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On Geometric Preduals of Jet Spaces on Closed Subsets of $\mathbb{R}^n$

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Abstract. Let $C_b^{k,\omega}(\mathbb{R}^n)$ be the Banach space of C^k functions on $\mathbb{R}^n$ bounded together with all derivatives of order $\leq k$, where the derivatives of order k have moduli of continuity majorization by $c\,\omega$, $c \in \mathbb{R}_+$, for some $\omega \in C(\mathbb{R}_+)$. For a closed set $S \subset \mathbb{R}^n$ the jet space $J_b^{k,\omega}(S)$ is the Banach space of vector functions whose components are partial derivatives of functions in $C_b^{k,\omega}(\mathbb{R}^n)$ evaluated at points of S equipped with the corresponding quotient norm. The geometric predual $G_J^{k,\omega}(S)$ of $J_b^{k,\omega}(S)$ is the minimal closed subspace of the dual $\big(C_b^{k,\omega}(\mathbb{R}^n)\big)^*$ containing the evaluation functionals of all partial derivatives of order $\leq k$ at points in S. In the paper we study some geometric properties of spaces $G_J^{k,\omega}(S)$ related to the classical Whitney problems.

Résumé. Soit $C_b^{k,\omega}(\mathbb{R}^n)$ l'espace de Banach des fonctions C^k sur $\mathbb{R}^n$ bornées avec toutes les dérivées d'ordre k, où les dérivés d'ordre k ont des modules de continuités majorés par $c\,\omega$, $c \in \mathbb{R}_+$, pour quelques $\omega \in C(\mathbb{R}_+)$. Pour un ensemble fermé $S \subset \mathbb{R}^n$ l'espace de jet $J_b^{k,\omega}(S)$ est l'espace de Banach des fonctions vectorielles dont les composantes sont des dérivées partielles des fonctions en $C_b^{k,\omega}(\mathbb{R}^n)$ évaluées aux points de S équipés de la norme du quotient correspondante. Le prédual géométrique $G_J^{k,\omega}(S)$ de $J_b^{k,\omega}(S)$ est le sous-espace minimal fermé du dual $\big(C_b^{k,\omega}(\mathbb{R}^n)\big)^*$ contenant les fonctionnelles d'évaluation de toutes les dérivées partielles d'ordre $\leq k$ aux points de S. Dans cet article, nous étudions certaines propriétés géométriques des espaces $G_J^{k,\omega}(S)$ liées aux problèmes classiques de Whitney.

1. Introduction Let $C_b^{k,\omega}(\mathbb{R}^n)$ be the space of C^k functions on $\mathbb{R}^n$ bounded with all derivatives up to order k, where the derivatives of order k have moduli of continuity $O(\omega)$ for some $\omega : \mathbb{R}_+ \to \mathbb{R}_+$. Let $C_b^{k,\omega}(S) := C_b^{k,\omega}(\mathbb{R}^n)|_S$ be the trace space on a closed set $S \subset \mathbb{R}^n$. In [3], in connection with the Whitney problems regarding the characterization of trace spaces of C^k functions, the first author initiated a study of the Banach space $G_b^{k,\omega}(S)$, the minimal subspace of the dual space $(C_b^{k,\omega}(\mathbb{R}^n))^*$ containing all evaluation functionals at points of S (see, e.g.,

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[6] and [9] for similar spaces in the context of Lipschitz function theory). In particular, the following facts were established:

(1) The dual space $(G_b^{k,\omega}(S))^*$ is isomorphic to $C_b^{k,\omega}(S)$;
(2) $G_b^{k,\omega}(S)$ has the bounded approximation property

(see, e.g., [5] for the corresponding definition).

Due to property (1), $G_b^{k,\omega}(S)$ is called the *geometric predual* of $C_b^{k,\omega}(S)$.

Let $C_0^{k,\omega}(\mathbb{R}^n)$ be the minimal closed subspace of $C_b^{k,\omega}(\mathbb{R}^n)$ containing all C^∞ functions with compact supports and $C_0^{k,\omega}(S) := C_0^{k,\omega}(\mathbb{R}^n)|_S$ be the corresponding trace space. Suppose that S has a weak Markov property, e.g., is the closure of an open subset of $\mathbb{R}^n$ (see [2] for more examples) and $\lim_{t\to 0^+} \frac{t}{\omega(t)} = 0$, then

(3) $G_b^{k,\omega}(S)$ is isomorphic to $(C_0^{k,\omega}(S))^*$ and consequently has the metric approximation property.

Properties (1) and (3) imply the *two stars theorem* for weak Markov sets S asserting that the second dual space $(C_0^{k,\omega}(S))^{**}$ is isomorphic to $C_b^{k,\omega}(S)$.

In the present paper we continue this line of research and obtain results analogous to (1)–(3) for geometric preduals $G_J^{k,\omega}(S) \subset G_b^{k,\omega}(\mathbb{R}^n)$ of jet spaces $J_b^{k,\omega}(S)$ related to $C_b^{k,\omega}(S)$.

