

PIECEWISE CONTRACTIONS AND b -ADIC EXPANSIONS

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ABSTRACT. Let $I = [0, 1)$, $b \in \{2, 3, \dots\}$ and $f : I \rightarrow I$ be an injective piecewise $\frac{1}{b}$ -affine map, that is, assume that there exists a partition of I into intervals I_1, \dots, I_n such that $f(x) - f(y) = \frac{1}{b}(x - y)$ for all $x, y \in I_i$ and $1 \leq i \leq n$. In this note, we study the δ -parameter family of maps $f_\delta = R_\delta \circ f$, where $R_\delta : x \mapsto \{x + \delta\}$. More precisely, we show that the set \mathcal{N} of parameters δ for which f_δ has only natural f_δ -codings with maximal complexity is a non-empty set with Hausdorff dimension 0. We also show that for all $\delta \in \mathcal{N}$, the map f_δ is topologically semiconjugate to a minimal n -interval exchange transformation satisfying Keane's i.d.o.c. condition.

RÉSUMÉ. Soit $I = [0, 1)$, $b \in \{2, 3, \dots\}$ et $f : I \rightarrow I$ une fonction injective $\frac{1}{b}$ -affine par morceaux, c'est-à-dire, supposons qu'il existe une partition de I en intervalles I_1, \dots, I_n telle que $f(x) - f(y) = \frac{1}{b}(x - y)$ pour tous $x, y \in I_i$ et $1 \leq i \leq n$. Dans cette note, nous étudions la famille de fonctions $f_\delta = R_\delta \circ f$, où $R_\delta : x \mapsto \{x + \delta\}$. Plus précisément, nous montrons que l'ensemble \mathcal{N} de paramètres δ pour lesquels f_δ a seulement f_δ -codages naturelles avec complexité maximale est un ensemble non-vide de dimension de Hausdorff 0. Nous montrons aussi que pour tous $\delta \in \mathcal{N}$, la fonction f_δ est topologiquement semi-conjugué à un échange de n intervalles minimal satisfaisant à la condition i.d.o.c. de Keane.

1. Introduction Let $I = [0, 1)$ denote the unit interval. A map $f : I \rightarrow I$ is a *piecewise contraction of n intervals* (n -PC) if there are $0 < \lambda < 1$, $n \geq 2$ and $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ such that $|f(x) - f(y)| \leq \lambda|x - y|$ for all $x, y \in [x_{i-1}, x_i)$ and $1 \leq i \leq n$. If there exist $d_1, \dots, d_n \in \mathbb{R}$ such that $f(x) = \lambda x + d_i$ for all $x \in [x_{i-1}, x_i)$, then we call f a *piecewise λ -affine map*. We assume that $\mathcal{D}(f) = \{x_1, \dots, x_{n-1}\}$ is the discontinuity set of f . The set $G = I \setminus f(I)$ is called the *gap set*. An infinite word $\theta = \theta_0\theta_1\dots$ over the alphabet $\{1, \dots, n\}$ is the *natural f -coding* of $x \in I$ if $\theta_i = \epsilon(f^i(x))$ for all $i \geq 0$, where $\epsilon : I \rightarrow \{1, \dots, n\}$ is defined by $\epsilon(y) = i$ if $y \in [x_{i-1}, x_i)$. We denote by $L_k(\theta) = \{\theta_\ell\theta_{\ell+1}\dots\theta_{\ell+k-1} : \ell \geq 0\}$ the set of subwords of θ of length k . The *complexity function* of θ is the map $p_\theta : \mathbb{N} \rightarrow \mathbb{N}$ defined by $p_\theta(k) = \#L_k(\theta)$, where $\#$ stands for cardinality. It is known (see [4] or [11, Corollary 2.4.(i)]) that if $f : I \rightarrow I$ is an injective n -PC and θ is a natural f -coding, then there exists $k_0 \in \mathbb{N}$ such that $p_\theta(k) \leq (n - 1)k + 1$ for all $k \geq k_0$. The aim of this note is to

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study the set of injective n -PCs $f : I \rightarrow I$ having the following property: every natural f -coding θ has the maximal complexity function $p_\theta(k) = (n-1)k + 1$ for all $k \geq 1$. By [11, Corollary 2.4.(ii)] and by Keane's Irrationality Criterion, such set is at least as big as the set of n -IETs (see Definition 2.1) with irrational length vectors.

