write $|B_k|$ for the cardinality of the set B_k . Now, for $\pi = \{B_1, \dots, B_r\} \in \text{NC}(n)$, with $\text{NC}(n)$ symbolising the set of

all *non-crossing* partitions of $\{1, \dots, n\}$, define

$$[\text{coef}(n); \pi](f) := \prod_{i=1}^r [\text{coef}(|B_i|)](f).$$

Then the *combinatorial convolution* \star of power series, as introduced by Speicher and Nica [13], [14], is constructed as follows.

For $f, g \in \text{Aut}(\mathcal{O})$ and $n \geq 1$, define the \star -convolution coefficient-wise by

$$(1) \quad \frac{1}{n!} \frac{d^n}{dz^n} (f \star g) \Big|_{z=0} := \sum_{\pi \in \text{NC}(n)} [\text{coef}(n); \pi](f) \cdot [\text{coef}(n); K(\pi)](g),$$

where $K(\pi)$ denotes the *Kreweras complement* of a non-crossing partition $\pi \in \text{NC}(n)$. The above prescription (1) for \star introduces an associative and commutative group structure on $\text{Aut}(\mathcal{O})$ with the neutral element given by the series $\text{Id}(z) = z$.

REMARK. It is appropriate to think of the binary operation \star as a product on Witt vectors, *cf.* [7].

For $f \in \text{Aut}(\mathcal{O})$ let $f^{-1}(z)$ denote the compositional inverse of f , *i.e.*, $(f^{-1} \circ f)(z) = z$. The \mathcal{F} -transform (‘Fourier’) [13], [14] is defined as

$$(2) \quad \mathcal{F}(f)(z) := \frac{f^{-1}(z)}{z},$$

and it gives a map $\mathcal{F}: \text{Aut}(\mathcal{O}) \rightarrow \mathbb{C}[[z]]^\times$.

THEOREM 2.1 ([13], [14]). *The \mathcal{F} -transform, as defined in (2), is a group isomorphism between the infinite Lie group $(\text{Aut}(\mathcal{O}), \star)$ and $(\mathbb{C}[[z]]^\times, \cdot)$.*

It follows from (2) and Theorem 2.1 that as sets

$$\text{Aut}_+(\mathcal{O}) = \mathcal{F}^{-1}(\Lambda(\mathbb{C})),$$

and that $\text{Aut}_+(\mathcal{O})$ is also a Lie subgroup of $\text{Aut}(\mathcal{O})$ with respect to the binary operation given by the combinatorial convolution \star .

Let $\mu = \mu_a$ be a distribution of a free random variable a in some non-commutative probability space (A, ϕ) , and let $(k_n(\mu))_{n \in \mathbb{N}}$ denote the numerical sequence of *free cumulants* associated with μ_a , *cf.* [15]. We require that $\mu(X) = k_1 \neq 0$. Define the formal power series

$$(3) \quad \Phi_\mu(z) := \sum_{n=1}^{\infty} k_n(\mu) z^n \in \text{Aut}(\mathcal{O}).$$

Then the S -transform can be expressed as [13], [14]:

$$S_\mu(z) = \frac{\Phi_\mu^{-1}(z)}{z} \in \mathbb{C}[[z]]^\times,$$

where $\Phi_\mu^{-1}(z)$ stands for the compositional inverse of Φ_μ .

The relation between the analytical and the combinatorial description of the S -transform can be summarised as

$$(4) \quad \begin{array}{ccc} (\Sigma^\times, \boxtimes) & & (\text{Aut}(\mathcal{O}), \star) \\ & \searrow S & \swarrow \mathcal{F} \\ & (\mathbb{C}[[z]]^\times, \cdot) & \end{array}$$

where S and \mathcal{F} are group isomorphisms [13], [14], [15].

3. Multiplicative sequences and genera. For simplicity, here we restrict our considerations to the field of complex numbers, instead of a more general \mathbb{Q} -algebra R . Let us recall some facts [1], [9], [10].

As in [9], let $Q(z)$ be an arbitrary (formal) power series in $\Lambda(\mathbb{C})$. Then there is an associated *multiplicative sequence* $\{K_j\}_{j \in \mathbb{N}}$ with $K(1+z) = Q(z)$. Conversely, any multiplicative sequence $\{K_j\}_{j \in \mathbb{N}}$ is completely determined by its characteristic power series $Q(z) = K(1+z)$.

PROPOSITION 3.1 ([9]). *The ring $\Lambda(\mathbb{C})$ is in one-to-one correspondence with the set of multiplicative sequences $\{K_j\}_{j \in \mathbb{N}}$.*

Let MU^* denote the complex cobordism ring. Then a complex *genus* φ is a unital ring homomorphism $\varphi: MU^* \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{C}$, i.e., with $\varphi(1) = 1$.

For every $Q(z) \in \Lambda(\mathbb{C})$ one can define a *genus* φ_Q corresponding to the power series $Q(z)$, [9], [10]. In this relation, $Q(z)$ is called the *characteristic power series of φ* .