2. Basic Definitions

We use the standard notations of differential analysis. In particular, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ denotes a multi-index and $|\alpha| := \sum_{i=1}^n \alpha_i$. Also for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$x^\alpha := \prod_{i=1}^n x_i^{\alpha_i} \quad \text{and} \quad D^\alpha := \prod_{i=1}^n D_i^{\alpha_i}, \text{ where } D_i := \frac{\partial}{\partial x_i}.$$

Let ω be a nonnegative function on $(0, \infty)$ (referred to as *modulus of continuity*) satisfying the following conditions

(i) $\omega(t)$ and $t/\omega(t)$ are nondecreasing functions on $(0, \infty)$;
(ii) $\lim_{t\to 0^+} \omega(t) = 0$.

Definition 2.1. $C_b^{k,\omega}(\mathbb{R}^n)$ is the Banach space of functions $f \in C^k(\mathbb{R}^n)$ with norm

$$\|f\|_{C_b^{k,\omega}(\mathbb{R}^n)} := \max\left(\|f\|_{C_b^k(\mathbb{R}^n)}, |f|_{C_b^{k,\omega}(\mathbb{R}^n)}\right),$$

where

$$\|f\|_{C_b^k(\mathbb{R}^n)} := \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha f(x)|$$

and

$$|f|_{C_b^{k,\omega}(\mathbb{R}^n)} := \max_{|\beta|=k} \sup_{x\ne y\in\mathbb{R}^n} \frac{|D^\beta f(x) - D^\beta f(y)|}{\omega(\|x-y\|)}.$$

Here $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

DEFINITION 2.2. For a closed subset $S \subset \mathbb{R}^n$, $C_b^{k,\omega}(S)$ denotes the trace space of restrictions to S of functions in $C_b^{k,\omega}(\mathbb{R}^n)$ equipped with the trace norm, i.e., $\|f\|_{C_b^{k,\omega}(S)} := \inf\{\|g\|_{C_b^{k,\omega}(\mathbb{R}^n)} : g \in C_b^{k,\omega}(\mathbb{R}^n),\ g|_S = f\}$.

The jet space $J_b^{k,\omega}(S)$ consists of vector functions $\vec{f} = (f_\alpha)_{|\alpha|\le k}$ on S such that $(D^\alpha g)|_S = f_\alpha$ for some $g \in C_b^{k,\omega}(\mathbb{R}^n)$ and all α, equipped with the norm $\|\vec{f}\|_{J_b^{k,\omega}(S)}$ that equals the infimum of norms $\|g\|_{C_b^{k,\omega}(\mathbb{R}^n)}$ over the set of all such g.

It is readily seen that both $(C_b^{k,\omega}(S), \|\cdot\|_{C_b^{k,\omega}(S)})$ and $J_b^{k,\omega}(S)$, equipped with the natural vector space structure and this norm, are Banach spaces.

The classical Whitney-Glaeser theorem (see, e.g., [4]) gives necessary and sufficient conditions for a vector function $\vec{f} = (f_\alpha)_{|\alpha|\le k}$ on S to lie in $J_b^{k,\omega}(S)$. Specifically, one associates with $\vec{f}$ the family $\{T_x^k \vec{f}\}_{x\in S}$ of polynomials on $\mathbb{R}^n$ by the following formula

$$T_x^k \vec{f}(z) := \sum_{\alpha\in\mathbb{Z}_+^n\,:\,|\alpha|\le k} f_\alpha(x)\,\frac{(z-x)^\alpha}{\alpha!}, \qquad x \in S,\ z \in \mathbb{R}^n. \tag{2.1}$$

THEOREM. *(Whitney-Glaeser) Vector function $\vec{f}$ lies in $J_b^{k,\omega}(S)$ iff there exists a constant C such that for all $\alpha \in \mathbb{Z}_+$, $|\alpha| \le k$, and all $x, y \in S$, $z \in \{x, y\}$*

$$|f_\alpha(x)| \le C \quad \text{and} \quad |D^\alpha(T_x^k \vec{f} - T_y^k \vec{f})(z)| \le C\|x-y\|^{k-|\alpha|}\omega(\|x-y\|). \tag{2.2}$$

The smallest such C determines a norm $\|\vec{f}\|'_{J_b^{k,\omega}(S)}$ equivalent to $\|\vec{f}\|_{J_b^{k,\omega}(S)}$ with the constants of equivalence depending on k and n only.

3. Geometric Predual of a Jet Space

3.1. Definition Let us consider the map $D_S : C_b^{k,\omega}(\mathbb{R}^n) \to J_b^{k,\omega}(S)$,

$$D_S(f)(s) := ((D^\alpha f)(s)), \quad s \in S,\ f \in C_b^{k,\omega}(\mathbb{R}^n). \tag{3.1}$$

According to Definition 2.2, D_S is a linear surjection of norm one such that the adjoint map $D_S^* : (J_b^{k,\omega}(S))^* \to (C_b^{k,\omega}(\mathbb{R}^n))^*$ is an isometry.

