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Unimodal sequences show that Lambert W is Bernstein

G. A. Kalugin and D. J. Jeffrey

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Abstract. We consider a sequence of polynomials appearing in expressions for the derivatives of the Lambert W function. The coefficients of each polynomial are shown to form a positive sequence that is log-concave and unimodal. This property implies that the positive real branch of the Lambert W function is a Bernstein function.

Résumé. Nous considérons une séquence de polynômes que l'on retrouve dans l'expression des dérivées de la fonction Lambert W. Nous montrons que les coefficients de chaque polynôme forment une séquence positive qui est log-concave et unimodale. Cette propriété implique que la branche réelle positive de la fonction Lambert W est une fonction de Bernstein.

1. Introduction. The Lambert W function was defined and studied in [5]. It is a multivalued function having branches $W_k(z)$, each of which obeys $W_k \exp(W_k) = z$. The principal branch W_0 maps the set of positive reals to itself, and is the only branch considered here. Therefore we omit the subscript 0 for brevity. The n-th derivative of W is given implicitly by

$$\frac{d^n W(x)}{dx^n} = \frac{\exp\bigl(-nW(x)\bigr)p_n\bigl(W(x)\bigr)}{\bigl(1+W(x)\bigr)^{2n-1}} \quad \text{for } n \geq 1, \tag{1.1}$$

where the polynomials $p_n(w)$ satisfy $p_1(w) = 1$, and the recurrence relation

$$p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1 + w)p_n'(w) \quad \text{for } n \geq 1. \tag{1.2}$$

In [6], the first 5 polynomials were given explicitly:

$$p_1(w) = 1, \quad p_2(w) = -2 - w, \quad p_3(w) = 9 + 8w + 2w^2,$$
$$p_4(w) = -64 - 79w - 36w^2 - 6w^3,$$
$$p_5(w) = 625 + 974w + 622w^2 + 192w^3 + 24w^4,$$

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and the polynomial coefficients were listed in [11, A042977]. These initial cases suggest the conjecture that each polynomial $(-1)^{n-1}p_n(w)$ has all positive coefficients, and if this be true, then $dW(x)/dx$ is a completely monotonic function [12]. We here prove the conjecture and prove in addition that the coefficients are unimodal and log-concave.

2. **Formulae for the coefficients.**

In view of the conjecture, we write

$$p_n(w) = (-1)^{n-1}\sum_{k=0}^{n-1}\beta_{n,k}w^k. \tag{2.1}$$

We now give several theorems regarding the coefficients.

THEOREM 2.1. *The coefficients $\beta_{n,k}$ defined in (2.1) obey the recurrence relations*

$$\beta_{n,0} = n^{n-1}, \quad \beta_{n,1} = 3n^n - (n+1)^n - n^{n-1}, \tag{2.2}$$

$$\beta_{n,n-1} = (n-1)!, \quad \beta_{n,n-2} = (2n-2)(n-1)!, \tag{2.3}$$

$$\beta_{n+1,k} = (3n-k-1)\beta_{n,k} + n\beta_{n,k-1} - (k+1)\beta_{n,k+1}, \quad 2 \le k \le n-3. \tag{2.4}$$

PROOF. By substituting (2.1) into (1.2) and equating coefficients. □

THEOREM 2.2. *An explicit expression for the coefficients $\beta_{n,k}$ is*

$$\beta_{n,k} = \sum_{m=0}^{k}\frac{1}{m!}\binom{2n-1}{k-m}\sum_{q=0}^{m}\binom{m}{q}(-1)^q(q+n)^{m+n-1}. \tag{2.5}$$

PROOF. We rewrite (1.1) in the form

$$p_n\big(W(x)\big) = \big(1+W(x)\big)^{2n-1}e^{nW(x)}\frac{d^nW(x)}{dx^n}.$$

From the Taylor series of $W(x)$ around $x=0$, given in [5], we obtain

$$\frac{d^nW(x)}{dx^n} = \sum_{m=n}^{\infty}\frac{(-m)^{m-1}}{(m-n)!}x^{m-n}.$$

Substituting this into the expression of p_n, using $x = We^W$ and changing the index of summation, we obtain the equation

$$p_n(w) = (1+w)^{2n-1}\sum_{s=0}^{\infty}(-1)^{n+s-1}(n+s)^{n+s-1}\frac{w^s}{s!}e^{(n+s)w}. \tag{2.6}$$