DEFINITION 1.1. An n -PC $f : I \rightarrow I$ has only natural f -codings with maximal complexity if every natural f -coding θ has the complexity function $p_\theta(k) = (n-1)k + 1$ for all $k \geq 1$.

Notice that when $n = 2$, a 2-PC $f : I \rightarrow I$ has only natural f -codings with maximal complexity if and only if every natural f -coding is a Sturmian word.

The relation between piecewise contractions and b -adic expansions of real numbers is revealed when we consider piecewise λ -affine maps with $\lambda = \frac{1}{b}$. This is the case of the one-parameter family of n -PCs defined next. As usual, $R_\delta : I \rightarrow I$ denotes the rotation by δ , that is, $R_\delta(x) = \{x + \delta\}$ for all $x \in I$.

DEFINITION 1.2 (The family of maps f_δ). Given integers $b \geq n \geq 2$ and real numbers $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$, let $f : I \rightarrow I$ be an injective piecewise $\frac{1}{b}$ -affine map defined by

$$f(x) = \frac{1}{b}x + d_i \quad \text{for all } x \in [x_{i-1}, x_i) \quad \text{and } 1 \leq i \leq n,$$

where $d_1, \dots, d_n \in \mathbb{R}$ are such that $d_i - d_{i+1} = 1$ for some $1 \leq i \leq n-1$. We assume that the connected components G_1, \dots, G_m of $G := I \setminus f(I)$ have the same length L . For each $\delta \in \mathbb{R}$, let $f_\delta : I \rightarrow I$ denote the map

$$(1) \quad f_\delta(x) = (R_\delta \circ f)(x) \quad \text{for all } x \in I.$$

Remark 1.1. In Definition 1.2, the condition $d_i - d_{i+1} = 1$ means that the interval map $f : I \rightarrow I$ is induced by a piecewise affine self-map of the unit circle $\hat{f} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$. In fact, such condition is equivalent to the following conditions: $\lim_{\epsilon \rightarrow 0^+} f(x_i - \epsilon) = 1$ and $f(x_i) = 0$.

Example 1.2. The maps $f : I \rightarrow I$ defined below satisfy the hypotheses of Definition 1.2.

$$(a) \quad f(x) = \begin{cases} \frac{1}{2}x + \frac{2}{3} & \text{if } x \in \left[0, \frac{2}{3}\right) \\ \frac{1}{2}x - \frac{1}{3} & \text{if } x \in \left[\frac{2}{3}, 1\right) \end{cases}, \quad (b) \quad f(x) = \begin{cases} \frac{1}{4}x + \frac{17}{40} & \text{if } x \in \left[0, \frac{2}{5}\right) \\ \frac{1}{4}x + \frac{4}{5} & \text{if } x \in \left[\frac{2}{5}, \frac{4}{5}\right) \\ \frac{1}{4}x - \frac{1}{5} & \text{if } x \in \left[\frac{4}{5}, 1\right) \end{cases}.$$

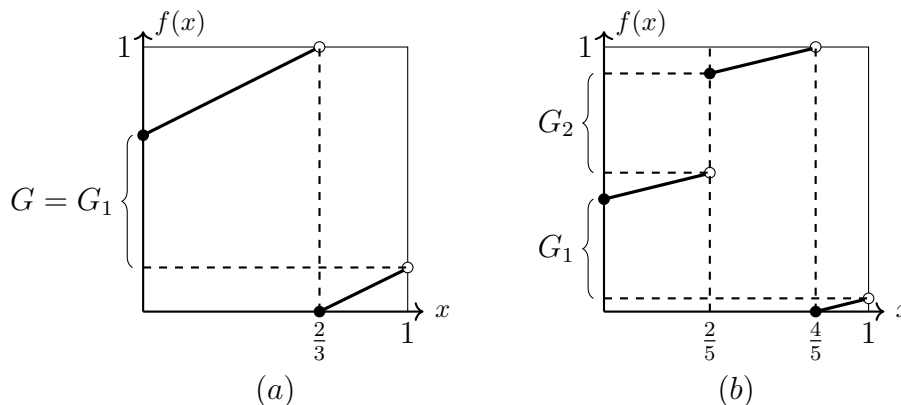


Figure 1: Examples of functions satisfying Definition 1.2

The corresponding graphs were plotted in Figure 1.

