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Renormalization of Bi-cubic Circle Maps

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Abstract. We develop a renormalization theory for analytic homeomorphisms of the circle with two cubic critical points. We prove a renormalization hyperbolicity theorem. As a basis for the proofs, we develop complex *a priori* bounds for multi-critical circle maps.

Résumé. On développe une théorie de renormalisation pour les homéomorphismes analytiques du cercle à deux points critiques cubiques. On démontre un théorème d'hyperbolicité dans le cadre de renormalisation. Comme base des démonstrations, on développe des bornes complexes *a priori* pour les applications du cercle dans lui-même aux points critiques multiples.

1. **Introduction** In this paper we develop a renormalization theory for analytic critical circle maps with several critical points. The principal result is a renormalization hyperbolicity theorem for *bi-cubic* circle maps, that is, analytic homeomorphisms of the circle with two critical points, both of which are of cubic type. We construct periodic orbits for renormalization for such maps (Theorem 2.8), and show (Theorem 8.3) that these orbits are hyperbolic, with *two* unstable directions.

The proof of Theorem 2.8 uses a new version of our complex *a priori* bounds [14]. The proof of the bounds is of an independent importance, and we give it for *arbitrary* multicritical circle maps, not just bi-cubic ones (Theorem 3.2).

2. Preliminaries

2.1. Multicritical circle maps and commuting pairs A multicritical circle map $f : \mathbb{T} \to \mathbb{T}$ is a C^3-smooth homeomorphism of the circle $\mathbb{T}$ which has finitely many critical points $c_0, \ldots, c_l$ such that:

$$f(x) - f(c_j) = a_j(x - c_j)^{k_j}\phi_j(x), \text{ for } x \approx c_j, \tag{2.1}$$

where $k_j \in 2\mathbb{Z} + 1$ and ϕ_j is a local C^3-diffeomorphism. We will say that a critical point described by the equation (2.1) has order k_j.

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The same definition becomes much simpler in the analytic category (C^ω): an analytic multicritical circle map is an analytic homeomorphism of the circle with at least one critical point.

A discussion of multicritical circle maps in the real setting has recently been carried out in [6].

In what follows, we will always place one of the critical points of a multicritical circle map at 0.

As discussed in some detail in [15], the space of critical circle maps is ill-suited to define renormalization. The pioneering works on the subject ([11] and [5]) circumvented this difficulty by replacing critical circle maps with different objects:

DEFINITION 2.1. A C^r*-smooth (or* C^ω*) multicritical commuting pair* $\zeta = (\eta, \xi)$ consists of two C^r-smooth (or C^ω) orientation preserving interval homeomorphisms $\eta : I_\eta \to \eta(I_\eta)$, $\xi : I_\xi \to \xi(I_\xi)$, where

(I) $I_\eta = [0, \xi(0)]$, $I_\xi = [\eta(0), 0]$;
(II) Both η and ξ have homeomorphic extensions to interval neighborhoods V_η, V_ξ of their respective domains *with the same degree of smoothness*, that is C^r (or C^ω), which commute, $\eta \circ \xi = \xi \circ \eta$;
(III) $\xi \circ \eta(0) \in I_\eta$;
(IV) each of the maps η and ξ has finitely many critical points in V_η, V_ξ respectively, each of them of an odd integer order;
(V) the point 0 is a critical point for both η and ξ.

We note that the commutation condition (II) ensures that η and ξ have the same order of the critical point at 0.

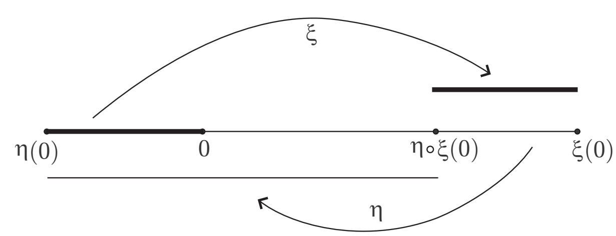

Figure 1: A commuting pair

Given a multicritical commuting pair $\zeta = (\eta, \xi)$ we can regard the interval $I = [\eta(0), \xi \circ \eta(0)]$ as a circle, identifying $\eta(0)$ and $\xi \circ \eta(0)$ and define $F_\zeta : I \to I$

by

$$F_\zeta = \begin{cases} \eta \circ \xi(x) \text{ for } x \in [\eta(0), 0] \\ \eta(x) \text{ for } x \in [0, \xi \circ \eta(0)] \end{cases} \tag{2.2}$$

The mapping ξ extends to a C^r- (or C^ω-) diffeomorphism of open neighborhoods of $\eta(0)$ and $\xi \circ \eta(0)$. Using it as a local chart we turn the interval I into a closed one-dimensional manifold M. Condition (II) above implies that the mapping F_ζ projects to a well-defined C^3-smooth homeomorphism $F_\zeta : M \to M$. Identifying M with the circle by a diffeomorphism $\phi : M \to \mathbb{T}$ we recover a multicritical circle mapping

$$f^\phi = \phi \circ F_\zeta \circ \phi^{-1}. \tag{2.3}$$

The critical circle mappings corresponding to two different choices of ϕ are conjugated by a diffeomorphism, and thus we recovered a C^r- (or C^ω) smooth conjugacy class of circle mappings from a multicritical commuting pair.

2.2. Commuting pairs and renormalization of critical circle maps The *height* $\chi(\zeta)$ of a multicritical commuting pair $\zeta = (\eta, \xi)$ is equal to r, if

$$0 \in [\eta^r(\xi(0)), \eta^{r+1}(\xi(0))].$$

If no such r exists, we set $\chi(\zeta) = \infty$, in this case the map $\eta|I_\eta$ has a fixed point. For a pair ζ with $\chi(\zeta) = r < \infty$ one verifies directly that the mappings $\eta|[0, \eta^r(\xi(0))]$ and $\eta^r \circ \xi|I_\xi$ again form a commuting pair.

Given a commuting pair $\zeta = (\eta, \xi)$ we will denote by $\widetilde{\zeta}$ the pair $(\widetilde{\eta}|\widetilde{I_\eta}, \widetilde{\xi}|\widetilde{I_\xi})$ where tilde means rescaling by a linear factor $\lambda = \frac{1}{|I_\eta|}$.

DEFINITION 2.2. We say that a real commuting pair $\zeta = (\eta, \xi)$ is *renormalizable* if $\chi(\zeta) < \infty$. The *renormalization* of a renormalizable commuting pair $\zeta = (\eta, \xi)$ is the commuting pair

$$\mathcal{R}\zeta = (\widetilde{\eta^r \circ \xi}|\widetilde{I_\xi},\ \widetilde{\eta}|[0, \widetilde{\eta^r(\xi(0))}]).$$

The non-rescaled pair $(\eta^r \circ \xi|I_\xi, \eta|[0, \eta^r(\xi(0))])$ will be referred to as the *pre-renormalization* $p\mathcal{R}\zeta$ of the commuting pair $\zeta = (\eta, \xi)$. Suppose $\{\zeta_i\}_{i=1}^{k-1}$ is a sequence of renormalizable pairs such that $\zeta_0 = \zeta$ and $\zeta_i = p\mathcal{R}\zeta_{i-1}$. We call $\zeta_k = p\mathcal{R}\zeta_{k-1}$ the k-th pre-renormalization of ζ; and $\widetilde{\zeta_k}$ the k-th renormalization of ζ and write

$$\zeta_k = p\mathcal{R}^k\zeta,\ \widetilde{\zeta_k} = \mathcal{R}^k\zeta.$$

For a pair ζ we define its *rotation number* $\rho(\zeta) \in [0, 1]$ to be equal to the continued fraction $[r_0, r_1, \ldots]$ where $r_i = \chi(\mathcal{R}^i\zeta)$. In this definition $1/\infty$ is understood as 0, hence a rotation number is rational if and only if only finitely many renormalizations of ζ are defined; if $\chi(\zeta) = \infty$, $\rho(\zeta) = 0$. Thus defined, the rotation number of a commuting pair can be viewed as a rotation number in the usual sense:

PROPOSITION 2.1. *The rotation number of the mapping F_ζ is equal to $\rho(\zeta)$.*

There is an advantage in defining $\rho(\zeta)$ using a sequence of heights, since it removes the ambiguity in prescribing a continued fraction expansion to *rational* rotation numbers in a dynamically natural way.

Let f be a multicritical circle map with a critical point at 0. Suppose that the continued fraction of $\rho(f)$ contains at least $n+1$ terms; as usual, let p_k/q_k be the k-th convergent of $\rho(f)$ for $k \leq n+1$, and let

$$I_n \equiv [0, f^{q_n}(0)].$$

One obtains a multicritical commuting pair from the pair of maps $(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$ as follows. Let $\bar{f}$ be the lift of f to the real line satisfying $\bar{f}'(0) = 0$, and $0 < \bar{f}(0) < 1$. For each $m > 0$ let $\bar{I}_m \subset \mathbb{R}$ denote the closed interval adjacent to zero which projects down to the interval I_m. Let $\tau : \mathbb{R} \to \mathbb{R}$ denote the translation $x \mapsto x + 1$. Let $\eta : \bar{I}_m \to \mathbb{R}$, $\xi : \bar{I}_{m+1} \to \mathbb{R}$ be given by $\eta \equiv \tau^{-p_{m+1}} \circ \bar{f}^{q_{m+1}}$, $\xi \equiv \tau^{-p_m} \circ \bar{f}^{q_m}$. Then the pair of maps $(\eta|\bar{I}_m, \xi|\bar{I}_{m+1})$ forms a multicritical commuting pair corresponding to $(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$. Henceforth, we shall abuse notation and simply denote this commuting pair by

$$(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1}). \tag{2.4}$$

The m-th renormalization of f at the critical point 0 is the rescaled pair (2.4):

$$(\widetilde{f^{q_{m+1}}}|\widetilde{I_m}, \widetilde{f^{q_m}}|\widetilde{I_{m+1}}). \tag{2.5}$$

Denote $G(x)$ the Gauss map

$$G(x) = \left\{ \frac{1}{x} \right\}.$$

It is easy to verify that for a renormalizable pair ζ

$$\rho(\mathcal{R}\zeta) = G(\rho\zeta). \tag{2.6}$$

2.3. Dynamical partitions and real a priori bounds Following the convention introduced by Sullivan [12], we say that for an infinitely renormalizable mapping f (or a pair ζ) a quantity is "beau" (which translates as "bounded and eventually universally (bounded)") if it is bounded for all renormalizations $\mathcal{R}^n f$ ($\mathcal{R}^n\zeta$), and the bound becomes universal (that is, independent of the map) for $n \geq n_0$ for some $n_0 \in \mathbb{N}$.