Let $\delta_{(s,\alpha)} \in (J_b^{k,\omega}(S))^*$ be the evaluation functional at the point (s, α), $s \in S$, $|\alpha| \le k$, i.e.,

$$\delta_{(s,\alpha)}(\vec{f}) := f_\alpha(s).$$

By $G_J^{k,\omega}(S) \subset (J_b^{k,\omega}(S))^*$ we denote the minimal closed subspace containing all functionals $\delta_{(s,\alpha)}$, $s \in S$, $|\alpha| \le k$, equipped with the induced norm. Consider a bounded linear map $I_S : J_b^{k,\omega}(S) \to (G_J^{k,\omega}(S))^*$,

$$I_S(\vec{f})(v) := v(\vec{f}), \quad v \in G_J^{k,\omega}(S). \tag{3.2}$$

Proposition 3.1. *The map I_S is an isometric isomorphism between $J_b^{k,\omega}(S)$ and $(G_J^{k,\omega}(S))^*$.*

In what follows, we call $G_J^{k,\omega}(S)$ the *geometric predual* of $J_b^{k,\omega}(S)$. Such spaces naturally appear in the context of Whitney extension problems for smooth functions (see [10]).

Let $\delta_s^\alpha \in (C_b^{k,\omega}(\mathbb{R}^n))^*$, $s \in S$, $|\alpha| \le k$, be functionals given by the formulas

$$\delta_s^\alpha(f) := (D^\alpha f)(s).$$

Then for all α and s

$$D_S^*(\delta_{(s,\alpha)}) = \delta_s^\alpha. \tag{3.3}$$

Therefore, D_S^* maps $G_J^{k,\omega}(S)$ isometrically onto the minimal subspace of $(C_b^{k,\omega}(\mathbb{R}^n))^*$ containing all functionals δ_s^α, $s \in S$, $|\alpha| \le k$. In particular, $D_S^*(G_J^{k,\omega}(S)) \subset G_b^{k,\omega}(\mathbb{R}^n)$ because each δ_s^α belongs to the geometric predual space.

As a corollary of the Whitney-Glaeser theorem we obtain the following desciption of balls in $G_J^{k,\omega}(S)$. Let $B_{G_J^{k,\omega}(S)}$ be a closed unit ball in $G_J^{k,\omega}(S)$ and $B'_{G_J^{k,\omega}(S)}$ be the balanced closed convex hull of the set of vectors $\delta_{(s,\alpha)}$, $s \in S$, and

$$\frac{\delta_{(x,\alpha)} - \sum_{|\beta| \le k-|\alpha|}(x-y)^\beta/\beta! \cdot \delta_{(y,\alpha+\beta)}}{\|x-y\|^{k-\alpha}\omega(\|x-y\|)}, \quad x, y \in S,\ |\alpha| \le k.$$

Proposition 3.2. *There exist $C_1, C_2 \in \mathbb{R}_+$ depending on k, n only such that*

$$C_1 B'_{G_J^{k,\omega}(S)} \subset B_{G_J^{k,\omega}(S)} \subset C_2 B'_{G_J^{k,\omega}(S)}.$$

3.2. Approximation property Given a closed subset $S \subset \mathbb{R}^n$ let $\mathcal{W}_S := \{Q\}$ be the Whitney cover by cubes of $\mathbb{R}^n \setminus S$, $\{\varphi_Q\}_Q \subset C^\infty(\mathbb{R}^n)$ be a smooth partition of unity subordinate to $\mathcal{W}_S$ and $s_Q \in S$ be such that $\operatorname{dist}(s_Q, Q) = \operatorname{dist}(S, Q)$ (see, e.g., [4, p. 95–96]).

Theorem 3.3. *There is a bounded linear projection P of $G_J^{k,\omega}(\mathbb{R}^n)$ onto a subspace isometrically isomorphic to $G_J^{k,\omega}(S)$ whose norm is bounded by a constant depending on k and n such that*

$$P(\delta_{(x,\alpha)}) := \begin{cases} \delta_{(x,\alpha)} & \text{if} \quad x \in S \\ \displaystyle\sum_{|Q| \le 1} \sum_{|\beta| \le k} \frac{D^\alpha\big((x-s_Q)^\beta \varphi_Q(x)\big)}{\beta!} \delta_{(s_Q,\beta)} & \text{if} \quad x \in \mathbb{R}^n \setminus S; \end{cases} \tag{3.4}$$

here $|Q|$ stands for the Lebesgue measure of Q.

Thus $G_J^{k,\omega}(S)$ is isometrically isomorphic to a complemented subspace of $G_J^{k,\omega}(\mathbb{R}^n)$.