Let $Q(z) \in \Lambda(\mathbb{C})$ and set $f(z) := z/Q(z)$ which gives an element in $\text{Aut}_+(\mathcal{O})$, i.e., a power series starting with z . Let $f^{-1}(z)$ be the (formal) inverse with respect to composition. Then f^{-1} is called the *logarithm of the genus φ_Q* and it has the following representation [1], [10]:

$$(5) \quad \ell_Q(z) := \text{Log}_Q(z) := f^{-1}(z) = \sum_{n=1}^{\infty} \frac{\varphi_Q(\mathbb{C}\mathbb{P}^{n-1})}{n} z^n \in \text{Aut}_+(\mathbb{C}),$$

where $\mathbb{C}\mathbb{P}^n$ is complex projective space of complex dimension n , with the convention that $\mathbb{C}\mathbb{P}^0 = 1$.

PROPOSITION 3.2 ([9], [10]). *There is a bijection between the set \mathfrak{G}_c of complex genera φ and the power series $Q(z) \in \Lambda(\mathbb{C})$.*

Let us consider the multiplicative subgroup $\Lambda(\mathbb{C}) \subset \mathbb{C}[[z]]^\times$. Then we have the following.

THEOREM 3.3. *Let $\mu \in \Sigma_1$ be the distribution of a non-commutative random variable a , and let $\varphi \in \mathfrak{G}_c$ be a complex genus. Then the associated power series of free cumulants $\Phi_\mu(z)$, cf. (3), is equal to the logarithm of the genus $\ell_\varphi(z)$, cf. (5), if and only if the S -transform $S_\mu(z)$ and the characteristic power series $Q_\varphi(z)$ are multiplicatively inverse to each other, i.e.,*

$$(6) \quad S_\mu(z) = Q_\varphi(z)^{-1}.$$

PROOF. If $\Phi_\mu(z) = \ell_\varphi(z)$ for μ and φ satisfying the assumptions above, then for

$$S_\mu(z) = \frac{\Phi_\mu^{-1}(z)}{z} \quad \text{and} \quad Q_\varphi(z) = \frac{z}{\ell_\varphi^{-1}(z)}$$

we have

$$S_\mu(z) \cdot Q_\varphi(z) = 1. \quad \square$$

The above gives a direct relation between complex genera φ and distributions μ coming from C^* -algebras. Further, by the Gelfand–Naimark–Segal theorem, one obtains concrete realisations as C^* -algebras of bounded operators on Hilbert spaces.

The relation (6) is noteworthy, cf., e.g., [8].

COROLLARY 3.4. *If $S_\mu(z)$ and $Q_\varphi(z)$ are multiplicatively inverse to each other for a given distribution μ and a genus φ , then they satisfy, up to a sign in the definition, the relationship between the total Segre and Chern classes.*

COROLLARY 3.5. *The set of complex genera \mathfrak{G}_c can be endowed with a combinatorial group law \star_H , induced by non-crossing partitions, and an isomorphism \mathcal{H} , such that the following diagram commutes:*

$$(7) \quad \begin{array}{ccc} (\mathfrak{G}_c, \star_H) & \xrightarrow{\text{“Segre-Chern”}} & (\Sigma_1, \boxtimes) \\ & \searrow \mathcal{H} & \swarrow \mathcal{F} \\ & (\Lambda(\mathbb{C}), \cdot) & \end{array}$$

4. The Hopf algebra of rooted trees and the KP-hierarchy. Denote the Lie algebra of $\text{Aut}_+(\mathcal{O})$ by $\text{Der}_+(\mathcal{O}) = z^2\mathbb{C}[[z]]\partial_z$ and let $\mathfrak{U}\text{Der}_+(\mathcal{O})$ be the corresponding enveloping algebra. We note that $\text{Der}_+(\mathcal{O})$ corresponds to the positive part of the Witt algebra, as represented, e.g., by the vector fields $l_n := -z^{n+1}\partial_z$. The following is known.

THEOREM 4.1 ([2]). *Let LN be the Landweber–Novikov algebra. Then $\text{LN} \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the enveloping algebra of $\text{Der}_+(\mathcal{O})$.*

For $n \geq 1$, the operators $\frac{1}{n!} \frac{d^n}{dz^n}$ define coordinate functions on the Lie group $\text{Aut}(\mathcal{O})$, cf. [1], [2], [5], and [6]. For $f \in \text{Aut}(\mathcal{O})$ let

$$a_n(f) := \frac{1}{n!} \frac{d^n f(z)}{dz^n} \Big|_{z=0}.$$

The Leibniz formula defines a coproduct Δ on the polynomial algebra $\mathbb{C}[a_1, a_2, \dots]$, cf. [1], [2]. By restricting to the subgroup $\text{Aut}_+(\mathcal{O})$, one obtains a graded and connected Hopf algebra F , which is dual to $\text{Aut}_+(\mathcal{O})$, and which is usually called the *Faà di Bruno* Hopf algebra, cf. [5], [6]. This Hopf algebra corresponds to the maximal commutative, but not co-commutative, Hopf sub-algebra of the *Connes–Moscovici* Hopf algebra [4], which has a description in terms of rooted trees, cf. [3], [6].