DEFINITION 2.3. Let f be a multicritical circle map which is at least n-times renormalizable. The collection of intervals

$$\mathcal{P}_n(f) = \{I_n, f(I_n), \ldots, f^{q_{n+1}-1}(I_n)\} \cup \{I_{n+1}, f(I_{n+1}), \ldots, f^{q_n-1}(I_n+1)\}$$

is called the *n-th dynamical partition of f*.

The following standard fact is easy to verify:

PROPOSITION 2.2. *The intervals of* $\mathcal{P}_n(f)$ *are disjoint except at the endpoints. Moreover, they cover the whole circle* $\mathbb{T}$.

Yoccoz [17] demonstrated:

THEOREM 2.3. *Suppose* $\rho(f) \notin \mathbb{Q}$. *Then*

$$\max_{J \in \mathcal{P}_n(f)} |J| \underset{n\to\infty}{\longrightarrow} 0.$$

This implies, in particular:

THEOREM 2.4. *Suppose* f *is a muticritical circle map with* $\rho(f) \notin \mathbb{Q}$. *Then* f *is topologically conjugate to the rigid rotation* $R_{\rho(f)}$.

A stronger statement is also true [4]:

THEOREM 2.5. *Suppose* f *has an irrational rotation number. Then the iterate* $f^{q_n}|_{I_{n+1}}$ *decomposes as*

$$f^{q_n}|_{I_{n+1}} = \psi_{m+1} \circ p_m \circ \psi_m \circ p_{m-1} \circ \cdots \circ \psi_1 \circ p_0 \circ \psi_0$$

where:

- $m \le l+1$, *where* l *is the number of the non-zero critical points of* f;
- p_j *is a power law* $p_j(x) = x^{b_j}$, $b_j \in 2\mathbb{Z}+1$;
- ψ_j *is an interval diffeomorphism with beau-bounded distortion.*

Let ϕ be a conjugacy $\phi \circ f \circ \phi^{-1} = R_{\rho(f)}$. As before, denote $c_0 = 0, c_1, \ldots, c_l$ the critical points of f, listed in the counter-clockwise order. Let k_j denote the order of the critical point c_j, and set

$$\mathcal{K}(f) = \{k_0, \ldots, k_l\}.$$

The points $\phi(c_j)$ divide the circle into $l+1$ arcs, let $a_0, a_1, \ldots, a_l$ be their lengths listed in the counter-clockwise order, starting from the point $\phi(0)$ and set

$$\mathcal{A}(f) = \{a_0, \ldots, a_l\}.$$

We say that $\mathcal{C}(f) = (\rho(f), \mathcal{K}(f), \mathcal{A}(f))$ is *the marked combinatorial type of* f. We say that f and g have *the same (unmarked) combinatorial type* if $\mathcal{C}(f) = \sigma\mathcal{C}(g)$, where σ is a cyclical permutation, and $\sigma\mathcal{C}(g) \equiv (\rho(f), \sigma\mathcal{K}(g), \sigma\mathcal{A}(g))$.

For a commuting pair ζ with $\rho(\zeta) \notin \mathbb{Q}$ we will defined the marked combinatorial type

$$\mathcal{C}(\zeta) = \mathcal{C}(f^{\phi}), \text{ where } f^{\phi} \text{ is given by (2.3),}$$

and similarly for the unmarked combinatorial type.

To define the n-th dynamical partition $\mathcal{P}_n(\zeta)$ for an n-times renormalizable commuting pair $\zeta = (\eta, \xi)$, we simply take the n-th dynamical partition of a map f^ϕ as in (2.3), and pull it back to $I_\eta \cup I_\xi$.

Both for multicritical circle maps and for commuting pairs we will denote by $\overline{\mathcal{P}_n}$ the set of boundary points of the n-th dynamical partition.

Theorem 2.5 implies:

THEOREM 2.6. *Suppose f has an irrational rotation number. Let J_1, J_2 be two adjacent intervals of the n-th dynamical partition $\mathcal{P}_n(f)$. Then J_1, J_2 are K-commensurable where $K = K(l, \max_{0\leq j\leq l} k_j)$ is beau.*

2.4. Periodic renormalization types for bi-critical circle maps Let us now specialize to the case when the map f has only two critical points: 0 and c with orders k_0 and k_1 respectively. Suppose $\rho(f)$ is irrational. We say that $\rho(f)$ is of periodic type if it is a periodic point of the Gauss map G. Let p be the period of $\rho(f)$, and assume that the orbit of c_1 never falls into the critical point 0. That ensures that for each $j \in \mathbb{N}$ the pair of maps $p\mathcal{R}^j f$ has exactly two critical points: 0 and $c_j \neq 0$ with orders k_0, k_1.

DEFINITION 2.4. We say that f is of a *periodic marked combinatorial type* with period n, where $p|n$, if $\mathcal{C}(f) = \mathcal{C}(R^n f)$.

An equivalent way of defining a periodic marked combinatorial type uses dynamical partitions as follows; it will be useful in the applications.

DEFINITION 2.5. Consider the 2-nd dynamical partition of the renormalization $\mathcal{R}^{j+m}(f)$ for $0 \leq m \leq p-1$. It has v_m intervals, let us enumerate them $P_1, \ldots, P_{v_m}$ from left to right. Let $i = w_{j,m}$ denote the number of the interval P_i which contains c_{j+m}. We say that f is of *a periodic marked combinatorial type* if:

(1) the numbers

$$w_m \equiv w_{j,m} \tag{2.7}$$

are independent of j;

(2) there exists $N \leq m$ such that for any j the two critical points of $\mathcal{R}^j f$ are separated by an interval of $\mathcal{P}_N(\mathcal{R}^j f)$.

The equivalence of the two definitions is a straightforward consequence of Theorem 2.3, and will be left to the reader.

Another straightforward consequence of Theorem 2.3 is the following:

PROPOSITION 2.7. *Suppose f and g both are of a periodic marked combinatorial type. Suppose $\rho(f) = \rho(g)$ and f and g have the same values of w_m (2.7). Then*

$$\mathcal{C}(f) = \mathcal{C}(g).$$

2.5. Renormalization theorems for bi-cubic critical circle maps In the following two statements we assume that the critical circle map has two critical points, both of which are of order 3. We call such maps *bi-cubic*.

Theorem 2.8. *Let $\mathcal{C}$ be a periodic marked combinatorial type with period m. Then there exists a commuting pair $\zeta_* \in C^\omega$ such that the following properties hold:*

(1) $\mathcal{C}(\zeta_*) = \mathcal{C}$;
(2) $\mathcal{R}^m(\zeta_*) = \zeta_*$;
(3) *for all $\zeta \in C^\omega$ such that $\mathcal{C}(\zeta) = \mathcal{C}$ we have*

$$\mathcal{R}^{mj}\zeta \underset{j\to\infty}{\longrightarrow} \zeta_*$$

in the uniform sense.

The next theorem talks about the hyperbolicity properties of the renormalization operator at the periodic point ζ_*. We will see in section 6 how to make a *canonical* choice of the analytic diffeomorphism $\phi = \phi_\zeta$ in (2.3), so that the correspondence

$$f \mapsto \zeta \equiv \mathcal{R}^N f \mapsto \phi_\zeta \circ F_\zeta \circ \phi_\zeta^{-1} \tag{2.8}$$

becomes a well-defined renormalization operator on the space of bi-cubic circle maps, rather than commuting pairs. We will then show:

Theorem 2.9. *There exists a Banach manifold structure $\mathbb{C}_U$ on the space of analytic bi-cubic critical circle maps, such that the following holds. The Banach structure is compatible with the topology of uniform convergence. The renormalization operator defined by (2.8) is an analytic operator, defined on an open set in this Banach manifold. The point $f_* = \phi_{\zeta_*} \circ \zeta_* \circ \phi_{\zeta_*}^{-1}$ is a hyperbolic periodic point of this operator, with two unstable eigendirections.*

3. Holomorphic Commuting Pairs Below we define holomorphic extensions of analytic commuting pairs which generalize de Faria's definition from [2] for the case of multicritical circle maps. We say that a multicritical commuting pair $\zeta = (\eta|_{I_\eta}, \xi|_{I_\xi})$ extends to a *holomorphic commuting pair* $\mathcal{H}$ if there exist three simply-connected $\mathbb{R}$-symmetric domains $D, U, V \subset \mathbb{C}$ whose intersections with the real line are denoted by $I_U = U \cap \mathbb{R}$, $I_V = V \cap \mathbb{R}$, $I_D = D \cap \mathbb{R}$, and a simply connected $\mathbb{R}$-symmetric Jordan domain Δ, such that

- the endpoints of I_U, I_V are critical points of η, ξ respectively;
- $\bar{D}$, $\bar{U}$, $\bar{V} \subset \Delta$; $\bar{U} \cap \bar{V} = \{0\} \subset D$; the sets $U \setminus D$, $V \setminus D$, $D \setminus U$, and $D \setminus V$ are nonempty, connected, and simply-connected; $I_\eta \subset I_U \cup \{0\}$, $I_\xi \subset I_V \cup \{0\}$;
- the sets $U \cap \mathbb{H}, V \cap \mathbb{H}, D \cap \mathbb{H}$ are Jordan domains;

- the maps η and ξ have analytic extensions to U and V respectively, so that η is a branched covering map of U onto $(\Delta \setminus \mathbb{R}) \cup \eta(I_U)$, and ξ is a branched covering map of V onto $(\Delta \setminus \mathbb{R}) \cup \xi(I_V)$, with all the critical points of both maps contained in the real line;
- the maps $\eta\colon U \to \Delta$ and $\xi\colon V \to \Delta$ can be further extended to analytic maps $\hat{\eta}\colon U \cup D \to \Delta$ and $\hat{\xi}\colon V \cup D \to \Delta$, so that the map $\nu = \hat{\eta} \circ \hat{\xi} = \hat{\xi} \circ \hat{\eta}$ is defined in D and is a branched covering of D onto $(\Delta \setminus \mathbb{R}) \cup \nu(I_D)$ with only real branched points.