REMARK 3.4. Equation (3.4) shows that $\operatorname{ran} P$ is the smallest closed subspace of $G_J^{k,\omega}(\mathbb{R}^n)$ containing all $\delta_{(s,\alpha)}$, $s \in S$, $|\alpha| \le k$.

As a corollary we obtain the following result.

THEOREM 3.5. *$G_J^{k,\omega}(S)$ has the λ-approximation property with λ depending on k, n and $\lim_{t\to\infty} 1/\omega(t)$ only.*

Let us recall that a Banach space X is said to have the λ-*approximation property* for some $\lambda \in [1,\infty)$ if for any $\varepsilon > 0$ and any compact $K \subset X$, there exists a linear operator of finite rank $T : X \to X$ of norm $\le \lambda$ such that $\|Tx - x\|_X \le \varepsilon$ for all $x \in K$. If $\lambda = 1$, then X is said to have the *metric approximation property*, for more details see, e.g., [5]. Since $G_J^{k,\omega}(S)$ is a separable Banach space, Theorem 3.5 and the classical result of Pełczyński [8] imply that $G_J^{k,\omega}(S)$ is isomorphic to a complemented subspace of a separable Banach space with a basis.

3.3. Two Stars Theorem Let $C_0^{k,\omega}(\mathbb{R}^n)$ be the subspace of functions $f \in C_b^{k,\omega}(\mathbb{R}^n)$ such that for all $|\alpha| \le k$

(i)
$$\lim_{\|x\|\to\infty} D^\alpha f(x) = 0;$$

(ii)
$$\lim_{\|x-y\|\to 0} \frac{D^\alpha f(x) - D^\alpha f(y)}{\omega(\|x-y\|)} = 0.$$

It was proved in [3, Cor. 1.13] that $C_0^{k,\omega}(\mathbb{R}^n)$ is the minimal closed subspace of $C_b^{k,\omega}(\mathbb{R}^n)$ containing all C^∞ functions with compact supports.

DEFINITION 3.6. We define $J_0^{k,\omega}(S)$ as the subspace of vector functions $\vec{f} = (f_\alpha)_{|\alpha|\le k}$ on S such that $D^\alpha g|_S = f_\alpha$ for some $g \in C_0^{k,\omega}(\mathbb{R}^n)$ and all α with $|\alpha| \le k$. We equip it with the quotient norm

$$\|\vec{f}\|_{J_0^{k,\omega}(S)} = \inf_{g\in C_0^{k,\omega}(\mathbb{R}^n):D^\alpha g|_S = f_\alpha} \|g\|_{C_b^{k,\omega}(\mathbb{R}^n)}.$$

THEOREM 3.7. *Suppose that*

$$\lim_{t\to 0^+} \frac{t}{\omega(t)} = 0. \tag{3.5}$$

Then $(J_0^{k,\omega}(S))^*$ is isomorphic to $G_J^{k,\omega}(S)$, isometrically if $\lim\limits_{t\to\infty} \omega(t) = \infty$.

As a direct corollary we obtain the 'Two Stars Theorem':

Corollary 3.8. *If ω satisfies (3.5), then the second dual of $J_0^{k,\omega}(S)$ is isomorphic to $J_b^{k,\omega}(S)$, isometrically if $\lim_{t\to\infty}\omega(t)=\infty$.*

Remark 3.9. (1) Condition (3.5) is necessary as for $\omega(t)=t$, $C_0^{k,\omega}(\mathbb{R}^n)$ consists of constant functions and so the conclusion of Theorem 3.7 fails.

(2) Since under condition (3.5) $G_J^{k,\omega}(S)$ is dual and by Theorem 3.5 has the λ-approximation property, it has the metric approximation property[1] (see [7, Ch. I]).

(3) As was mentioned in the introduction an analog of Theorem 3.7 is valid for spaces $G_b^{k,\omega}(S)$ and $C_0^{k,\omega}(S)\,(\subset C_b^{k,\omega}(S))$ for weak Markov sets S (because for such sets these spaces are isomorphic to $G_J^{k,\omega}(S)$ and $J_0^{k,\omega}(S)$, respectively). For general closed sets $S\subset\mathbb{R}^n$ the question whether under condition (3.5) $(J_0^{k,\omega}(S))^*$ is isomorphic to $G_J^{k,\omega}(S)$ is open. In a forthcoming paper we prove that the answer is positive if and only if the higher-order tangent bundles for $C_b^{k,\omega}(S)$ and $C_0^{k,\omega}(S)$ coincide (see [1] for definitions of such bundles for spaces $C_b^k(S)$). As a result, we prove that $(J_0^{k,\omega}(S))^*$ is isomorphic to $G_J^{k,\omega}(S)$ for a wide class of ω satisfying (3.5).