Much study was devoted to the family f_δ when $n = 2$ and $b \in (1, \infty)$, by using an approach based on Rotation Number Theory. In particular, by Bugeaud [1,2] and by Bugeaud and Conze [3] we know that the set \mathcal{N} of parameters δ for which f_δ has only natural f_δ -codings of maximal complexity has Lebesgue measure 0. More recently, Laurent and Nogueira [8] proved that \mathcal{N} has Hausdorff dimension 0. Janson and Öberg [6] improved that result even further: they supplied gauge functions that provide finer information, including both an upper and a lower bound on the “size” of \mathcal{N} . Concerning $n \geq 2$ and $b \in (1, \infty)$, the author, Nogueira and Rosales [10] showed that the set \mathcal{N} has Lebesgue measure 0. In this note, we show that for all integers $b \geq n \geq 2$, the set \mathcal{N} has Hausdorff dimension 0.

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2. Statement of the Results Our first result concerns the family of maps f_δ introduced in Definition 1.2. We recall that the set $G = I \setminus f(I)$ is called the gap set of the map f . It follows from Remark 1.1 that $0 < \inf G < \sup G < 1$ and that f_δ has $n - 1$ discontinuities for all $\delta \in (-\inf G, 1 - \sup G)$.

Now we state our first result.

THEOREM 2.1. *The set $\mathcal{N} \subset (-\inf G, 1 - \sup G)$ of parameters δ for which $f_\delta = R_\delta \circ f$ has only natural f_δ -codings with maximal complexity has Hausdorff dimension 0.*

The approach we use to prove Theorem 2.1 is based on Number Theory. The idea is to characterize the numbers in the set \mathcal{N} in terms of their b -adic expansions. Then we use results obtained by Mauduit and Moreira [9] that give the Hausdorff dimension of sets of numbers with b -adic expansions of entropy zero.

DEFINITION 2.1 (n -IET). A bijection $T : I \rightarrow I$ is an n -Interval Exchange Transformation (n -IET) if there exist numbers $0 = y_0 < y_1 < \dots < y_{n-1} < y_n = 1$ and $c_1, \dots, c_n \in \mathbb{R}$ such that $T(y) = y + c_i$ for all $y \in [y_{i-1}, y_i)$ and $1 \leq i \leq n$, where $\mathcal{D}(T) = \{y_1, \dots, y_{n-1}\}$ is the discontinuity set of T . We say that T satisfies the *i.d.o.c.* (infinite distinct orbit condition) if $O_T(y_i) = \{y_i, T(y_i), \dots\}$, $1 \leq i \leq n-1$, are infinite pairwise disjoint sets.

Keane [7] proved that if an n -IET T satisfies the i.d.o.c., then all orbits of T are dense.

Our second result clarifies what does it mean for an n -PC to have natural codings with maximal complexity.

THEOREM 2.2 (Structure Theorem). *Let $n \geq 2$ and $g : I \rightarrow I$ be an injective n -PC with discontinuity set $\mathcal{D}(g) = \{x_1, \dots, x_{n-1}\}$. The following statements are equivalent:*

- (i) *g has only natural g -codings with maximal complexity;*
- (ii) *g is topologically semiconjugate to an n -IET satisfying the i.d.o.c.*

It is worth mentioning that Ferenczi and Zamboni [5] studied n -IETs whose natural codings θ have complexity function $p_\theta(k) = (n-1)k + 1$ for every $k \geq 1$, which is the case of n -IETs satisfying the i.d.o.c.

3. Preparatory Lemmas and Proof of Theorem 2.2 In what follows, let $n \geq 2$ and $g : I \rightarrow I$ be an injective n -PC with discontinuity set $\mathcal{D}(g) = \{x_1, \dots, x_{n-1}\}$ and gap set $G = I \setminus g(I)$. We say that g has a *connection* if there exist $k \geq 1$ and $1 \leq i, j \leq n$ such that $g^k(x_i) = x_j$.