We shall identify a holomorphic pair $\mathcal{H}$ with a triple of maps $\mathcal{H} = (\eta, \xi, \nu)$, where $\eta\colon U \to \Delta$, $\xi\colon V \to \Delta$ and $\nu\colon D \to \Delta$. We shall also call ζ the *commuting pair underlying* $\mathcal{H}$, and write $\zeta \equiv \zeta_{\mathcal{H}}$. When no confusion is possible, we will use the same letters η and ξ to denote both the maps of the commuting pair $\zeta_{\mathcal{H}}$ and their analytic extensions to the corresponding domains U and V.

The sets $\Omega_{\mathcal{H}} = D \cup U \cup V$ and $\Delta \equiv \Delta_{\mathcal{H}}$ will be called *the domain* and *the range* of a holomorphic pair $\mathcal{H}$. We will sometimes write Ω instead of $\Omega_{\mathcal{H}}$, when this does not cause any confusion.

We can associate to a holomorphic pair $\mathcal{H}$ a piecewise defined map $S_{\mathcal{H}}\colon \Omega \to \Delta$:

$$S_{\mathcal{H}}(z) = \begin{cases} \eta(z), & \text{if } z \in U, \\ \xi(z), & \text{if } z \in V, \\ \nu(z), & \text{if } z \in \Omega \setminus (U \cup V). \end{cases}$$

De Faria [2] calls $S_{\mathcal{H}}$ the *shadow* of the holomorphic pair $\mathcal{H}$.

We can naturally view a holomorphic pair $\mathcal{H}$ as three triples

$$(U, \xi(0), \eta),\ (V, \eta(0), \xi),\ (D, 0, \nu).$$

We say that a sequence of holomorphic pairs converges in the sense of Carathéodory convergence, if the corresponding triples do. We denote the space of triples equipped with this notion of convergence by $\mathbf{H}$.

We let the modulus of a holomorphic commuting pair $\mathcal{H}$, which we denote by $\mathrm{mod}(\mathcal{H})$ to be the modulus of the largest annulus $A \subset \Delta$, which separates $\mathbb{C} \setminus \Delta$ from $\overline{\Omega}$.

DEFINITION 3.1. For $\mu \in (0,1)$ let $\mathbf{H}(\mu) \subset \mathbf{H}$ denote the space of holomorphic commuting pairs $\mathcal{H} : \Omega_{\mathcal{H}} \to \Delta_{\mathcal{H}}$, with the following properties:

(1) $\mathrm{mod}(\mathcal{H}) \geq \mu$;
(2) $|I_\eta| = 1$, $|I_\xi| \geq \mu$ and $|\eta^{-1}(0)| \geq \mu$;
(3) $\mathrm{dist}(\eta(0), \partial V_{\mathcal{H}})/\operatorname{diam} V_{\mathcal{H}} \geq \mu$ and $\mathrm{dist}(\xi(0), \partial U_{\mathcal{H}})/\operatorname{diam} U_{\mathcal{H}} \geq \mu$;
(4) the domains $\Delta_{\mathcal{H}}$, $U_{\mathcal{H}} \cap \mathbb{H}$, $V_{\mathcal{H}} \cap \mathbb{H}$ and $D_{\mathcal{H}} \cap \mathbb{H}$ are $(1/\mu)$-quasidisks.
(5) $\mathrm{diam}(\Delta_{\mathcal{H}}) \leq 1/\mu$;

Let the *degree* of a holomorphic pair $\mathcal{H}$ denote the maximal topological degree of the covering maps constituting the pair. Denote $\mathbf{H}^K(\mu)$ the subset of $\mathbf{H}(\mu)$

consisting of pairs whose degree is bounded by K. The following is an easy generalization of Lemma 2.17 of [16]:

Lemma 3.1. *For each* $K \geq 3$ *and* $\mu \in (0,1)$ *the space* $\mathbf{H}^K(\mu)$ *is sequentially compact.*

We say that a real commuting pair $\zeta = (\eta, \xi)$ with an irrational rotation number has *complex* a priori *bounds*, if there exists $\mu > 0$ such that all renormalizations of $\zeta = (\eta, \xi)$ extend to holomorphic commuting pairs in $\mathbf{H}(\mu)$. The existence of complex *a priori* bounds is a key analytic issue of renormalization theory.

Definition 3.2. For a set $S \subset \mathbb{C}$, and $r > 0$, we let $N_r(S)$ stand for the r-neighborhood of S in $\mathbb{C}$. For each $r > 0$ we introduce a class $\mathcal{A}_r$ consisting of pairs (η, ξ) such that the following holds:

- η, ξ are real-symmetric analytic maps defined in the domains

$$N_r([0,1]) \text{ and } N_{r|\eta(0)|}([0,\eta(0)])$$

respectively, and continuous up to the boundary of the corresponding domains;
- the pair

$$\zeta \equiv (\eta|_{[0,1]}, \xi|_{[0,\eta(0)]})$$

is a multicritical commuting pair.

For simplicity, if ζ is as above, we will write $\zeta \in \mathcal{A}_r$. But it is important to note that viewing our multicritical commuting pair ζ as an element of $\mathcal{A}_r$ imposes restrictions on where we are allowed to iterate it. Specifically, we view such ζ as undefined at any point $z \notin N_r([0,\xi(0)]) \cup N_r([0,\eta(0)])$ (even if ζ can be analytically continued to z). Similarly, when we talk about iterates of $\zeta \in \mathcal{A}_r$ we iterate the restrictions $\eta|_{N_r([0,\xi(0)])}$ and $\xi|_{N_{r|\eta(0)|}([0,\eta(0)])}$. In particular, we say that the first and second elements of $p\mathcal{R}\zeta = (\eta^r \circ \xi, \eta)$ are defined in the maximal domains, where the corresponding iterates are defined in the above sense.

For a domain $\Omega \subset \mathbb{C}$, we denote by $\mathfrak{D}(\Omega)$ the complex Banach space of analytic functions in Ω, continuous up to the boundary of Ω and equipped with the sup norm. Consider the mapping $i\colon \mathcal{A}_r \to \mathfrak{D}(N_r([0,1])) \times \mathfrak{D}(N_r([0,1]))$ defined by

$$i(\eta,\xi) = (\eta, h \circ \xi \circ h^{-1}), \quad \text{where} \quad h(z) = z/\eta(0). \tag{3.1}$$

The map i is injective, since $\eta(0)$ completely determines the rescaling h. Hence, this map induces a metric on $\mathcal{A}_r$ from the direct product of sup-norms on $\mathfrak{D}(N_r([0,1])) \times \mathfrak{D}(N_r([0,1]))$. This metric on $\mathcal{A}_r$ will be denoted by $\operatorname{dist}_r(\cdot,\cdot)$.

We will denote $\mathcal{A}_r^L$ the subset of $\mathcal{A}_r$ consisting of pairs whose degree is bounded by L.

Theorem 3.2 (**Complex bounds for periodic type**). *Let* $\mathcal{C}$ *be a periodic combinatorial type and* $L \geq 3$. *There exists a constant* $\mu > 0$ *such that the following holds. For every positive real number* $r > 0$ *and every pre-compact family*

$S \subset \mathcal{A}_r^L$ of multicritical commuting pairs, there exists $N = N(r, S) \in \mathbb{N}$ such that if $\zeta \in S$ is a commuting pair with $\mathcal{C}(\zeta) = \mathcal{C}$, then $p\mathcal{R}^n\zeta$ restricts to a holomorphic commuting pair $\mathcal{H}_n : \Omega_n \to \Delta_n$ with $\Delta_n \subset N_r(I_\eta) \cup N_r(I_\xi)$. Furthermore, the range Δ_n is a Euclidean disk, and the appropriate affine rescaling of $\mathcal{H}_n$ is in $\mathbf{H}(\mu)$.

In the next section we will give a proof of this theorem which generalizes our proof in [14]. For simplicity of notation, we will assume that the pair ζ is a pre-renormalization of a multicritical circle map f, that is

$$\zeta = (f^{q_n}, f^{q_{n+1}}).$$

This will allow us to write explicit formulas for long compositions of terms η and ξ, which will greatly streamline the exposition. Theorem 3.2 follows from Theorem 4.1 which we state and prove in the following section.

4. Complex *a priori* Bounds

4.1. Power law estimate Let f be an analytic multicritical circle map with a periodic combinatorial type $\mathcal{C}$ as above. Let U be a $\mathbb{T}$-symmetric annulus which is contained in the domain of analyticity of f. Suppose 0 is a critical point of f of an odd integer order $k = 2m + 1$ and denote $\mathcal{R}^n f$ the n-th renormalization of f at 0 as above. Note that in what follows, rather than working with the map of the circle f, we will work with the appropriately translated lifts of f (2.4), so all of our estimates will be for maps defined in a neighborhood of the real line rather than $\mathbb{T}$.

Given an interval $J \subset \mathbb{R}$, let $\mathbb{C}_J \equiv \mathbb{C} \setminus (\mathbb{R} \setminus \mathrm{J})$ denote the plane slit along two rays. Let $\overline{\mathbb{C}_J}$ denote the completion of this domain in the path metric in $\mathbb{C}_J$ (which means that we add to $\mathbb{C}_J$ the banks of the slits).

The following theorem is a generalization of the main estimate of [14]:

THEOREM 4.1. *There exist positive constants R_0, b, c depending only on $\mathcal{C}$ such that the following holds. Let f be as above, and $R \geq R_0$. Then there exists $n_0 \in \mathbb{N}$ which depends on f and R such that for all $n \geq n_0$ we have the following.*

Denote

$$\mathcal{R}^n(f) \equiv (\eta, \xi).$$

There exist subdomains $V_\eta \supset \mathring{I}_\eta$, $V_\xi \supset \mathring{I}_\xi$ such that the map η is a branched covering

$$V_\eta \longrightarrow \mathbb{D}_R \cap \mathbb{C}_{\eta(I_\eta)},$$

and similarly for ξ. Furthermore, for any $z \in V_\eta \cup V_\xi$ we have

$$|\mathcal{R}^n f(z)| \geq c|z|^k + b, \tag{4.1}$$

where the left-hand side stands for η on V_η and ξ on V_ξ.