4. Proof of Proposition 3.1

Recall that $D_S: C_b^{k,\omega}(\mathbb{R}^n)\to J_b^{k,\omega}(S)$ is the quotient map that determines $J_b^{k,\omega}(S)$ and whose kernel consists of functions f such that $\big((D^\alpha f)(s)\big)_{|\alpha|\le k}$ vanishes for all $s\in S$ (see (3.1)). In turn, the adjoint map D_S^* maps $G_J^{k,\omega}(S)$ isometrically onto the minimal subspace of $G_b^{k,\omega}(\mathbb{R}^n)$ containing all functionals δ_s^α, $s\in S$, $|\alpha|\le k$ (see (3.3)). We set $L:=D_S^*|_{G_J^{k,\omega}(S)}$ and consider the adjoint map $L^*:(G_b^{k,\omega}(\mathbb{R}^n))^*\to(G_J^{k,\omega}(S))^*$. According to [3, Prop. 2.2], $(G_b^{k,\omega}(\mathbb{R}^n))^*$ is naturally identified with $C_b^{k,\omega}(\mathbb{R}^n)$. Under this identification for each $f\in C_b^{k,\omega}(\mathbb{R}^n)$ and all $s\in S$, $|\alpha|\le k$ we obtain

$$(L^*(f))(\delta_{(s,\alpha)})=f(L(\delta_{(s,\alpha)}))=f(D_S^*(\delta_{(s,\alpha)}))=f(\delta_s^\alpha)=(D^\alpha f)(s). \tag{4.6}$$

This implies that $\ker L^*=\ker D_S$. Hence, there exists a bounded linear map $\tilde L: J_b^{k,\omega}(S)\to(G_J^{k,\omega}(S))^*$ such that $L^*=\tilde L\circ D_S$. Moreover, $\|\tilde L\|=\|L^*\|=1$.

Furthermore, by the Hahn-Banach theorem for each $v\in(G_J^{k,\omega}(S))^*$ there exists $f_v\in C_b^{k,\omega}(\mathbb{R}^n)$ such that

$$L^*(f_v)=v \quad\text{and}\quad \|f_v\|_{C_b^{k,\omega}(\mathbb{R}^n)}=\|v\|_{(G_J^{k,\omega}(S))^*}.$$

Thus, $\tilde L(D_S(f_v))=v$ and

$$\|v\|_{(G_J^{k,\omega}(S))^*}\le\|\tilde L\|\cdot\|D_S(f_v)\|_{J_b^{k,\omega}(S)}=\|D_S(f_v)\|_{J_b^{k,\omega}(S)}$$
$$\le\|f_v\|_{C_b^{k,\omega}(\mathbb{R}^n)}=\|v\|_{(G_J^{k,\omega}(S))^*}.$$

[1] i.e., 1-approximation property

The latter shows that $\tilde{L} : J_b^{k,\omega}(S) \to (G_J^{k,\omega}(S))^*$ is an isometric isomorphism.

Finally, according to (4.6),

$$(\tilde{L}(\vec{f}))(\delta_{(s,\alpha)}) = f^{\alpha}(s) = (I_S(\vec{f}))(\delta_{(s,\alpha)}) \text{ for all } \vec{f} \in J_b^{k,\omega}(S),\ s \in S,\ |\alpha| \le k.$$

This implies that $\tilde{L} = I_S$ and completes the proof of the proposition.

5. Proof of Proposition 3.2

Clearly, $\|\delta_{(s,\alpha)}\|_{[J_b^{k,\omega}(S)]^*} \le 1$ for all $s \in S$, $|\alpha| \le k$. Moreover, by the Whitney-Glaeser theorem (see (2.2)) there is a constant $C_2 \ge 1$ depending on k, n only such that for every $\vec{f} \in J_b^{k,\omega}(S)$ and $x, y \in S$,

$$\left| \left(\frac{\delta_{(x,\alpha)} - \sum_{|\beta| \le k-|\alpha|} (x-y)^{\beta}/\beta! \cdot \delta_{(y,\alpha+\beta)}}{\|x-y\|^{k-\alpha}\omega(\|x-y\|)} \right) (\vec{f}) \right|$$

$$:= \frac{|f_{\alpha}(x) - \sum_{|\beta| \le k-|\alpha|} (x-y)^{\beta}/\beta! \cdot f_{\alpha+\beta}(y)|}{\|x-y\|^{k-\alpha}\omega(\|x-y\|)} \le C_2 \|\vec{f}\|_{J_b^{k,\omega}(S)}.$$

Thus functionals $\delta_{(s,\alpha)}$ and $\frac{\delta_{(x,\alpha)} - \sum_{|\beta| \le k-|\alpha|} (x-y)^{\beta}/\beta! \cdot \delta_{(y,\alpha+\beta)}}{\|x-y\|^{k-\alpha}\omega(\|x-y\|)}$ lie in $C_2 B_{G_b^{k,\omega}(S)}$, i.e., $B'_{G_J^{k,\omega}(S)} \subset C_2 B_{G_b^{k,\omega}(S)}$.