We expand the right side around $w=0$ and equate coefficients of w. □

REMARK 2.3. The polynomials $p_n(w)$ can be expressed in terms of the diagonal Poisson transform $\mathbf{D}_n[f_s; z]$ defined in [10], namely, by (2.6)

$$p_n(w) = (-1)^{n-1}(1+w)^{2(n-1)}\mathbf{D}_n[(n+s)^{n-1}; -w]. \tag{2.7}$$

THEOREM 2.4. *The coefficients can equivalently be expressed either in terms of shifted r-Stirling numbers of the second kind* $\left\{ {n+r \atop m+r} \right\}_r$ *defined in* [3],

$$\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n-1}{k-m} \left\{ {2n-1+m \atop n+m} \right\}_n, \tag{2.8}$$

or in terms of Bernoulli polynomials of higher order $B_n^{(z)}(\lambda)$ *defined in* [9],

$$\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n-1}{k-m} \binom{m+n-1}{n-1} B_{n-1}^{(-m)}(n), \tag{2.9}$$

or in terms of the forward difference operator Δ [7, p. 188],

$$\beta_{n,k} = \sum_{m=0}^{k} \binom{2n-1}{k-m} \frac{(-1)^m}{m!} \Delta^m n^{m+n-1}.$$

PROOF. We convert (2.5) using identities found in [3] and [8] respectively.

$$\left\{ {n+r \atop m+r} \right\}_r = \frac{1}{m!} \sum_{q=0}^{m} (-1)^{m-q} \binom{m}{q} (q+r)^n$$

and

$$B_n^{(-m)}(r) = \frac{n!}{(m+n)!} \sum_{q=0}^{m} (-1)^{m-q} \binom{m}{q} (q+r)^{m+n}. \qquad \square$$

3. Properties of the coefficients.

We now give theorems regarding the properties of the $\beta_{n,k}$. We recall the following definitions [13]. A sequence $c_0, c_1, \ldots, c_n$ of real numbers is said to be *unimodal* if for some $0 \le j \le n$ we have $c_0 \le c_1 \le \cdots \le c_j \ge c_{j+1} \ge \cdots \ge c_n$, and it is said to be *logarithmically concave* (or log-concave for short) if $c_{k-1}c_{k+1} \le c_k^2$ for all $1 \le k \le n-1$. We prove that for each fixed n, the $\beta_{n,k}$ are unimodal and log-concave with respect to k. Since a log-concave sequence of positive terms is unimodal [16], it is convenient to start with the log-concavity property.

THEOREM 3.1. *For fixed* $n \ge 3$ *the sequence* $\{k!\,\beta_{n,k}\}_{k=0}^{n-1}$ *is log-concave.*

PROOF. Using (2.5) we can write

$$k!\,\beta_{n,k} = (2n-1)!\sum_{m=0}^{k}\binom{k}{m}x_m y_{k-m},$$

where

$$x_m = \sum_{j=0}^{m}\binom{m}{j}a_j, \quad a_j = (-1)^j(n+j)^{m+n-1}, \tag{3.1}$$

and $y_m = 1/(2n-1-m)!$. Since the binomial convolution preserves the log-concavity property [14], [15], it is sufficient to show that the sequences $\{x_m\}$ and $\{y_m\}$ are log-concave. We have

$$\begin{aligned} a_{j-1}a_{j+1} &= (-1)^{j-1}(n+j-1)^{m+n-1}(-1)^{j+1}(n+j+1)^{m+n-1} \\ &= (-1)^{2j}\big((n+j)^2-1\big)^{m+n-1} < (-1)^{2j}(n+j)^{2(m+n-1)} = a_j^2. \end{aligned}$$

Thus the sequence $\{a_j\}$ is log-concave and so is $\{x_m\}$, being itself a binomial convolution. The sequence $\{y_m\}$ is log-concave because

$$\begin{aligned} y_{m-1}y_{m+1} &= \frac{1}{(2n-1-m+1)!}\frac{1}{(2n-1-m-1)!} \\ &= \frac{2n-1-m}{2n-1-m+1}\frac{1}{(2n-1-m)!}\frac{1}{(2n-1-m)!} < y_m^2. \end{aligned}$$ □

Now we prove that the coefficients $\beta_{n,k}$ are positive. The following two lemmas are useful.