Finally, the constant n_0 can be chosen to depend only on R in any family of maps which is pre-compact in the compact-open topology on U.

Fix an integer $M > 0$ and $n \geq M$. Let

$$H_m \equiv [f^{q_{m+1}}(0), f^{q_m - q_{m+1}}(0)],$$

and let D_m denote the Euclidean disc $D(H_m)$. Assume that $D_M \subset U$. Consider the inverse orbit:

$$J_0 \equiv f^{q_{n+1}}(I_n), J_{-1} \equiv f^{q_{n+1}-1}(I_n), \ldots, J_{-(q_{n+1}-1)} \equiv f(I_n). \tag{4.2}$$

For a point $z \in D_M$ we say that

$$z_0 \equiv z, z_{-1}, \ldots, z_{-(q_{n+1}-1)} \tag{4.3}$$

is a *corresponding* inverse orbit if each $z_{-(k+1)}$ is obtained by applying to z_{-k} a univalent inverse branch of $f|_U$ where U is a sub-interval of $J_{-(k+1)}$.

We derive Theorem 4.1 from the following assertion:

LEMMA 4.2. *There exists $M > 0$ which depends only on f, and can be chosen uniformly in a compact family, such that for all $n > M$ and any $z \in \mathbb{C}_{f^{q_{n+1}}(I_n)} \cap D_M$ the following estimate holds:*

$$\frac{\operatorname{dist}(z_{-(q_{n+1}-1)}, f(I_n))}{|f(I_n)|} \leq C \frac{\operatorname{dist}(z, I_n)}{|I_n|}, \tag{4.4}$$

where z_n is as in (4.3) and C is a universal constant.

4.2. Poincaré neighborhoods By symmetry, J is a hyperbolic geodesic in $\mathbb{C}_J$. Consider the *hyperbolic neighborhood* of J of radius r that is the set of all points in $\mathbb{C}_J$ whose hyperbolic distance to J is less than r. One verifies directly that a hyperbolic neighborhood is the union of two Euclidean discs symmetric to each other with respect to the real axis, with a common chord J. We will denote such a neighborhood $D_\theta(J)$, where $\theta = \theta(r)$ is the outer angle the boundaries of the two discs form with the real axis. An elementary computation yields:

$$r = \log \cot(\theta/4). \tag{4.5}$$

Note, that in particular the Euclidean disc $D(J) \equiv D_{\pi/2}(J)$ can be interpreted as a hyperbolic neighborhood of J. These hyperbolic neighborhoods were introduced into the subject by Sullivan [12]. They are a key tool for proving complex bounds due to the following version of the Schwarz Lemma:

LEMMA 4.3. *Let us consider two intervals $J' \subset J \subset \mathbb{R}$. Let $\phi : \mathbb{C}_J \to \mathbb{C}_{J'}$ be an analytic map such that $\phi(J) \subset J'$. Then for any $\theta \in (0, \pi)$, $\phi(D_\theta(J)) \subset D_\theta(J')$.*

The above invariance statement becomes quasi-invariance if the domain of ϕ is not the whole double-slit plane, yet is large enough relative to the size of J. Let us state this as a Lemma below. Denote $\mathbb{D}_R = \{|z| < R\}$.

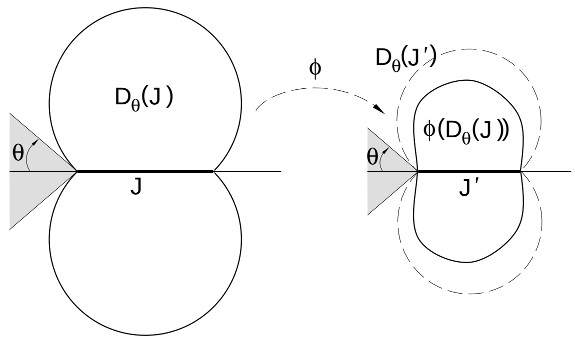

Figure 2: An illustration of Lemma 4.3

LEMMA 4.4. *For every sufficiently large $R > 0$ there exists $\theta(R) > 0$ with $\theta(R) \to 0$ and $(R\theta(R))^{-1} \to 0$ as $R \to \infty$ such that the following holds. Let*

$$\phi : \mathbb{D}_R \cap \mathbb{C}_{[-1,1]} \to \mathbb{C}$$

be a univalent and real-symmetric function satisfying $\phi(\pm 1) = \pm 1$. Then for every $\theta \geq \theta(R)$ we have

$$\phi(D_\theta([-1,1])) \subset D_{(1-R^{-1-\delta})\theta}([-1,1]),$$

where $\delta \in (0,1)$ is a fixed constant.

While superficially similar to [3, Lemma 2.4], it is a much stronger statement: we do not assume that ϕ is conformal in all of $\mathbb{D}_R$ (since this will clearly not be the case in our applications). Before proving this statement, let us formulate a lemma (see [13, Lemma3]):

LEMMA 4.5. *Suppose Ω is a Jordan subdomain of $\mathbb{D}$ such that*

$$\partial\Omega \subset \{|z| \geq 1 - \epsilon\} \text{ where } \epsilon > \frac{1}{2}.$$

There exists $C > 0$ such that the following holds. Let $h : \Omega \to \mathbb{D}$ be the Riemann mapping normalized by $h(0) = 0$ and $h'(0) > 0$. Then

$$|h^{-1}(z) - z| < C\epsilon,$$

for all $|z| < 1/2$.

Combining this with the Cauchy Integral Formula, we obtain

$$g'(0) < 1 + D\epsilon, \tag{4.6}$$

where D is a universal constant.

PROOF OF LEMMA 4.4. We start with an exercise in conformal mapping. Let $G(z)$ denote the branch of $\sqrt{z}$ which is defined in $\mathbb{C} \setminus \mathbb{R}_{<0}$ and preserves $\mathbb{R}_{>0}$; and let

$$F(z) = \frac{1-z}{1+z}.$$

Then, the composition $\Psi \equiv -F \circ G \circ F$ is a conformal mapping of $\mathbb{C}_{[-1,1]}$ onto $\mathbb{D}$ which preserves the interval $[-1,1]$. Easy estimates show that the image of $\partial\mathbb{D}_R$ under Ψ is contained in ϵ-neighborhoods of $\pm i$ with $\epsilon = O(R^{-1})$. Let us denote

$$\Omega \equiv \Psi(\mathbb{D}_R \cap \mathbb{C}_{[-1,1]}).$$

The hyperbolic metric of Ω at a point a is obtained by the pull-back of the Euclidean metric by the composition $h_a \circ M_a$ where M_a is the Möbius map $(z-a)/(1-\bar{a}z)$ and h_a is the conformal map of $\Omega_a \equiv M_a(\Omega)$ onto the unit disk with $h_a(0) = 0$, $h'_a(0) > 0$.

Fix $K > 0$. For z with $\text{dist}_{\bar{\mathbb{D}}}(z, [-1,1]) < K$,

$$\partial\Omega_a \subset \{|z| > \epsilon(a)\} \text{ with } \epsilon(a) = O(R^{-1})$$

(this estimate could be much improved for z near ± 1, but we will not require it). Using (4.6), we see that K-neighborhood of $[-1,1]$ in the hyperbolic metric of Ω is contained in the K'-neighborhood of $[-1,1]$ in the hyperbolic metric of $\mathbb{D}$ with

$$K' = K(1 + O(R^{-1})).$$

Together with 4.5 and the Schwarz Lemma, this implies the claim.

□

Easy induction combined with real *a priori* bounds implies (cf. [3, Lemma 2.5]):

LEMMA 4.6. *For each $n \geq 1$ there exist $K_n \geq 1$ and $\theta_n > 0$ with $K_n \to 1$ and $\theta_n \to 0$ as $n \to \infty$ such that the following holds. Let $\theta \geq \theta_n$, and let $0 \leq i < j \leq q_{n+1}$ be such that the restriction*

$$f^{j-i} : f^i(I_n) \to f^j(I_n)$$

is a diffeomorphism on the interior. Then the inverse branch $f^{-(j-i)}|_{f^j(I_n)}$ is well-defined over $D_\theta(f^j(I_n))$ and maps it univalently into the Poincaré neighborhood $D_{\theta/K_n}(f^i(I_n))$.

4.3. Proof of complex a priori bounds Let $J = [a,b]$, $a < b$. For a point $z \in \overline{\mathbb{C}}_J$, the *angle between z and J*, $\widehat{(z,J)}$ is the least of the angles between the intervals $[a,z]$, $[b,z]$ and the corresponding rays $(a,-\infty]$, $[b,+\infty)$ of the real line, measured in the range $0 \leq \theta \leq \pi$. In what follows, a beau bound $\epsilon > 0$ on the angle will appear; we will say that a point z_{-i} of (4.3) *ϵ-jumps* if $\widehat{(z_{-i}, J_{-i})} > \epsilon$. We will also say that such points have a *good angle*.

The next statement shows that for points with a good angle the normalized distance to the intervals of (4.2) cannot increase by more than a fixed multiple. It is taken from [14]. However, its proof is somewhat modified, since for a multicritical circle map the orbit (4.2) may pass through critical values.

LEMMA 4.7. *We work with the notations of Lemma* 4.2. *Fix* $\epsilon_1 > 0$, $B > 0$. *Let* $J = J_{-i}$, $J' = J_{-k}$ *be two intervals of the inverse orbit (*4.2*) for* $k > i$, *and let* z, z' *be the corresponding points of (*4.3*). Assume that* $\widehat{(z, J)} \geq \epsilon_1$ *and* $\operatorname{dist}(z, J) \geq B|J|$. *Then*

$$\frac{\operatorname{dist}(z', J')}{|J'|} \leq C \frac{\operatorname{dist}(z, J)}{|J|}$$

for some constant $C = C(\epsilon_1, B)$

PROOF. Suppose, f has l critical points, not counting 0. Since the orbit (4.2) forms a part of the dynamical partition of the circle of level n, there are at most l critical points for the iterate $f^{k-i}|_{J'}$. Hence, there exists a sub-interval $K' \subset J'$ which is commensurable with J' such that $f^{k-i} : K' \mapsto K$ is a diffeomorphism. By the same reasoning as in [14], the smallest closed hyperbolic neighborhood $\operatorname{cl} D_\theta(K)$ enclosing z satisfies $\operatorname{diam} D_\theta(K) \leq C(\epsilon, B) \operatorname{dist}(z, K)$. The claim now follows by Lemma 4.6. □

We now formulate an analogue of Lemma 4.2 of [14].