LEMMA 3.1. *Assume that g has only natural g -codings with maximal complexity. Then g has no connection and $\mathcal{D}(g) \cap \bigcup_{k=0}^{\infty} g^k(G) = \emptyset$.*

PROOF. Let $\mathcal{A} = \{1, \dots, n\}$ and $\mathcal{P} = \{I_1, \dots, I_n\}$ be the partition associated with g , where $I_i = [x_{i-1}, x_i)$. It is elementary that $g^{-1}(I_i)$, $1 \leq i \leq n$, is either empty or the union of finitely many intervals. Moreover, $g^{-\ell}(\mathcal{P})$ is a partition

of I for every $\ell \geq 0$, implying that each member of the family

$$\begin{aligned} \mathcal{P}_k &= \bigvee_{\ell=0}^{k-1} g^{-\ell}(\mathcal{P}) \\ &= \left\{ I_{i_0} \cap g^{-1}(I_{i_1}) \cap \cdots \cap g^{-(k-1)}(I_{i_{k-1}}) : i_0, i_1, \dots, i_{k-1} \in \mathcal{A} \right\}, \quad k \geq 1, \end{aligned}$$

is either empty or the union of pairwise disjoint intervals. Let θ be the natural g -coding of a point $x \in I$. Then the k -word $i_0 i_1 \dots i_{k-1}$ occurs in θ at the position $j \geq 0$ if and only if $g^j(x) \in I_{i_0} \cap g^{-1}(I_{i_1}) \cap \cdots \cap g^{-(k-1)}(I_{i_{k-1}})$. Therefore, $p_\theta(k)$ is bounded above by the number M_k of non-empty members in \mathcal{P}_k , there is, $p_\theta(k) \leq M_k$ for all $k \geq 1$. Moreover, the following equation holds:

$$(2) \quad M_k = 1 + \sum_{\ell=0}^{k-1} n_\ell,$$

where $n_0 = n - 1$ and

$$(3) \quad n_\ell = \# \left(g^{-\ell}(\{x_1\}) \cup \dots \cup g^{-\ell}(\{x_{n-1}\}) \setminus \bigcup_{p=0}^{\ell-1} g^{-p}(\{x_1\}) \cup \dots \cup g^{-p}(\{x_{n-1}\}) \right)$$

gives the number of new division points at the ℓ -th step towards the construction of \mathcal{P}_k . In particular, since $n_\ell \leq n - 1$ and since θ is a natural g -coding with maximal complexity, we have by (2) that

$$(n - 1)k + 1 = p_\theta(k) \leq M_k \leq (n - 1)k + 1.$$

Hence, $M_k = (n - 1)k + 1$ and $n_\ell = n - 1$ for all $\ell \geq 0$. By (3), g has no connection and $\mathcal{D}(g) \cap \bigcup_{k=0}^{\infty} g^k(G) = \emptyset$, otherwise $n_\ell < n - 1$ for some ℓ . \square

LEMMA 3.2. *Assume that g has only natural g -codings with maximal complexity and write its gap $G = G^{(1)} \cup \dots \cup G^{(m)}$ as the union of its connected components. Then, the collection $\{g^k(G^{(j)}) : 1 \leq j \leq m, k \geq 0\}$ is formed by pairwise disjoint intervals and*

$$\text{Leb} \left(\bigcup_{j=1}^m \bigcup_{k=0}^{\infty} g^k(G^{(j)}) \right) = 1,$$

where *Leb* means Lebesgue measure.

PROOF. By Lemma 3.1, we have that $g^k(G^{(j)})$ is an interval for all $1 \leq j \leq m$ and $k \geq 0$. Since $g^{-1}(G) = \emptyset$ and g is injective, we have that $\{g^k(G^{(j)}) : 1 \leq j \leq m, k \geq 0\}$ is formed by pairwise disjoint intervals. Let $0 < c < 1$ be the contraction rate of g and $A = \bigcup_{j=1}^m \bigcup_{k=0}^{\infty} g^k(G^{(j)})$. We have that $I \setminus A = \bigcap_{k \geq 0} g^k(I)$, thus $\text{Leb}(I \setminus A) \leq c^k \text{Leb}(I)$ for any $k \geq 0$, which yields $\text{Leb}(A) = 1$. \square