LEMMA 3.2. *If a positive sequence* $\{k!\,c_k\}_{k\geq 0}$ *is log-concave, then*
(i) $\{(k+1)c_{k+1}/c_k\}$ *is non-increasing;*
(ii) $\{c_k\}$ *is log-concave;*
(iii) *the terms* c_k *satisfy*

$$c_k c_m \geq \binom{k+m}{k}c_0 c_{k+m} \quad (0 \leq m \leq k+1). \tag{3.2}$$

PROOF. The statements (i) and (ii) are obvious. To prove (iii) we apply a method used in [1]. Specifically, by (i) we have for $0 \leq p \leq k$

$$\frac{c_{p+1}}{c_p} \geq \frac{k+p+1}{p+1}\frac{c_{k+p+1}}{c_k}.$$

Apply the last inequality for $p = 0, 1, 2, \ldots, m$ with $m \leq k+1$, and form the products of all left-hand and right-hand sides. As a result, after cancellation we obtain

$$\frac{c_m}{c_0} \geq \frac{k+1}{1}\frac{k+2}{2}\cdots\frac{k+m}{m}\frac{c_{k+m}}{c_k},$$

which is equivalent to (3.2). □

LEMMA 3.3. *If the coefficients $\beta_{n,k}$ are positive, then for fixed $n \geq 3$ they satisfy*

$$\frac{(k+1)\beta_{n,k+1}}{\beta_{n,k}} < n-1. \tag{3.3}$$

PROOF. Proof By Theorem 3.1 and under the assumption of lemma, for fixed $n \geq 3$ the sequence $\{k!\,\beta_{n,k}\}_{k=0}^{n-1}$ meets the conditions of Lemma 3.2. Applying the inequality (3.2) with $m = 1$ to this sequence gives $(k+1)\beta_{n,k+1}/\beta_{n,k} \leq \beta_{n,1}/\beta_{n,0}$. Then the lemma follows, since owing to (2.2)

$$\begin{aligned} \frac{\beta_{n,1}}{\beta_{n,0}} &= \frac{3n^n - (n+1)^n - n^{n-1}}{n^{n-1}} \\ &= 3n - n\Big(1 + \frac{1}{n}\Big)^n - 1 < 3n - 2n - 1 = n - 1. \end{aligned}$$ □

THEOREM 3.4. *The coefficients $\beta_{n,k}$ are positive.*

PROOF. We prove the statement by induction on n. It is true for $n \leq 5$ (see Section 1). Assume that for some fixed n all the members of the sequence $\{\beta_{n,k}\}_{k=0}^{n-1}$ are positive. Since $\beta_{n+1,0} = (n+1)^n > 0$ and $\beta_{n+1,n} = n! > 0$ by (2.2) and (2.3), we only need to consider $k = 1, 2, \ldots, n-1$.

Substituting inequalities

$$\beta_{n,k+1} < (n-1)\beta_{n,k}/(k+1) \text{ and } \beta_{n,k-1} > k\beta_{n,k}/(n-1),$$

which follow from (3.3), in the recurrence (2.4) immediately gives the result

$$\beta_{n+1,k} > (3n-k-1)\beta_{n,k} + n\frac{k}{n-1}\beta_{n,k} - (k+1)\frac{n-1}{k+1}\beta_{n,k} = \Big(2n + \frac{k}{n-1}\Big)\beta_{n,k} > 0.$$

Thus the proof by induction is complete. □

COROLLARY 3.5. *The sequence $\{\beta_{n,k}\}_{k=0}^{n-1}$ is unimodal for $n \geq 3$.*

PROOF. By Theorem 3.4 the sequence $\{\beta_{n,k}\}_{k=0}^{n-1}$ is positive, therefore by Theorem 3.1 and Lemma 3.2(ii) it is log-concave and, hence, unimodal. □

4. Relation to Carlitz's numbers.

There is a relation between the coefficients $\beta_{n,k}$ and numbers $B(\kappa, j, \lambda)$ introduced by Carlitz in [4]. Comparing the formula (2.8) with [4, (6.3)] and taking into account that he uses the notation $R(n,m,r) = \left\{ {n+r \atop m+r} \right\}_r$, we find

$$\beta_{n,k} = (-1)^k B(n-1, n-1-k, n). \tag{4.1}$$

It follows that for $n \geq 3$, the sequence $\{B(n-1,k,n)\}_{k=0}^{n-1}$ is log-concave together with $\{\beta_{n,k}\}_{k=0}^{n-1}$.