LEMMA 4.8. *Let* $J \equiv J_{-k}, J_{-k-q_{m+1}} \equiv J'$ *be two consecutive returns of the backward orbit (*4.2*) to* I_m, *for* $m \geq M$, *and let* ζ *and* ζ' *be the corresponding points of the orbit (*4.3*). Suppose* $\zeta \in D_m$, *then either* $\zeta' \in D_m$, *or* $\widehat{(\zeta', J')} > \epsilon$, *and* $\operatorname{dist}(\zeta', J') < C|I_m|$, *where the quantifiers* ϵ *and* C *are beau.*

PROOF. The proof is a very slight modification of the proof of Lemma 4.2 of [14], the only difference being that the iterate $f^{q_{m+1}}$ restricted to H_m may have at most l critical points in the interior. We refer the reader to Figure 3 for an illustration.

Let

$$\tilde{H}_m = f^{-(q_m - 1)}(H_m)$$

and let $\tilde{D}_m$ be the connected component of $f^{-(q_m-1)}(D_m)$ which intersects the real line over the interior of $\tilde{H}_m$. Note that the iterate f^{q_m-1} has at most l critical points in $\tilde{H}_m$ and they divide it into commensurable parts. Hence, by Lemma 4.6, the domain $\tilde{D}_m \subset D_{\theta_1}(\tilde{H}_m)$ where θ_1 is beau.

To fix the ideas, suppose that $f^{-q_m+1}(0) < f(0)$. Let $\hat{H}_m$ denote the interval $[f^{q_{m+1}}(0), 0]$ and let $\hat{D}_m$ be the domain which intersects the real line over the interior of $\hat{H}_m$ and such that

$$f : \hat{D}_m \to \tilde{D}_m \setminus \mathbb{R}_{\geq f(0)}$$

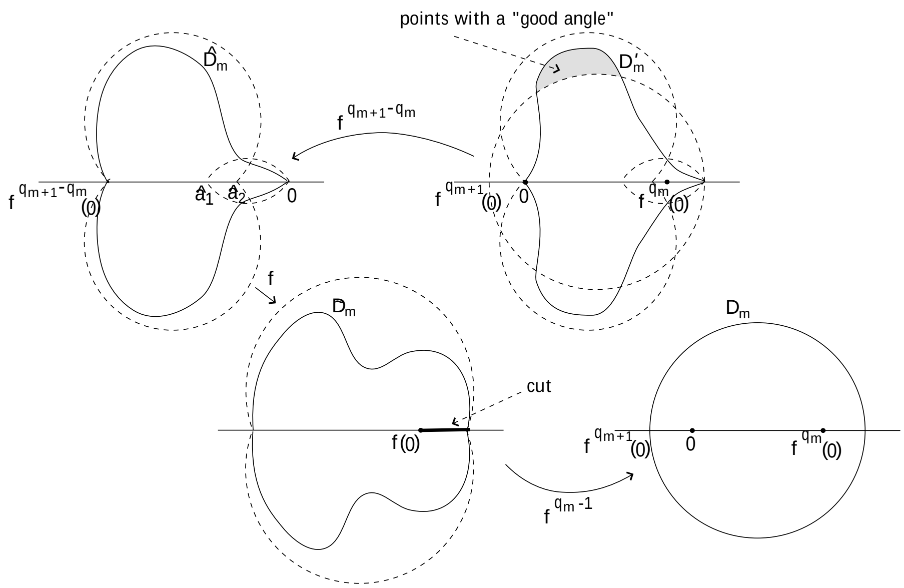

Figure 3: An illustration to the proof of Lemma 4.8

is a branched covering (this is a precise way of saying that $\hat{D}_m$ is a pull-back of $\tilde{D}_m$ by the "branch" of f^{-1} defined in $\mathbb{C}_{[f^{q_{m+1}-q_m+1}(0),f(0)]}$). Since f has a critical point of order $k \geq 3$ at 0, the boundary of $\hat{D}_m$ forms angles $\pi/k \leq \pi/3$ with the negative real axis. In particular, there exist $\tilde{a}_1$, $\tilde{a}_2$ in $\hat{H}_m$ which divide it in commensurable parts, so that

$$\hat{D}_m \subset D_{\theta_2}([f^{q_{m+1}-q_m}(0), \tilde{a}_2]) \cup D_{5\pi/8}([\tilde{a}_1, 0]),$$

where θ_2 is beau.

Again using Lemma 4.6 for the pull-back D'_m of $\hat{D}_m$ by the iterate $f^{q_{m+1}-q_m}$ along the real line, we see that there exist a_1, a_2 which divide $[0, f^{q_m-q_{m+1}}(0)]$ into commensurable sub-intervals such that

$$D'_m \subset D_{\theta_3}([0, a_2]) \cup D_{13\pi/24}([a_1, f^{q_m-q_{m+1}}(0)]),$$

where θ_3 is beau, and the proof is completed.

□

The next statement is a direct analogue of Lemma 4.4 of [14]. The proof from [14] is again adapted *mutatis mutandis* to our situation, in the same way as the preceding proof. We will leave the details to the reader.

LEMMA 4.9. *Let J be the last return of the backward orbit (4.2) to the interval I_m before the first return to I_{m+1}, and let J' and J'' be the first two returns of (4.2) to I_{m+1}. Let ζ, ζ', ζ'' be the corresponding moments in the backward orbit (4.3), $\zeta = f^{q_m}(\zeta')$, $\zeta' = f^{q_{m+2}}(\zeta'')$.*

Suppose $\zeta \in D_m$. Then either $\widehat{(\zeta'', I_{m+1})} > \epsilon$ and $\operatorname{dist}(\zeta'', J'') < C|I_{m+1}|$, or $\zeta'' \in D_{m+1}$; where the constants C and ϵ are beau.

PROOF OF LEMMA 4.2. We start with a point $z \in D_M$. Consider the largest m such that D_m contains z. We will carry out induction in m. Let $P_0, \ldots, P_{-k}$ be the consecutive returns of the backward orbit (4.2) to the interval I_m until the first return to I_{m+1}, and denote by $z = \zeta_0, \ldots, \zeta_{-k} = \zeta'$ the corresponding points of the orbit (4.3). Note that the intervals P_{-i} are commensurable with J_0. By Lemma 4.8 ζ_{-i} either ϵ-jumps and $\operatorname{dist}(\zeta_{-i}, P_{-i}) \leq C|I_m|$, or $\zeta' \in D_m$.

In the former case we are done by Lemma 4.7. In the latter case consider the point ζ'' which corresponds to the second return of the orbit (4.2) to I_{m+1}. By Lemma 4.9, either $\widehat{(\zeta'', I_{m+1})} > \epsilon$, and $\operatorname{dist}(\zeta'', I_{m+1}) \leq C|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.

In the first case we are done again by Lemma 4.7. In the second case, the argument is completed by induction in m.

□

5. Proof of Theorem 2.8

5.1. Quasiconformal conjugacies between holomorphic pairs Let us first quote the following (see [4,8]):

THEOREM 5.1. *Suppose ζ_1, ζ_2 are two commuting pairs with*

$$\rho(\zeta_1) = \rho(\zeta_2) \notin \mathbb{Q} \text{ and } \mathcal{C}(\zeta_1) = \mathcal{C}(\zeta_2).$$

Then for $n \geq 1$, the renormalizations $\mathcal{R}^n\zeta_i$ are K_n-quasi-symmetrically conjugate, and the constant K_n is beau.

Combining this with Theorem 3.2 and using the standard pull-back argument (see e.g. [2,14]), we obtain:

THEOREM 5.2. *Let $\mathcal{C}$ be a periodic combinatorial type and $L \geq 3$. There exists a constant $K > 0$ such that the following holds. For every positive real number $r > 0$ and every pre-compact family $S \subset \mathcal{A}_r^L$ of critical commuting pairs, there exists $N = N(r, S) \in \mathbb{N}$ and $K > 1$ such that the following holds. Let $\zeta_1, \zeta_2 \in S$ be two commuting pairs with*

$$\rho(\zeta_1) = \rho(\zeta_2) \notin \mathbb{Q} \text{ and } \mathcal{C}(\zeta_1) = \mathcal{C}(\zeta_2) = \mathcal{C},$$

and $\mathcal{H}_n^i$ is the holomorphic commuting pair extension of $p\mathcal{R}^n\zeta_i$ constructed in Theorem 3.2, then $\mathcal{H}_n^1$ is K-quasiconformally conjugate to $\mathcal{H}_n^2$.

For bi-cubic critical circle maps we can refine this statement further by showing that the quasiconformal conjugacy is actually conformal on the filled Julia set:

Theorem 5.3. *Suppose f is a bi-cubic critical circle map, $\rho(f) \notin \mathbb{Q}$, and let $\mathcal{H}$ be a holomorphic pair extension of $\mathcal{R}^n f$. Then any Beltrami differential μ which is $\mathbb{R}$-symmetric, $\mathcal{H}$-invariant, and with support $supp(\mu) \subset K(\mathcal{H})$ is necessarily trivial:*

$$\mu \underset{a.e.}{=} 0.$$

Theorem 2.8 follows from the two above statements and McMullen's Tower Rigidity argument (see [3, 16]). The proof of Theorem 5.3 will occupy the rest of this section.