Similarly, by the Whitney-Glaeser theorem there exists $C_1 > 0$, depending on k, n only, and for every $\vec{f} \in J_b^{k,\omega}(S)$ with $\|\vec{f}\| = 1$ there exist $x, y \in S$, $x \ne y$, such that

(5.7)
$$\frac{1}{C_1} \le \max \left\{ \max_{|\alpha| \le k} |f_{\alpha}(x)|, \max_{|\alpha| \le k} \frac{|f_{\alpha}(x) - \sum_{|\beta| \le k-|\alpha|} (x-y)^{\beta}/\beta! \cdot f_{\alpha+\beta}(y)|}{\|x-y\|^{k-\alpha}\omega(\|x-y\|)} \right\}.$$

Assume that there is $v \in B_{G_b^{k,\omega}(S)} \setminus C_1 B'_{G_b^{k,\omega}(S)}$. Then by the Hanh-Banach theorem and Proposition 3.1 we can find $\vec{f} \in J_b^{k,\omega}(S)$ with $\|\vec{f}\|_{J_b^{k,\omega}(S)} = 1$ such that

$$\sup_{v' \in C_1 B'_{G_b^{k,\omega}(S)}} |v'(\vec{f})| < v(\vec{f}) < 1.$$

This contradicts inequality (5.7) with such $\vec{f}$. Hence, $B_{G_b^{k,\omega}(S)} \subset C_1 B'_{G_b^{k,\omega}(S)}$.

The proof of the proposition is complete.

6. Proof of Theorem 3.3

We extend P given by (3.4) linearly to the linear space generated by $\delta_{(x,\alpha)}$, $x \in \mathbb{R}^n$, $|\alpha| \le k$. Specifically, if $\ell = \sum_{i=1}^{N} c_i \delta_{(x^i,\alpha^i)} \in G_J^{k,\omega}(\mathbb{R}^n)$, $x^i \in \mathbb{R}^n$, $|\alpha^i| \le k$, $N \in \mathbb{N}$, then we set

$$(6.8)\quad P(\ell) := \sum_{x^i \in S} c_i \delta_{(x^i,\alpha^i)} + \sum_{x^i \notin S} c_i \sum_{|\beta| \le k, |Q| \le 1} \frac{D^{\alpha^i}\left((x^i - s_Q)^{\beta} \varphi_Q(x^i)\right)}{\beta!} \delta_{(s_Q,\beta)}.$$

Since the family $\{\delta_{(x,\alpha)} : x \in \mathbb{R}^n,\ |\alpha| \le k\} \subset G_J^{k,\omega}(\mathbb{R}^n)$ is linearly independent and for each $x^i \notin S$ there are only finitely many φ_Q such that $(D^{\alpha^i}\varphi_Q)(x^i) \neq 0$ (because $\{\varphi_Q\}$ is the partition of unity subordinate to the Whitney cover of $\mathbb{R}^n \setminus S$), the above extension is well defined. It is also clear that $P^2(\ell) = P(\ell)$ for all such ℓ.

Further, let $\vec{f} \in J_b^{k,\omega}(\mathbb{R}^n)$. Then by (6.8) we get

$$
\begin{aligned}
&(P\ell)(\vec{f})\\
&= \sum_{x^i \in S} c_i \delta_{(x^i,\alpha^i)}(\vec{f}) + \sum_{x^i \notin S} c_i \sum_{|\beta| \le k, |Q| \le 1} \frac{D^{\alpha^i}\big((x^i - s_Q)^\beta \varphi_Q(x^i)\big)}{\beta!} \delta_{(s_Q,\beta)}(\vec{f})\\
&= \sum_{x^i \in S} c_i f^{\alpha^i}(x^i) + \sum_{x^i \notin S} c_i \sum_{|\beta| \le k, |Q| \le 1} \frac{D^{\alpha^i}\big((x^i - s_Q)^\beta \varphi_Q(x^i)\big)}{\beta!} f^\beta(s_Q).
\end{aligned}
$$

Moreover, if $\pi : J_b^{k,\omega}(\mathbb{R}^n) \to J_b^{k,\omega}(S)$ is the bounded linear surjection of norm ≤ 1 defined by restriction of jets on $\mathbb{R}^n$ to S, then

$$
(P\ell)(\vec{f}) = \ell(T(\pi\vec{f})), \tag{6.9}
$$

where for $\vec{g} \in J_b^{k,\omega}(S)$ the jet $T(\vec{g}) \in J_b^{k,\omega}(\mathbb{R}^n)$ is defined by (see (3.1))

(6.10)

$$
(T\vec{g})(x) = \begin{cases} \vec{g}(x) & \text{if} \quad x \in S \\ D_{\mathbb{R}^n}\left(\displaystyle\sum_{|Q| \le 1} \sum_{|\beta| \le k} \frac{(x - s_Q)^\beta \varphi_Q(x)}{\beta!} g^\beta(s_Q)\right) & \text{if} \quad x \in \mathbb{R}^n \setminus S. \end{cases}
$$