Using the property [4, (2.7)] that $\sum_{j=0}^{\kappa} B(\kappa,j,\lambda) = (2\kappa-1)!!$, we can compute $p_n(w)$ at the singular point where $W = -1$ (*cf.* (1.1)). Substituting $w = -1$ in (2.1) gives $p_n(-1) = (-1)^{n-1}(2n-3)!!$, and thus $w = -1$ is not a zero of $p_n(w)$.

We also note that the numbers $B(\kappa,j,\lambda)$ are polynomials of λ and satisfy a three-term recurrence [4, (2.4)]

$$B(\kappa,j,\lambda) = (\kappa+j-\lambda)B(\kappa-1,j,\lambda) + (\kappa-j+\lambda)B(\kappa-1,j-1,\lambda) \tag{4.2}$$

with $B(\kappa,0,\lambda) = (1-\lambda)^{\bar{\kappa}}$, $B(0,j,\lambda) = \delta_{j,o}$. This gives one more way to compute the coefficients $\beta_{n,k}$, specifically, for given n and k we find a polynomial $B(n-1,n-1-k,\lambda)$ using (4.2) and then set $\lambda = n$ to use (4.1).

5. **Concluding remarks.** It has been established that the coefficients of the polynomials $(-1)^{n-1}p_n(w)$ are positive, unimodal and log-concave. Since $W(x)$ is positive for all positive x, it follows from formula (1.1) and Theorem 3.4 that $(-1)^{n-1}(d/dx)^{(n-1)}W > 0$ for $n \geq 1$. This means that the derivative W' is completely monotonic and W itself is a Bernstein function [2].

Some additional identities can be obtained from the results above. For example, computing $\beta_{n,n-1}$ by (2.8) and comparing with (2.3) gives

$$\sum_{m=0}^{n-1}(-1)^m\binom{2n-1}{n-m-1}\left\{\begin{matrix}2n-1+m\\ n+m\end{matrix}\right\}_n = (n-1)!\,.$$

A relation between $\left\{\begin{smallmatrix}2n-1+m\\ n+m\end{smallmatrix}\right\}_n$ and $B_{n-1}^{(-m)}(n)$ can be obtained from (2.8) and (2.9), but this is a special case of [4, (7.5)]. Finally, it is interesting to note that (2.8) and (2.9) can be inverted. For fixed n, the sequence $(-1)^k\beta_{n,k}$ is a convolution of two sequences, and therefore its generating function $G(w)$ is a product of the generating functions of these two sequences. We can write $G(w) = (1-w)^{2n-1}F(w)$, where $F(w)$ represents a generating function either for the sequence $\left\{\begin{smallmatrix}2n-1+m\\ n+m\end{smallmatrix}\right\}_n$ or for the sequence $\binom{m+n-1}{n-1}B_{n-1}^{(-m)}(n)$ as appropriate. Now, since

$$F(w) = G(w)(1-w)^{-(2n-1)} = G(w)\sum_{k\geq 0}\binom{2n-2+k}{2n-2}w^k,$$

the inverse of, for example, (2.8) is

$$\left\{\begin{matrix}2n-1+m\\ n+m\end{matrix}\right\}_n = \sum_{k=0}^{n-1}(-1)^k\beta_{n,k}\binom{2n-2+m-k}{2n-2}. \tag{5.1}$$

After taking into account (4.1), we see this is a special case of [4, (2.9)].

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Department of Applied Mathematics, The University of Western Ontario, London, Ontario, Canada, N6A 5B7

e-mail: gkalugin@uwo.ca
djeffrey@uwo.ca