5.2. Zakeri's models for bi-cubic circle maps Note that for a bi-cubic critical circle map f the combinatorial type $\mathcal{C}(f)$ is expressed by a single number. Consider the family of degree 5 Blaschke products

$$B(z) = e^{2\pi i t} z^3 \left(\frac{z-p}{1-\bar{p}z}\right)\left(\frac{z-q}{1-\bar{q}z}\right), \text{ where } |p|>1,\ |q|>1. \tag{5.1}$$

The following is shown in [18]:

Theorem 5.4. *Any Blaschke product which is topologically conjugate to an element of the family (5.1) by a conjugacy which preserves 0, ∞, and S^1 has the form $B \circ R$ where B is an element of the family (5.1) and $R(z) = e^{ir}z$, $r \in \mathbb{R}$.*

Theorem 5.5. *Let $\rho \notin \mathbb{Q}$ and $c \in (0,1)$. Then there exists a unique Blaschke product $B(z)$ of type (5.1) such that the following is true:*

- *$\rho(B|_{S^1}) = \rho$;*
- *$B(z)$ is a bi-cubic critical circle map (one of the critical points is at $e^{2\pi i 0} = 1$) and the unmarked combinatorial type of B is given by $(\rho, \{3,3\}, \{c, 1-c\})$.*

We now prove:

Lemma 5.6. *Theorem 5.3 holds under the assumption that $f = B$ is as in Theorem 5.5.*

Proof. Extend μ by the dynamics of B to a B-invariant, S^1-symmetric Beltrami differential supported on

$$\overline{\cup_{k\geq 0} B^{-k}(K(\mathcal{H}))} = J(B)$$

(the equality is implied by the standard properties of Julia sets, cf. e.g. [9]). Complete it to a B-invariant Beltrami differential λ_1 in $\hat{\mathbb{C}}$ by setting it equal to σ_0 on the complement of the Julia set of B. For $t \in [0,1]$, set

$$\lambda_t = t\lambda_1,$$

so that $\lambda_0 = \sigma_0$. By Ahlfors-Bers-Boyarski Theorem, there exists a unique solution $\psi_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the Beltrami equation

$$\psi_t^* \sigma_0 = \lambda_t$$

which fixes the points 0, 1, and ∞ and continuously depends on t. By S^1-symmetry, the rational map

$$G \equiv \psi_t \circ B \circ \psi_t^{-1}$$

is a Blaschke product. By Theorem 5.4 and Theorem 5.5,

$$G = B.$$

Hence, for every t and m, the map ψ_t permutes the discrete set $B^{-m}(1)$. Since $\psi_0 = \mathrm{Id}$ and ψ_t continuously depends on t, ψ_t fixes the points in each $B^{-m}(1)$. By the standard properties of Julia sets,

$$\overline{\cup_{m \geq 0} B^{-m}(1)} = J(f).$$

Hence, for all $t \in [0, 1]$,

$$\psi_t|_{J(B)} = \mathrm{Id}.$$

By Bers Sewing Lemma,

$$\bar{\partial}\psi_t = t\lambda_1 = 0 \text{ a.e.}$$

□

PROOF OF THEOREM 5.3. By Theorem 5.5, and complex *a priori* bounds, there exists a critical circle map B such that the renormalization $\mathcal{R}^k(B)$ extends to a holomorphic commuting pair $\mathcal{G}$ which is K-quasiconformally conjugate to $\mathcal{H}$. Denote ψ such a conjugacy. The claim will follow from the above considerations if we can show that $\bar{\partial}\psi = 0$ a.e. on $K(\mathcal{H})$. Indeed, assume the contrary. Then $(\psi^{-1})^* \sigma_0$ is a non-trivial invariant Beltrami differential on $K(\mathcal{G})$, which contradicts Lemma 5.6. □

6. Cylinder Renormalization of Bi-cubic Maps

6.1. Some functional spaces Firstly, let us denote π the natural projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. For an equatorial annulus $U \subset \mathbb{C}/\mathbb{Z}$ let $\mathbf{A}_U$ be the space of bounded analytic maps $\phi : U \to \mathbb{C}/\mathbb{Z}$ continuous up to the boundary, such that $\phi(\mathbb{T})$ is homotopic to $\mathbb{T}$, equipped with the uniform metric. We shall turn $\mathbf{A}_U$ into a real-symmetric complex Banach manifold as follows. Denote $\tilde{U}$ the lift $\pi^{-1}(U) \subset \mathbb{C}$. The space of functions $\tilde{\phi} : \tilde{U} \to \mathbb{C}$ which are analytic, continuous up to the boundary, and 1-periodic, $\tilde{\phi}(z+1) = \tilde{\phi}(z)$, becomes a Banach space when endowed with the sup norm. Denote that space $\tilde{\mathbf{A}}_U$. For a function $\phi : U \to \mathbb{C}/\mathbb{Z}$ denote $\check{\phi}$ an arbitrarily chosen lift $\pi(\check{\phi}(\pi^{-1}(z))) = \phi$. Observe that $\phi \in \mathbf{A}_U$ if and only if $\tilde{\phi} = \check{\phi} - \mathrm{Id} \in \tilde{\mathbf{A}}_U$. We use the local homeomorphism between $\tilde{\mathbf{A}}_U$ and $\mathbf{A}_U$ given by

$$\tilde{\phi} \mapsto \pi \circ (\tilde{\phi} + \mathrm{Id}) \circ \pi^{-1}$$

to define the atlas on $\mathbf{A}_U$. The coordinate change transformations are given by $\tilde{\phi}(z) \mapsto \tilde{\phi}(z+n)+m$ for $n, m \in \mathbb{Z}$, therefore with this atlas $\mathbf{A}_U$ is a real-symmetric complex Banach manifold.

Let us denote $\mathbf{f}$ the critical circle map

$$\mathbf{f}(z) = z - \frac{1}{2\pi}\sin(2\pi z) \bmod \mathbb{Z}.$$

Definition 6.1. Let $\mathfrak{U} = (U_1, U_2, U_3)$ be an ordered triple of $\mathbb{R}/\mathbb{Z}$-symmetric equatorial annuli. We denote $\mathbf{C}_\mathfrak{U}$ the space of triples $\phi_i \in \mathbf{A}_{U_i}$, $i = 1, 2, 3$ such that:

- $\mathbf{f} \circ \phi_1(U_1) \Subset U_2$ and $\mathbf{f} \circ \phi_2 \circ \mathbf{f} \circ \phi_1(U_1) \Subset U_3$;
- $\phi_1(0) = 0$;
- and each ϕ_i is univalent in some neighborhood of the circle.

Clearly, $\mathbf{C}_\mathfrak{U}$ is an open subset of $\mathbf{A}_{U_1} \times \mathbf{A}_{U_2} \times \mathbf{A}_{U_3}$, and thus possesses a natural product Banach manifold structure. Note that the composition

$$g_{(\phi_i)} \equiv \phi_3 \circ \mathbf{f} \circ \phi_2 \circ \mathbf{f} \circ \phi_1 : U_1 \to U_3 \tag{6.1}$$

is a bi-cubic circle map in $\mathbf{A}_{U_1}$. We note:

Proposition 6.1. *The correspondence* $\Gamma : \mathbf{C}_\mathfrak{U} \to \mathbf{A}_{U_1}$*, given by*

$$(\phi_i) \mapsto g_{(\phi_i)}$$

is analytic.

Definition 6.2. Given an element $\mathbf{v} \in \mathbf{C}_\mathfrak{U}$ we say that it is *cylinder renormalizable* with period k if the following holds. Denote $f = \Gamma(\mathbf{v})$. Then,

- there exists $m \in \mathbb{N}$ such that the rotation number $\rho(f)$ has at least $m+1$ digits in its continued fraction expansion, and $k = q_m$;
- there exist repelling periodic points p_1, p_2 of f in U_1 with periods k and a simple arc l connecting them such that $f^k(l)$ is a simple arc, and $f^k(l) \cap l = \{p_1, p_2\}$;
- the iterate f^k is defined and univalent in the domain C_f bounded by l and $f^k(l)$, the corresponding inverse branch $f^{-k}|_{f^k(C_f)}$ univalently extends to C_f; and the quotient of $\overline{C_f \cup f^k(C_f)} \setminus \{p_1, p_2\}$ by the action of f^k is a Riemann surface conformally isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ (we will call a domain C_f with these properties a *fundamental crescent of* f^k);

- for a point $z \in \bar{C}_f$ with $\{f^j(z)\}_{j\in\mathbb{N}} \cap \bar{C}_f \neq \varnothing$, set $R_{C_f}(z) = f^{n(z)}(z)$ where $n(z) \in \mathbb{N}$ is the smallest value for which $f^{n(z)}(z) \in \bar{C}_f$. We further require that there exists a point c in the domain of R_{C_f} such that $f^m(c) = 0$ for some $m < n(c)$.

Denote $\hat{f}$ the projection of R_{C_f} to $\mathbb{C}/\mathbb{Z}$ with $c \mapsto 0$. Evidently, it is a bi-cubic circle map. We will say that $\hat{f}$ is a *cylinder renormalization* of f with period k.