One should note that the Faà di Bruno Hopf algebra has been discussed earlier in algebraic topology without this name, cf., e.g., [1], [2]. The graded dual Hopf algebra F' has a basis $\{a'_n\}_{n \geq 2}$ for its primitive elements. Define $b'_n := (n + 1)! a'_{n+1}$ for $n \geq 1$. Then the b'_n satisfy the following commutation relations [4], [5], [6]

$$(8) \quad [b'_n, b'_m] = (m - n)b'_{n+m},$$

which are exactly those of the positive part of the Witt algebra.

As a consequence of the Milnor–Moore theorem, F' is isomorphic to the enveloping algebra of the Lie algebra spanned by the b'_n with the commutators given in (8). Hence, we have the following.

THEOREM 4.2. *Let LN be the Landweber–Novikov algebra. Then $\text{LN} \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to F' , the graded dual of the Faà di Bruno Hopf algebra F . Therefore a combinatorial description in terms of rooted trees holds.*

Free probability is fundamentally related to the full Fock space, cf. [15], [13]. The following derivation of a link with conformal field theory (CFT) and the KP-hierarchy not only provides us with an even deeper understanding of this fact but it also closes the circle of ideas presented in this note.

A connection between complex cobordism and Fock spaces in CFT was established in [11], [12]. Further, in [11] it was proved that a bijection can be established between the set of multiplicative sequences over \mathbb{C} and the subset $1 + \partial_x^{-1} \mathbb{C}[[\partial_x^{-1}]]$, where ∂^{-1} is a pseudo-differential operator, of all *wave operators* with constant coefficients in the KP-hierarchy.

Now, the diagrams (4) and (7) together with [11] provide us with the next theorem.

THEOREM 4.3. *There is a bijection between the infinite Lie group (Σ_1, \boxtimes) in non-commutative probability and all the wave operators with constant complex coefficients in the KP-hierarchy.*

ACKNOWLEDGEMENTS. The second author gratefully acknowledges the hospitality of the Max Planck Institute for Mathematics (MPIfM) in Bonn. Both authors thank the MPIfM for the opportunity to collaborate on this project.

REFERENCES

1. J. F. Adams, *Stable Homotopy and Generalised Homology*. Chicago Lectures in Mathematics, University of Chicago Press, Chicago–London, 1974.
2. V. Buhstaber and A. Shokurov, *The Landweber–Novikov algebra and formal vector fields on the line*. Funktsional. Anal. i Prilozhen **12** (1978), 1–11; English translation: Functional Anal. Appl. **12** (1978), no. 3, 159–168 (1979).
3. A. Connes and D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*. Comm. Math. Phys. **199** (1998), 203–242.
4. A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*. Comm. Math. Phys. **198** (1998), 198–246.
5. H. Figueroa, J. Gracia-Bondía and J. Várilly, *Faà di Bruno Hopf algebras*. arxiv:math/0508337, 2005.
6. L. Foissy, *Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson–Schwinger equations*. Adv. Math. **218** (2008), 136–162.
7. R. Friedrich and J. McKay, in preparation, 2011.
8. W. Fulton, *Intersection Theory*. Second edition, Springer-Verlag, Berlin, 1998.
9. F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*. Ergebnisse der Mathematik, Springer-Verlag, Berlin, 1956.
10. F. Hirzebruch, Th. Berger and R. Jung, *Manifolds and Modular Forms*. In: Aspects of Mathematics **E20**, Friedr. Vieweg & Sohn, Braunschweig, 1992.
11. T. Katsura, Y. Shimizu and K. Ueno, *Complex cobordism ring and conformal field theory over \mathbf{Z}* . Math. Ann. **291** (1991), 551–571.
12. J. Morava, *On the complex cobordism as a Fock representation*. Lecture Notes in Math. **1418**, Springer, Berlin, 1990, 184–204.
13. A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*. London Mathematical Society Lecture Note Series **335**, Cambridge University Press, Cambridge, 2006.
14. R. Speicher, *Free probability theory and non-crossing partitions*. Sémin. Lothar. Combin. **B39c** (1997), 38pp (electronic).
15. D. Voiculescu, *Multiplication of certain non-commuting random variables*. J. Operator Theory **18** (1987), 223–235.

BMS, Humboldt University Berlin, Unter den Linden 6, D-10099 Berlin, Germany
e-mail: rolandf@mathematik.hu-berlin.de

Dept. Mathematics, Concordia University, Montreal, Canada H3G 1M8
e-mail: mac@mathstat.concordia.ca