PROOF OF THEOREM 2.2. (i) \implies (ii). Let $\omega(x) = \bigcap_{\ell=0}^{\infty} \overline{\bigcup_{k=\ell}^{\infty} \{g^k(x)\}}$ denote the ω -limit set of $x \in I$. By the Morse-Hedlund Theorem, g has no ultimately periodic natural g -coding. Then, by [11, Lemma 3.1], we have that $\omega(x)$ is a set with infinite cardinality. By [11, Theorem 3.5], g admits a non-atomic g -invariant Borel probability measure. Proceeding as in the proof of [11, Theorem 2.1], we conclude that g is topologically semiconjugate to a r -IET $T : I \rightarrow I$ with $r \leq n$. More precisely, by [11, Equation (8)], we have that r is the cardinality of the set:

$$J = \{1 \leq i \leq n : (x_{i-1}, x_i) \cap A \neq \emptyset\},$$

where A is the union of the ω -limit set of finitely many points. The fact that g has only natural g -codings with maximal complexity implies that the orbit of any point visits each interval (x_{i-1}, x_i) infinitely often, thus $r = n$ and that T satisfies the i.d.o.c.

(ii) \implies (i). Denote by $\mathcal{D}(g) = \{x_1, \dots, x_{n-1}\}$ and $\mathcal{D}(T) = \{y_1, \dots, y_{n-1}\}$ the discontinuity sets of g and T , respectively. Let $h : I \rightarrow I$ denote the topological semiconjugacy, thus $\mathcal{D}(T) \subseteq h(\mathcal{D}(g))$. Since $\mathcal{D}(g)$ and $\mathcal{D}(T)$ have the same cardinality n and since h is non-decreasing, we have that $y_i = h(x_i)$ for every $1 \leq i \leq n-1$. In particular, $h([x_{i-1}, x_i]) = [y_{i-1}, y_i]$ for all $1 \leq i \leq n$, where $x_0 = y_0 = 0$ and $x_n = y_n = 1$. This means that the natural f -coding θ of any $x \in I$ equals the natural T -coding θ of $h(x)$. Moreover, since T is an n -IET satisfying the i.d.o.c., we have that the complexity function of θ by T is $p_{\theta}(k) = (n-1)k + 1$ for every $k \geq 1$ (see [11, Lemma 6.2]). \square

4. Proof of Theorem 2.1 Throughout this section, let $f : I \rightarrow I$, $G = I \setminus f(I)$, $\mathcal{N} \subset (-\inf G, 1 - \sup G)$, $\delta \in \mathcal{N}$, $f_{\delta} : I \rightarrow I$ and $G^{(\delta)} = I \setminus f_{\delta}(I)$ be as in the statement of Theorem 2.1. The gap sets G and $G^{(\delta)}$ are the union of $1 \leq m \leq n-1$ connected components:

$$(4) \quad G = G^{(1)} \cup \dots \cup G^{(m)}, \quad G_{\delta} = G_{\delta}^{(1)} \cup \dots \cup G_{\delta}^{(m)}.$$

In what follows, we will keep $\delta \in \mathcal{N}$ fixed. We will use $|\cdot|$ to denote interval length.

LEMMA 4.1. *There exist infinite words $w^{(j)} = w_0^{(j)} w_1^{(j)} \dots$, $1 \leq j \leq m$, over the alphabet $\{0, 1\}$ such that $p_{w^{(j)}}(k) \leq (n-1)k + 2$ for every $k \geq 1$ and $1 \leq j \leq m$, and*

$$(5) \quad \delta = -d_1 + \sum_{j=1}^m \sum_{i=0}^{\infty} w_i^{(j)} b^{-i} |G^{(j)}|.$$

PROOF. Let $J = [0, d_1 + \delta)$. Let q_j be the middle-point of $G_{\delta}^{(j)}$ and $w^{(j)} = w_0^{(j)} w_1^{(j)} \dots$ be the infinity binary word defined by $w_i^{(j)} = \chi_J(f_{\delta}^i(q_j))$. Since $\delta \in (-\inf G, 1 - \sup G)$, we have that $f_{\delta}^{-1}(0) \in \mathcal{D}(f_{\delta})$. This combined with the fact that $d_1 + \delta = f_{\delta}(0)$ yield

$$(6) \quad f_{\delta}^{-2}(d_1 + \delta) \in \mathcal{D}(f_{\delta}).$$

On the other, since $\delta \in \mathcal{N}$, we have that the n -PC $g := f_\delta$ has only natural g -codings with maximal complexity. Moreover, its gap set is equal to G_δ . Hence, it follows immediately from Lemma 3.1 that

$$(7) \quad \mathcal{D}(f_\delta) \cap \bigcup_{i=0}^{\infty} f_\delta^i(G_\delta) = \emptyset.$$