The relation of the cylinder renormalization procedure to critical circle maps is easy to see:

PROPOSITION 6.2 ([15]). *Suppose f is a critical circle map with rotation number $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. Assume that it is cylinder renormalizable with period q_n. Then the corresponding renormalization $\hat{f}$ is also a critical circle map with rotation number $G^n(\rho(f))$. Also, $\mathcal{R}\hat{f}$ (in the sense of pairs) is analytically conjugate to $\mathcal{R}^{n+1}f$.*

The next property of cylinder renormalization is the following:

PROPOSITION 6.3 ([15]). *Let $f = \Gamma(\mathbf{v})$ be renormalizable with period $k = q_n$, with a cylinder renormalization $\hat{f}$ corresponding to the fundamental crescent C_f, and let W be any equatorial annulus compactly contained in the domain U_1 of f. Then there is an open neighborhood G of $\mathbf{v}$ such that every $\mathbf{u} \in G$ the map $g \equiv \Gamma(\mathbf{u})$ is renormalizable, with a fundamental crescent $C_g \subset U$ which depends continuously on $\mathbf{u}$ in the Hausdorff sense. Moreover, there exists a holomorphic motion $\chi_{\mathbf{u}} : \partial C_f \mapsto \partial C_g$ over G, such that $\chi_g(f(z)) = g(\chi_g(z))$. And finally, the renormalization $\hat{g}$ is contained in $\mathbf{A}_W$.*

To define cylinder renormalization as an operator acting on triples in $\mathbf{C}_{\mathcal{U}}$, let us first note the following simple statement:

PROPOSITION 6.4. *Set $\hat{\mathcal{U}} = (\hat{U}_1, \hat{U}_2, \hat{U}_3)$, and suppose, that there exists a triple $\hat{\mathbf{v}} \in \mathbf{C}_{\hat{\mathcal{U}}}$ such that $\Gamma(\hat{\mathbf{v}}) = \hat{f}$ as above. Then such an element $\hat{\mathbf{v}}$ is unique.*

The next proposition shows that $\mathbf{u} \mapsto \hat{g}$ is well-defined:

PROPOSITION 6.5. *In the notations of Proposition 6.3, let C_g' be a different family of fundamental crescents, which also depends continuously on $\mathbf{u} \in G$ in the Hausdorff sense. Then there exists an open neighborhood $G' \subset G$ of f, such that the cylinder renormalization of $\mathbf{u} \in G'$ corresponding to C_g' is also $\hat{g}$.*

A key property of the cylinder renormalization is the following:

PROPOSITION 6.6. *In the notations of Proposition 6.3, the dependence $\mathbf{u} \mapsto \hat{g}$ is an analytic map $G \to \mathbf{A}_W$.*

PROOF. By the Theorem of Bers and Royden [1], the holomorphic motion χ_g extends to a holomorphic motion $\chi_g : C_f \cup f^k(C_f) \to C_g \cup g^k(C_g)$ over a smaller open neighborhood G' with the same equivariance property. As shown in [10], the holomorphic motion induces an analytic family of quasiconformal maps

$$\Psi_g : \mathbb{C}/\mathbb{Z} = (C_f \cup f^k(C_f))/f^k \to \mathbb{C}/\mathbb{Z} = (C_g \cup g^k(C_g))/g^k.$$

Applying the theorem of Ahlfors and Bers, we see that the projection $\pi_g : \overline{C_g} \cup g^k(C_g) \to \mathbb{C}/\mathbb{Z}$ depends analytically on g. Let $D \subset C_f \cup f^{-k}(C_f)$ and $n \in \mathbb{N}$ be such that $W \Subset \pi_f(D)$, and $\hat{f} \in \mathbf{C}_W$ is the projection of the iterate $f^n|_D$. The iterate $g^n|_D$ projects to $\pi_g \circ g^n = \hat{g} \in \mathbf{C}_W$ for all g sufficiently close to f, and the claim follows. □

The following is an immediate corollary:

Proposition 6.7. *In the above notation, the correspondence* $\mathbf{v} \mapsto \hat{\mathbf{v}}$ *is a locally analytic operator from a neighborhood of* $\mathbf{v} \in \mathbf{C}_{\mathfrak{U}}$ *to* $\mathbf{C}_{\hat{\mathfrak{U}}}$.

Proposition 6.8. *Let f and g be two bi-critical circle maps which are analytically conjugate*

$$\phi \circ f = g \circ \phi$$

in a neighborhood U of the circle. Suppose that f is cylinder renormalizable with period $k = q_n$, and let C_f be a corresponding fundamental crescent of f. Suppose $C_f \cup f(C_f) \Subset U$. Then, g is also cylinder renormalizable with the same period and with a fundamental crescent $C_g = \phi(C_f)$; and denoting $\hat{f}$ and $\hat{g}$ the corresponding cylinder renormalizations of f and g respectively, we have

$$\hat{f} \equiv \hat{g}.$$

Proof. The domain C_g is a fundamental crescent by definition. Denote C'_f, C'_g the closures of the two fundamental crescents minus the endpoints. The analytic conjugacy ϕ projects to an analytic isomorphism

$$\hat{\phi} : C'_f \cup f^k(C'_f)/f^k \simeq \mathbb{C}/\mathbb{Z} \longrightarrow C'_g \cup g^k(C'_g)/g^k \simeq \mathbb{C}/\mathbb{Z},$$

which conjugates $\hat{f}$ to $\hat{g}$. By Liouville's Theorem, $\hat{\phi}$ is a translation; since it fixes the origin, it is the identity. □

6.2. Cylinder renormalization operator The cylinder renormalization of a bi-cubic commuting pair is defined in the same way as that of an analytic bi-cubic map.:

Definition 6.3. Let $\zeta = (\eta, \xi)$ be a bi-cubic holomorphic commuting pair which is at least n times renormalizable for some $n \in \mathbb{N}$. Denote D_η the domain of the pair ζ. Denote

$$\zeta_n = (\eta_n, \xi_n) \equiv p\mathcal{R}^n(\zeta) : D_{\eta_n} \to \mathbb{C}.$$

We say that ζ is cylinder renormalizable with period $k = q^n$ if:

- there exist two complex conjugate fixed points p_n^+, p_n^- of the map η_n with $p_n^\pm \in \pm\mathbb{H}$, and an $\mathbb{R}$-symmetric simple arc l connecting these points such that $l \subset D_{\eta_n}$ and $\eta_n(l) \cap l = \{p_n^+, p_n^-\}$;
- denoting C the $\mathbb{R}$-symmetric domain bounded by l and $\eta(l)$ we have $C \subset D_{\eta_n}$, and the quotient of $\overline{C \cup \eta(C)} \setminus \{p_n^+, p_n^-\}$ by the action of ζ_n is a Riemann surface homeomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ (that is, C is a fundamental crescent of η).

Clearly, the action of ζ induces a bi-critical circle map f with rotation number $\rho(\mathcal{R}^n\zeta)$ on the quotient

$$(\overline{C_\zeta \cup \eta(C_\zeta)} \setminus \{p_n^+, p_n^-\})/\zeta_n \simeq \mathbb{C}/\mathbb{Z};$$

we call f the cylinder renormalization of ζ.

THEOREM 6.9. *Fix a periodic marked combinatorial type $\mathcal{C}$, and let $f \in \mathbf{A}_U$ be a bi-cubic map of this type. Then, there exists $N = N(\mathcal{C}, U)$ such that the following holds. For every $n \geq N$, setting $k = q_n$, f is cylinder renormalizable with period k. Moreover, there exists a fundamental crescent $C_f \Subset U$ (which moves analytically with f) with period k such that $f^k(C_f) \subset U$, and, denoting $\hat{f}$ the corresponding cylinder renormalization of f, we have*

$$\hat{f} \in \mathbf{A}_U.$$

Similarly, let ζ be any bi-cubic holomorphic commuting pair of type $\mathcal{C}$. Then, there exists $k = k(\zeta) = q_n$ such that ζ is cylinder renormalizable with period k, and the cylinder renormalization is a bi-cubic map of type $\mathcal{C}$.

The proof is a direct consequence of complex *a priori* bounds, and follows *mutatis mutandis* the proof of Lemma 4.10 of [7], so we will not reproduce it here.

PROPOSITION 6.10. *Let $\mathcal{C}$ be a periodic marked type with period p and let ζ_* be a periodic point of $\mathcal{R}$ with this period. Then, there exist $n \in \mathbb{N}$, an equatorial annulus U_*, and a triple $\mathcal{U} = (U_*, U_2, U_3)$ such that the following holds:*

(1) *m such that ζ_* is cylinder renormalizable with period m and the renormalization $f_* \in \mathbf{A}_{U_*}$ has combinatorial type $\mathcal{C}$;*
(2) *the bi-cubic map f_* is cylinder renormalizable with period k; its renormalization is equal to f_* and is defined in $U_1 \Supset U_*$;*
(3) *the map f_* has a decomposition $\mathbf{v}_* \in \mathbf{C}_{\mathcal{U}}$.*

The last property implies that there exists a neighborhood $V(\mathbf{v}_*) \subset \mathbf{C}_{\mathcal{U}}$ such that cylinder renormalization with period k is an analytic operator to $\mathbf{C}_{\mathcal{U}}$. We call this operator *cylinder renormalization operator* corresponding to the type $\mathcal{C}$ and denote it

$$\mathcal{R}_{cyl} : V(\mathbf{v}_*) \longrightarrow \mathbf{C}_{\mathcal{U}}.$$

7. Constructing the Stable Manifold of $\mathbf{v}_*$ We will now specialize to real-symmetric maps, and consider the real slice $\mathbf{C}_{\mathcal{U}}^{\mathbb{R}}$ of the space $\mathbf{C}_{\mathcal{U}}$ consisting of maps with real symmetry, with the induced real Banach manifold structure (cf. [7,15]). We proceed with the above notation. Let ρ_* be the rotation number of the combinatorial class $\mathcal{C}$ and define

$$D_* = \{\mathbf{v} \in \mathbf{C}_{\mathcal{U}}^{\mathbb{R}},\ \text{such that } \rho(\Gamma(\mathbf{v})) = \rho_*\},$$

and

$$S_* = \{\mathbf{v} \in \mathbf{C}_{\mathcal{U}}^{\mathbb{R}},\ \text{such that } \mathcal{C}(\Gamma(\mathbf{v})) = \mathcal{C}\} \subset D_*.$$

As we have shown,

Proposition 7.1. *There exists a neighborhood Y of $\mathbf{v}_*$ in $\mathbf{C}_{\mathcal{U}}^{\mathbb{R}}$ such that for every $\mathbf{v} \in Y \cap S_*$ $\mathcal{R}_{cyl}^j(\mathbf{v})$ is defined for all j, and*

$$\mathcal{R}_{cyl}^j(\mathbf{v}) \longrightarrow \mathbf{v}_*$$

uniformly in Y.

