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THE SURPRISING POWER OF AVERAGING OVER GROUPS

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ABSTRACT. We highlight the surprising power of averaging via a few illuminating examples. Two of these problems involve characterizations of Hilbert space, and the third is a fundamental result in noncommutative geometry.

RÉSUMÉ. Nous soulignons le pouvoir surprenant de la moyenne par quelques exemples éclairants. Deux de ces problèmes concernent la caractérisation de l'espace de Hilbert, et le troisième est un résultat fondamental en géométrie non commutative.

1. **Introduction** This article is one on techniques, as opposed to results. More precisely, we showcase how the elementary concept of *averaging*, when applied cleverly, can be surprisingly powerful in a wide range of contexts. In general, one begins with some object of consideration and averages this object over a carefully chosen compact group G. The resulting average will be G-invariant in a suitable sense, often yielding useful information about the object of interest.

2. **A Motivating Example** As an introductory example, consider the following problem [4].

PROBLEM 1. *Let V be a two-dimensional normed real vector space with infinitely many linear isometries. Then V is a (real) Hilbert space.*

In fact, the result holds in higher dimensions as well, if we assume that for each two dimensional subspace W of V there are infinitely many linear isometries of V which send W into itself. We encourage readers to attempt this exercise themselves, before looking over the following key lemma.

LEMMA 1. *Let G denote the group of linear isometries of V. Then there exists a G-invariant inner product on V.*

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PROOF. Since V is finite dimensional, the group G is compact, so there exists a right-invariant Haar measure μ on G. Fix an arbitrary inner product $\langle \cdot, \cdot \rangle$ on V, and define the averaged inner product

$$\langle x, y \rangle_G := \int_{T \in G} \langle Tx, Ty \rangle \, \mathrm{d}\mu.$$

This yields a G-invariant inner product on V, as desired. $\square$

Since $\langle \cdot, \cdot \rangle_G$ is an inner product, we may identify its group of symmetries with the orthogonal group $O(2)$. We conclude that G is a subgroup of $O(2)$. In fact, by the compactness of G we may deduce that $G = O(2)$. Thus the norm on V is invariant under an action of $O(2)$, and must therefore be a Hilbert space norm.

Note carefully the core idea of the proof: we average a carefully chosen function f over a compact symmetry group G, obtaining a G-invariant version of f. The generalization of this idea to higher dimensions will be crucial in the following problem.

3. Higher-Dimensional Hilbert Space We will apply the averaging technique to the following characterization of Hilbert space, first considered by [1]. For a more detailed discussion of the problem, see [5].

Let V denote a finite-dimensional normed real vector space, and let K denote its closed unit ball. Then we have the following result:

PROBLEM 2 ("No-shadow" characterization of Hilbert space). *Suppose that given any two-dimensional subspace $W \subset V$, there exists a linear projection onto W that maps K into $K \cap W$. Then V is a Hilbert space if* $\dim V \geq 3$.

Although the full proof of this theorem is beyond the scope of this article, we will show how the averaging technique can be used to prove the following key lemma. This lemma was used by Busemann [2] to prove the above theorem, but his proof does not use the averaging technique.

LEMMA 2. *Let C be a compact, convex subset of $\mathbb{R}^n$ that spans $\mathbb{R}^n$, and let $G = \{T \in \mathrm{GL}_n(\mathbb{R}) : T(C) = C\}$. If for each $u, v \in \partial C$ there is a $T \in G$ such that $T(u) = v$, then C is an ellipsoid.*

PROOF. Using the fact that C is compact and spans $\mathbb{R}^n$, we can show that G is a compact group. As in Problem 1, we can fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ and define

$$\langle x, y \rangle_G := \int_{T \in G} \langle Tx, Ty \rangle \, \mathrm{d}\mu,$$

where μ is the right-invariant Haar measure on G. As above, this gives us a G-invariant inner product with an associated norm $|| \cdot ||_G$. For any $u, v \in \partial C$ we can find, by assumption, a $T \in G$ such that $Tu = v$. Since $|| \cdot ||_G$ is G-invariant, this shows that $||u||_G = ||v||_G$ and so $|| \cdot ||_G$ is constant on ∂C. Thus, C is a ball in the norm $|| \cdot ||_G$ and so C is an ellipsoid. $\square$

4. **A_θ is Simple** We now showcase an application of this technique a to a completely different flavour of problem from non-commutative geometry.

For $\theta \in \mathbb{R}$ irrational, let $A_\theta = C^*(u, v)$ denote the corresponding rotation algebra; i.e., the universal C*-algebra generated by two unitaries u, v satisfying the relation $vu = e^{2\pi i\theta}uv$. Then we have the following fundamental fact, first proved in [3]:

PROBLEM 3. *The irrational rotation algebra A_θ is simple.*

PROOF. Given any pair of unit complex numbers $(z_1, z_2) \in \mathbb{T}^2$, note that the unitaries $u' = z_1u$ and $v' = z_2v$ still satisfy the universal property of A_θ, so that $C^*(u, v) \cong C^*(z_1u, z_2v)$. Thus the endomorphism on $C^*(u, v)$ induced by sending $u \mapsto u'$, $v \mapsto v'$ is in fact an automorphism. This construction defines a continuous map $\phi : \mathbb{T}^2 \to \mathrm{Aut}(C^*(u, v))$.

Using the map ϕ, we define the averaging operator

$$\begin{aligned} \langle\cdot\rangle : A_\theta &\to A_\theta \\ a &\mapsto \int_{(z_1,z_2)\in\mathbb{T}^2} \phi(z_1, z_2)(a)\,\mathrm{d}\mu, \end{aligned}$$

where μ denotes the Haar measure on $\mathbb{T}^2$. Note that the map $a \mapsto \langle a\rangle$ is linear, continuous, and sends u^mv^n to 0 whenever at least one of $m, n \in \mathbb{Z}$ is non-zero. In particular, $\langle a\rangle$ is a (complex) scalar whenever $a \in A_\theta$ is a finite linear combination of elements of the form u^mv^n. By the universal property of A_θ, elements of this form are dense in A_θ, and so the image of $\langle\cdot\rangle$ is contained in $\mathbb{C}$ by continuity.

Now let I be a non-zero closed, two-sided *-ideal in A_θ, and consider any non-zero $a_0 \in I$. We may assume that a_0 is strictly positive by replacing it with $a_0^*a_0$ if necessary.

Observe that, for any $m, n \in \mathbb{Z}$, the inner automorphism $a \mapsto u^mv^nav^{-n}u^{-m}$ is precisely the automorphism $\phi(e^{2\pi im\theta}, e^{2\pi in\theta})$. Since θ is irrational, these automorphisms are dense in $\phi(\mathbb{T}^2)$. The ideal I and the set of strictly positive elements $A_\theta^+ \setminus \{0\}$ are preserved by inner automorphisms, and so, by continuity, they are preserved by all the automorphisms in $\phi(\mathbb{T}^2)$. Since both of these sets are convex, they are closed under $\langle\cdot\rangle$ as well.

In particular, since a_0 is a element of $I \cap (A_\theta^+ \setminus \{0\})$, the average $\langle a_0\rangle \in \mathbb{C}$ must be as well. We conclude that I contains a non-zero scalar, and so we must have $I = A_\theta$ as required. $\square$

This proof demonstrates the generality of this technique: an average over a group G is invariant under the action of G, placing heavy restrictions the properties of the average. Assuming the structure of interest is sufficiently linear, these conditions on the average often extend to the whole structure.

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