By definition T coincides with the Whitney-Glaeser extension operator (see, e.g., [4, p. 98]). In particular, $T : J_b^{k,\omega}(S) \to J_b^{k,\omega}(\mathbb{R}^n)$ is a bounded linear map whose norm depends on k and n only. Hence, for some $C_{k,n} \ge 1$

$$
\begin{aligned}
|(P\ell)(\vec{f})| = |\ell(T(\pi\vec{f}))| &\le C_{k,n}\|\ell\|_{G_J^{k,\omega}(\mathbb{R}^n)} \|\pi\vec{f}\|_{J_b^{k,\omega}(S)}\\
&\le C_{k,n}\|\ell\|_{G_J^{k,\omega}(\mathbb{R}^n)} \|\vec{f}\|_{J_b^{k,\omega}(\mathbb{R}^n)}.
\end{aligned}
$$

This implies that

$$
\|P\ell\|_{G_J^{k,\omega}(\mathbb{R}^n)} := \|P\ell\|_{[J_b^{k,\omega}(\mathbb{R}^n)]^*} \le C_{k,n}\|\ell\|_{G_J^{k,\omega}(\mathbb{R}^n)}.
$$

Hence, P extends to a continuous linear projection on $G_J^{k,\omega}(\mathbb{R}^n)$ with norm bounded by $C_{k,n}$. In turn, due to (6.8), $\operatorname{ran} P$ is the smallest closed subspace of $G_J^{k,\omega}(\mathbb{R}^n)$ containing all $\delta_{(s,\alpha)}$, $s \in S$, $|\alpha| \le k$.

Note that the adjoint map $\pi^*: [J_b^{k,\omega}(S)]^* \to [J_b^{k,\omega}(\mathbb{R}^n)]^*$ maps $G_J^{k,\omega}(S)$ into $\operatorname{ran} P$ as $\delta_{(s,\alpha)} \in G_J^{k,\omega}(S)$ are mapped by π^* in $\delta_{(s,\alpha)} \in G_J^{k,\omega}(\mathbb{R}^n)$. Moreover, by Definition 2.2

$$\|\pi^*(v)\|_{[J_b^{k,\omega}(\mathbb{R}^n)]^*} := \sup_{\|\vec{f}\|_{J_b^{k,\omega}(\mathbb{R}^n)}\le 1} |(\pi^*(v))(\vec{f})| = \sup_{\|\vec{f}\|_{J_b^{k,\omega}(\mathbb{R}^n)}\le 1} |v((\pi(\vec{f}))|$$

$$= \sup_{\|\vec{g}\|_{J_b^{k,\omega}(S)}\le 1} |v(\vec{g})| := \|v\|_{[J_b^{k,\omega}(S)]^*} \quad \text{for all} \quad v \in [J_b^{k,\omega}(S)]^*.$$

Hence, π^* maps $G_J^{k,\omega}(S)$ isometrically onto $\operatorname{ran} P$.

The proof of the theorem is complete.

7. Proof of Theorem 3.5

Theorem 1.8 in [3] asserts that the geometric predual space $G_b^{k,\omega}(\mathbb{R}^n)$ has the $\lambda(k,n,\omega)$-approximation property with $\lambda(k,n,\omega) = 1 + C\lim_{t\to\infty} 1/\omega(t)$ for some C depending on k, n only. As $G_b^{k,\omega}(\mathbb{R}^n)$ is isometrically isomorphic to $G_J^{k,\omega}(\mathbb{R}^n)$, the latter also has the $\lambda(k,n,\omega)$-approximation property. For a closed subset $S \subset \mathbb{R}^n$, consider the bounded linear projection P on $G_J^{k,\omega}(\mathbb{R}^n)$ of Theorem 3.3. Then $\operatorname{ran} P$ has the λ-approximation property with $\lambda = \|P\| \cdot \lambda(k,n,\omega) \le C_{k,n}\lambda(k,n,\omega)$, see, e.g., [3, Sec. 4.4] for a similar argument. Since $\operatorname{ran} P$ is isometrically isomorphic to $G_J^{k,\omega}(S)$, the latter space has the $\|P\| \cdot \lambda(k,n,\omega)$-approximation property as well.

This competes the proof of the theorem.