Below we shall demonstrate that the local stable set of $\mathbf{v}_*$ is a submanifold of $\mathbf{C}_{\mathcal{U}}^{\mathbb{R}}$:

Theorem 7.2. *There is an open neighborhood $W \subset \mathbf{C}_{\mathcal{U}}^{\mathbb{R}}$ of $\mathbf{v}_*$ such that $S_* \cap W$ is a smooth submanifold of $\mathbf{C}_{\mathcal{U}}^{\mathbb{R}}$ of codimension 2. Moreover, for $\mathbf{v} \in S_*$ denote $f = \Gamma(\mathbf{v})$, and let ϕ_f be the unique smooth conjugacy between f and f_* which fixes 0. Then the dependence $\mathbf{v} \mapsto \phi_f$ is smooth.*

Our first step will be to show that

Theorem 7.3. *The set D_* is a local submanifold of $\mathbf{C}_{\mathcal{U}}^{\mathbb{R}}$ of codimension 1.*

As before denote p_k/q_k the reduced form of the k-th continued fraction convergent of ρ_*. Furthermore, define D_k as the set of $\mathbf{v} \in \mathbf{C}_{\mathcal{U}}$ for which $\rho(g) = p_k/q_k$ and 0 is a periodic point of g with period q_k, where $g = \Gamma(\mathbf{v})$. As follows from the Implicit Function Theorem, this is a codimension 1 submanifold of $\mathbf{C}_{\mathcal{U}}$. Let us define a cone field $\mathcal{E}$ in the tangent bundle of $\mathbf{C}_{\mathcal{U}}$ given by the condition:

$$\mathcal{E} = \{\nu \in T\mathbf{C}_{\mathcal{U}} |\ \inf_{x \in \mathbb{R}} D\Gamma\nu(x) > 0\}.$$

Lemma 7.4. *Let $\mathbf{v} \in D_k$, and denote $T_{\mathbf{v}}D_k$ the tangent space to D_k at this point. Then $T_{\mathbf{v}}D_k \cap \mathcal{E} = \varnothing$.*

Proof. Let $\nu \in \mathcal{E}$ and suppose $\{\mathbf{v}_t\} \subset \mathbf{C}_{\mathcal{U}}$ is a one-parameter family such that

$$\mathbf{v}_t = \mathbf{v} + t\nu + o(t),$$

and set $f = \Gamma(\mathbf{v})$ and $f_t = \Gamma(\mathbf{v}_t)$, so that

$$\pi^{-1} \circ f_t \circ \pi = \pi^{-1} \circ f \circ \pi + tD\Gamma(\nu) + o(t).$$

Then for sufficiently small values of t, $\pi^{-1} \circ f_t \circ \pi > \pi^{-1} \circ f \circ \pi$. Hence $f_t^{q_k}(0) \neq f^{q_k}(0)$ and thus $f_t \notin D_k$. □

Elementary considerations of the Intermediate Value Theorem imply that for every k there exists a value of $\theta \in (0,1)$ such that the map $f_\theta = R_\theta \circ \Gamma(\mathbf{v}_*) \in D_k$. Moreover, if we denote θ_k the angle with the smallest absolute value satisfying this property, then $\theta_k \to 0$. Set

$$\mathbf{v}_k = (\phi_1, \phi_2, R_{\theta_k} \circ \phi_3).$$

For k large enough, $\mathbf{v}_k \in \mathbf{C}_{\mathcal{U}} \cap D_k$. Let $T_k = T_{\mathbf{v}_k} D_k$. Fix $v \in \mathcal{E}$. By Lemma 7.4 and the Hahn-Banach Theorem there exists $\epsilon > 0$ such that for every k there exists a linear functional $h_k \in (T\mathbf{C}_{\mathcal{U}})^*$ with $||h_k|| = 1$, such that $\operatorname{Ker} h_k = T_k$ and $h_k(v) > \epsilon$. By the Alaoglu Theorem, we may select a subsequence h_{n_k} weakly converging to $h \in (T\mathbf{C}_{\mathcal{U}})^*$. Necessarily $v \notin \operatorname{Ker} h$, so $h \not\equiv 0$. Set $T = \operatorname{Ker} h$.

Proof of Theorem 7.3. By the above, we may select a splitting $T\mathbf{C}_{\mathcal{U}} = T \oplus v \cdot \mathbb{R}$. Denote $p : T\mathbf{C}_{\mathcal{U}} \to T$ the corresponding projection, and let $\psi : \mathbf{C}_{\mathcal{U}} \to T\mathbf{C}_{\mathcal{U}}$ be a local chart at $\mathbf{v}_*$. Lemma 7.4 together with the Mean Value Theorem imply that $p \circ \psi : D_k \to T$ is an isomorphism onto the image, and there exists an open neighborhood $\mathcal{U}$ of the origin in T, such that $p \circ \psi(D_k) \supset \mathcal{U}$. Since the graphs D_k are analytic, we may select a C^1-converging subsequence D_{k_j} whose limit is a smooth graph G over $\mathcal{U}$. Necessarily, for every $g \in G$, $\rho(g) = \rho_*$. As we have seen above, every point $g \in D_*$ in a sufficiently small neighborhood of $\mathbf{v}_*$ is in G, and thus G is an open neighborhood in D_*. □

We now proceed to describing the manifold structure of $S_* \subset D_*$. Let $c_f \neq 0$ denote the non-zero critical point of f. Define a cone field $\mathcal{E}_1$ in $T\mathbf{C}_{\mathcal{U}}$ consisting of tangent vectors ν in $T_{\mathbf{v}}\mathbf{C}_{\mathcal{U}}$ such that setting $\upsilon = D\Gamma\nu$, we have

$$\upsilon'(c_f) > 0.$$

Furthermore, for $k > n+2$, let $D_{k,n} \subset D_k$ consist of maps f such that f has the same combinatorics as $\mathcal{E}$ up to the level n, and the n-th renormalization $\mathcal{R}^n(f)$ is uni-critical (that is, two critical orbits collide). Elementary considerations again imply that for every $f \in S_*$ there exists a sequence $\{f_{k,n(k)}\}$ converging to it.

Since the n-th renormalization of $f \in D_{k,n}$ is a uni-critical circle mapping, we can apply the results of [16] to conclude:

THEOREM 7.5. *All maps $f \in \Gamma(D_{k,n})$ are smoothly conjugate. Moreover, let $\hat{f}$ be an arbitrary base point in $D_{k,n}$. Then, for $f \in D_{k,n}$ the conjugacy ϕ_f which realizes*

$$\phi_f \circ f \circ \phi_f^{-1} = \hat{f}$$

and sends 0 to 0 can be chosen so that the dependence $f \mapsto \phi_f$ is analytic in S_n.

We note:

PROPOSITION 7.6. *The set $D_{k,n}$ is a local submanifold of D_k of codimension 1, and the cone field $\mathcal{E}_1$ lies outside of its tangent bundle.*

Proof. The first statement is a straightforward consequence of Implicit Function Theorem considerations. By Theorem 7.5, a vector field $\upsilon \in D\Gamma T_{\mathbf{v}} D_{k,n}$ satisfies the cohomological equation

$$\upsilon = \omega \circ f - f'\omega$$

where $f = \Gamma(\mathbf{v})$ and ω is a smooth vector field on the circle. In particular, $\upsilon'(c_f) = 0$. □

We also note:

Proposition 7.7. *The cone field $\mathcal{E}_1$ intersects $T_{\mathbf{v}} D_*$ at every point $\mathbf{v}$.*

Proof. Evidently, $\mathcal{E}_1$ has non-zero intersection with $\mathcal{E}$ and $-\mathcal{E}$ at every point $\mathbf{v}$. The claim follows. □

Applying the proof of Theorem 7.3 *mutatis mutandis* (replacing $D_k \to D_*$ with $D_{k,n(k)} \to S_*$), we obtain the statement of Theorem 7.2.

8. Hyperbolicity of $\mathbf{v}_*$

Let us denote

$$\mathcal{L} \equiv D\mathcal{R}_{cyl}|_{\mathbf{v}_*}.$$

We have (see [7]):

Proposition 8.1. *The linear operator $\mathcal{L}$ is compact.*

In this section we establish:

Theorem 8.2. *The linear operator $\mathcal{L}$ has two unstable eigendirections.*

Together with Theorem 7.2 this has the following corollary:

Theorem 8.3. *The renormalization fixed point $\mathbf{v}_*$ is hyperbolic. It has two unstable eigendirections. Its local stable manifold $W^s_{loc}(\mathbf{v}_*)$ coincides with S_* and is an analytic submanifold of codimension 2.*

The following was shown in [7] (see also [15]):

Proposition 8.4. *There exist constants $a_1 > 0$ and $\lambda_1 > 1$ such that for every $\nu \in \mathcal{E} \cap T_{\mathbf{v}} \mathbf{C}^{\mathbb{R}}_{\mathcal{U}}$ and $n \in \mathbb{N}$ we have*

$$||\mathcal{L}^n \nu|| > a_1 \lambda_1^n \inf_{x \in \mathbb{R}} D\Gamma\nu(x).$$

We also have:

Proposition 8.5. *There exist constants $a_2 > 0$ and $\lambda_2 > 1$ such that the following holds. Let $\nu \in T_{\mathbf{v}_*} D_* \cap \mathcal{E}_1$. Set $\upsilon = D\Gamma\nu$ and $\upsilon_n^{\pm} = D\Gamma\mathcal{L}^n(\pm\nu)$ for $n \in \mathbb{N}$. Then,*

$$\max(|(\upsilon_n^+)'(c_{\mathcal{R}^n_{cyl} f_*})|, |(\upsilon_n^-)'(c_{\mathcal{R}^n_{cyl} f_*})|) > a_2 \lambda_2^n |\upsilon'(c_{f_*})|.$$

PROOF. The composition $f \mapsto f^n$ transforms the vector field υ into $\hat{\upsilon}_n$. An easy induction shows that

$$\hat{\upsilon}_n'(c_{f_*}) = \frac{df_*^{n-1}}{dx}(f_*(c_{f_*}))\upsilon'(c_{f_*}).$$

By real *a priori* bounds, the derivative $\frac{df_*^{n-1}}{dx}(f_*(c_{f_*}))$ is uniformly bounded below by $C_0 > 0$ for $n = q_m$. Let Φ_n stand for the uniformizing coordinate of the fundamental crescent of the n-th cylinder renormalization of f_*. Then, either for $u = \upsilon_n^+$, or for $u = \upsilon_n^-$ (depending on the sign of the remainder terms), we can bound below the value of $|u'(c_{\mathcal{R}^n_{cyl} f_*})|$ by

$$\min_{x \in \mathbb{R}} \Phi_n'(x) C_0 |\upsilon'(c_{f_*})|.$$

By Koebe Distortion Theorem combined with real *a priori* bounds, $\min_{x\in\mathbb{R}} \Phi_n'(x)$ grows exponentially with n, and the claim follows.

$\square$

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