In this way, by (4), (6), (7) and by the injectivity of f_δ , we reach $d_1 + \delta \notin \bigcup_{j=1}^m \bigcup_{i=0}^{\infty} f_\delta^i(G_\delta^{(j)})$. Therefore, given $1 \leq j \leq m$ and $i \geq 0$, the interval $f_\delta^i(G_\delta^{(j)})$ is entirely contained in either J or $I \setminus J$. Moreover, applying Lemma 3.2 with $g := f_\delta$, we obtain that the intervals $f_\delta^i(G_\delta^{(j)})$, $1 \leq j \leq m$ and $i \geq 0$, are pairwise disjoint and their union is dense in I . Putting it all together, we conclude that the length of J equals the sum of the lengths of the intervals $f_\delta^i(G_\delta^{(j)})$ which are entirely contained in J . In quantitative terms, what we have is the following equalities:

$$(8) \quad d_1 + \delta = |J| = \sum_{j=1}^m \sum_{i=0}^{\infty} w_i^{(j)} \left| f^i(G_\delta^{(j)}) \right| = \sum_{j=1}^m \sum_{i=0}^{\infty} w_i^{(j)} b^{-i} \left| G_\delta^{(j)} \right|.$$

Since $\delta \in (-\inf G, 1 - \sup G)$, we have that $G + \delta \subset I$, which yields $\{G + \delta\} = G + \delta$. Hence,

$$G_\delta = I \setminus R_\delta(f(I)) = R_\delta(I \setminus f(I)) = R_\delta(G) = \{G + \delta\} = G + \delta.$$

In this way, by (4), $G_\delta^{(j)} = G^{(j)} + \delta$ for all $1 \leq j \leq m$. This shows that

$$(9) \quad \left| G_\delta^{(j)} \right| = \left| G^{(j)} \right| \quad \text{for all } 1 \leq j \leq m.$$

Replacing (9) in (8) yield (5).

Let us now prove that $p_{w^{(j)}}(k) \leq (n-1)k + 2$ for all $k \geq 1$. Let $\mathcal{P}_\delta = \{I_\delta^{(1)}, \dots, I_\delta^{(n)}\}$ be the partition into intervals associated with the injective n -PC f_δ . Since $d_1 + \delta = f_\delta(0)$, we have that $f_\delta^{-1}(J)$ is the union of finitely many members of \mathcal{P}_δ . Let $\eta : \{1, 2, \dots, n\} \rightarrow \{0, 1\}$ the map defined by $\eta(i) = 1$ if $f_\delta(I_\delta^{(i)}) \subset J$ and $\eta(i) = 0$ if $f_\delta(I_\delta^{(i)}) \subset I \setminus J$. As before, let q_j be the middle-point of $G_\delta^{(j)}$ and let $\theta^{(j)} = \theta_0^{(j)} \theta_1^{(j)} \dots$ denote the natural f_δ -coding of q_j . Then $w_{i+1}^{(j)} = \eta(\theta_i^{(j)})$ for all $i \geq 0$ and $1 \leq j \leq m$. Hence, $w^{(j)} = w_0^{(j)} \eta(\theta_0^{(j)}) \eta(\theta_1^{(j)}) \dots$. Writing $w^{(j)} = w_0^{(j)} v$, then $v = \eta(\theta_0^{(j)}) \eta(\theta_1^{(j)}) \dots$. In this way, for each $k \geq 1$, the map η induces a surjective map from $L_k(\theta^{(j)})$ onto $L_k(v)$, implying that

$$p_v(k) = \#L_k(v) \leq \#L_k(\theta^{(j)}) = p_{\theta^{(j)}}(k) = (n-1)k + 1.$$