8. Proof of Theorem 3.7

Let $\pi : J_b^{k,\omega}(\mathbb{R}^n) \to J_b^{k,\omega}(S)$ and $\pi_0 : J_0^{k,\omega}(\mathbb{R}^n) \to J_0^{k,\omega}(S)$ be the surjective bounded linear maps induced by restrictions of jets on $\mathbb{R}^n$ to S, $i_S : J_0^{k,\omega}(S) \to J_b^{k,\omega}(S)$ and $i : J_0^{k,\omega}(\mathbb{R}^n) \to J_b^{k,\omega}(\mathbb{R}^n)$ be the inclusion maps and $T : J_b^{k,\omega}(S) \to J_b^{k,\omega}(\mathbb{R}^n)$ be the Whitney-Glaeser extension operator defined by (6.10). Observe that T is universal in the sense that it maps $J_0^{k,\omega}(S)$ into $J_0^{k,\omega}(\mathbb{R}^n)$ as well. Then we have the following commutative diagram

$$\begin{array}{ccc} (J_0^{k,\omega}(S))^* & \xleftarrow{\ i_S^*\ } & (J_b^{k,\omega}(S))^* \\ \pi_0^*\big\downarrow\big\uparrow T^* & & \pi^*\big\downarrow\big\uparrow T^* \\ (J_0^{k,\omega}(\mathbb{R}^n))^* & \xleftarrow[\ i^*\ ]{} & (J_b^{k,\omega}(\mathbb{R}^n))^*. \end{array} \tag{8.11}$$

We claim that i_S^* maps $G_J^{k,\omega}(S)$ isomorphically onto $(J_0^{k,\omega}(S))^*$. Since i_S^* is a bounded linear map, it suffices to show that $i_S^*|_{G_J^{k,\omega}(S)}$ is a bijection onto $(J_0^{k,\omega}(S))^*$ (then the claim will follow from the inverse mapping theorem).

Injectivity of $i_S^*|_{G_J^{k,\omega}(S)}$. Let $g \in G_J^{k,\omega}(S)$ be such that $i_S^*(g) = 0$. It was established in the proof of Theorem 3.3 that $P = \pi^* \circ (T^*|_{G_J^{k,\omega}(\mathbb{R}^n)})$ (see (6.9)) and π^* maps $G_J^{k,\omega}(S)$ isometrically onto $\operatorname{ran} P$. Hence, $T^*|_{\operatorname{ran} P} = (\pi^*)^{-1}$ is an isometry between $\operatorname{ran} P$ and $G_J^{k,\omega}(S)$. In particular, there is $h \in \operatorname{ran} P\,(\subset G_J^{k,\omega}(\mathbb{R}^n))$ such that $g = T^*(h)$. Since $\pi_0^*(i_S^*(T^*(h))) = \pi_0^*(i_S^*(g)) = 0$, we have by commutativity of (8.11)

$$i^*(h) = \pi_0^*(i_S^*(T^*(h))) = 0.$$

Moreover, the proof of Theorem 1.12 (1) in [3] shows that i^* maps $G_J^{k,\omega}(\mathbb{R}^n)$ isomorphically onto $(J_0^{k,\omega}(\mathbb{R}^n))^*$. Hence, $h = 0$ and so $g = 0$ as well.

Surjectivity of $i_S^*|_{G_J^{k,\omega}(S)}$. Let $\phi \in (J_0^{k,\omega}(S))^*$. Since $T^* \circ \pi_0^*$ is the identity map on $(J_0^{k,\omega}(S))^*$,

$$\phi = (T^* \circ \pi_0^*)(\phi).$$

As mentioned before $i^* : G_J^{k,\omega}(\mathbb{R}^n) \to (J_0^{k,\omega}(\mathbb{R}^n))^*$ is an isomorphism and so there is $g \in G_J^{k,\omega}(\mathbb{R}^n)$ such that $i^*(g) = \pi_0^*(\phi)$. Hence, due to commutativity of (8.11)

$$\phi = (T^* \circ i^*)(g) = (i_S^* \circ T^*)(g).$$

So for $h = T^*(g) \in G_J^{k,\omega}(S)$ we have $\phi = i_S^*(h)$.

This completes the proof of the first part of the theorem.

Next, let us show that if

$$\lim_{t\to\infty} \omega(t) = \infty, \tag{8.12}$$

then i_S^* maps $G_J^{k,\omega}(S)$ isometrically onto $(J_0^{k,\omega}(S))^*$.

In fact, it was proved in [3, Thm. 1.12 (1)] that under condition (8.12) i^* maps $G_J^{k,\omega}(\mathbb{R}^n)$ isometrically onto $(J_0^{k,\omega}(\mathbb{R}^n))^*$. Moreover, by definitions of spaces $J_b^{k,\omega}(S)$ and $J_0^{k,\omega}(S)$ maps π^* and π_0^* are isometric embeddings. From here by commutativity of (8.11) we obtain for each $g \in G_J^{k,\omega}(S)$

$$\|i_S^*(g)\|_{(J_0^{k,\omega}(S))^*} = \|(\pi_0^* \circ i_S^*)(g)\|_{(J_0^{k,\omega}(\mathbb{R}^n))^*} = \|(i^* \circ \pi)(g)\|_{(J_0^{k,\omega}(\mathbb{R}^n))^*}$$

$$= \|\pi(g)\|_{\operatorname{ran} P} = \|g\|_{G_J^{k,\omega}(S)}.$$

Hence, $i^*|_{G_J^{k,\omega}(\mathbb{R}^n)}$ is an isometry as required.

The proof of the theorem is complete.

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