Therefore,

$$p_{w^{(j)}}(k) \leq 1 + p_v(k) = (n-1)k + 2.$$

□

In what follows, $\mathcal{A} = \{0, 1, \dots, b-1\}$. To each infinite word $w = w_0w_1\dots$ over the alphabet \mathcal{A} , we can associate naturally the real number x in $[0, 1]$ whose representation in base b is given by w . More precisely, $x = \rho(w)$, where $\rho : \mathcal{A}^{\mathbb{N}} \rightarrow [0, 1]$ is the function defined by

$$\rho(w) = \sum_{i=0}^{\infty} w_i b^{-(i+1)} = b^{-1} \sum_{i=0}^{\infty} w_i b^{-i}.$$

The *entropy* of an infinite word $w = w_0w_1\dots$ over the alphabet \mathcal{A} is the number $E(w) \log b$, where

$$E(w) = \lim_{k \rightarrow \infty} \frac{\log_b(p_w(k))}{k}.$$

We will need the following result.

THEOREM 4.2 (Mauduit-Moreira [9, Proposition 1]). *For each integer $b \geq 2$, the set $\{\rho(w) : w \in \mathcal{A}^{\mathbb{N}} \text{ and } E(w) = 0\}$ has Hausdorff dimension 0.*

Given two infinite words $u = u_0u_1\dots$ and $v = v_0v_1\dots$ over the alphabet $\{0, 1\}$, we denote by $u+v$ the infinite word over the alphabet $\{0, 1, 2\}$ whose i -th letter is $u_i + v_i$. Likewise, the sum of m infinite binary words is an infinite word over the alphabet $\{0, 1, \dots, m\} \subseteq \mathcal{A}$ because $m \leq n-1 \leq b-1$, where m is the number of connected components of the gap set G (which has the same number of connected components as G_δ).

LEMMA 4.3. *Let $w^{(1)}, \dots, w^{(m)}$ be the binary words in the statement of Lemma 4.1. Then, $w = w^{(1)} + \dots + w^{(m)}$ is an infinite word over the alphabet $\{0, 1, \dots, b-1\}$ with $E(w) = 0$.*

PROOF. Since $m \leq b-1$ and also because each word $w^{(j)}$ is binary, we have that $w = w^{(1)} + \dots + w^{(m)}$ is an infinite word over the alphabet $\{0, 1, \dots, b-1\}$. The operation of adding m words componentwisely yields a surjective map from $L_k(w^{(1)}) \times \dots \times L_k(w^{(m)})$ onto $L_k(w)$. Then, by Lemma 4.1, for every $k \geq 1$,

$$p_w(k) \leq \prod_{j=1}^m p_{w^{(j)}}(k) \leq [(n-1)k + 2]^m.$$

Moreover,

$$E(w) = \lim_{k \rightarrow \infty} \frac{\log_b(p_w(k))}{k} \leq m \lim_{k \rightarrow \infty} \frac{\log_b[(n-1)k + 2]}{k} = 0.$$

□

PROOF OF THEOREM 2.1. Let $\mathcal{A} = \{0, 1, \dots, b-1\}$ and

$$\mathcal{N}_1 = \left\{ \sum_{j=1}^m \sum_{i=0}^{\infty} w_i^{(j)} b^{-i} : w^{(j)} = w_0^{(j)} w_1^{(j)} \dots \in \{0, 1\}^{\mathbb{N}} \right. \\ \left. \text{and } p_{w^{(j)}}(k) \leq (n-1)k + 2, \forall k, j \right\}.$$

It follows from Lemma 4.3 that

$$\mathcal{N}_1 \subset \left\{ \sum_{i=0}^{\infty} w_i b^{-i} : w = w_0 w_1 \dots \in \mathcal{A}^{\mathbb{N}} \text{ and } E(w) = 0 \right\}.$$

In other words, $\mathcal{N}_1 \subset \{b\rho(w) : w \in \mathcal{A}^{\mathbb{N}} \text{ and } E(w) = 0\}$. By Theorem 4.2, the set \mathcal{N}_1 has Hausdorff dimension 0. By Lemma 4.1, since $|G^{(j)}| = L$ for all $1 \leq j \leq m$ (see Definition 1.2), we have that $\mathcal{N} \subset -d_1 + L\mathcal{N}_1$. In this way, \mathcal{N} has Hausdorff dimension 0. \square

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