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# The Cuntz Semigroup of the Tensor Product of C*-algebras

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Abstract. We calculate the Cuntz semigroup of the tensor product of two C*-algebras, restricting attention to the case that the Cuntz semigroup, both for the given algebras and for the tensor product, is given by affine functions. We show that the answer is the universal Cuntz category tensor product of [2].

Résumé. On démontre que, dans certains cas, le semigroupe de Cuntz du produit tensoriel de deux C*-algèbres est le produit tensoriel dans la catégorie de Cuntz.

**1.** The Cuntz semigroup, first introduced forty years ago ([7]), has proven relatively recently to be an important invariant for C*-algebras (see, e.g., [18], [21], and [22]).

In this paper, following on [2], we consider the question of how the Cuntz semigroup behaves under forming tensor products of C*-algebras. In the finite-dimensional case, as the Cuntz semigroup is just the Murray-von Neumann semigroup, with an infinite element adjoined to each minimal direct summand (a copy of $\mathbb{N}$), the Cuntz semigroup of the tensor product C*-algebra is exactly the tensor product of the Cuntz semigroups, in a simple, purely algebraic sense (tensor product as semigroups, with the order relation given by addition just as in the component ordered semigroups). More generally, in [2], the Cuntz semigroup of the tensor product of an arbitrary C*-algebra with a separable AF algebra was shown to be the universal Cuntz category tensor product ordered semigroup (as defined and constructed in [2], with respect to the Cuntz category as defined in [6]), and the general question of when this is true (not always!) was formulated as Problem 6.4.11 of [2]. (Also the case that both component algebras are separable and one of them is unital, simple, nuclear, and purely infinite was dealt with in [2].)

In the present article, we deal with the case that $A$ and $B$ and also $A \otimes B$ are simple, exact, Jiang-Su stable (absorbing the Jiang-Su C*-algebra tensorially—see e.g. [22]), and stably projectionless. (If in addition either $A$ or $B$ is nuclear then $A \otimes B$ is necessarily simple if $A$ and $B$ are, but in any case the minimal tensor product is simple ([20]), and that is what we shall consider.) (We do not know if either the minimal or maximal tensor product must be stably projectionless if both $A$ and $B$ are.)

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By Theorem 6.6 of [11], in the setting mentioned, the Cuntz semigroups are just certain cones of extended positive real-valued affine functions on cones, strictly positive except at zero, with pointwise addition and order. The cones that these functions are defined (and lower semicontinuous) on, in the case of simple C*-algebras, are ordinary simplical cones—i.e., topological cones with a compact convex base consisting of a Choquet simplex. (Equivalently, topological cones with a compact convex base that are lattice cones, in a natural sense.)

As we shall show, the cones in question, the tracial cones (densely defined lower semicontinuous traces) of the two given C*-algebras and of their tensor product, are related in that the universal tensor product of simplex bases for the two given cones (see Section 7 of [19]) is a simplex base for the third cone. (This will be integral to the proof of the Cuntz semigroup tensor product calculation.)

As it happens, cones with a base that is just some compact convex set in a locally convex space still give rise, through the appropriate cones of affine functions on them (see Section 2), to objects in the Cuntz category.

The (universal) Cuntz tensor product of two such Cuntz objects is presumably still the appropriate cone of affine functions on the universal tensor product of the compact convex sets (as defined in Section 7 of [15]), but we shall not address that question here.

**2.** Our main result may be formulated (in an abstract setting—without mentioning C*-algebras) as follows. If $K$ is a compact convex subset of a locally convex topological vector space then (cf. [18], Section 6) we shall denote by $\mathrm{LAff}_+(K)$ the semigroup (cone) of lower semicontinuous extended positive real-valued affine functions on $K$, strictly positive except for the zero function, which can be expressed as the pointwise supremum of an increasing sequence of continuous, finite-valued such functions. (As a consequence of Corollary I.1.4 of [1], if $K$ is metrizable, then (as an approximation argument shows) the last property is automatic—any strictly positive lower semicontinuous function is such a supremum.) It is easy to see that $\mathrm{LAff}_+(K)$, considered as an ordered semigroup with the pointwise addition and order, is an object in the Cuntz category introduced in [6]. (It need only be checked that the relation of compact containment, $f << g$, in $\mathrm{LAff}_+(K)$ is determined by the continuous finite-valued functions—let us call this subcone $\mathrm{Aff}_+(K)$—in the following way: $f << g$ holds if and only if there exist $k \in \mathrm{Aff}_+(K)$ and $\varepsilon > 0$ such that $f \leq k \leq (1+\varepsilon)k \leq g$.) (The "if" is immediate. To see the "only if", note that if $f << g$, then writing $g = \sup g_n$ with $(g_n)$ an increasing sequence in $\mathrm{Aff}_+(K)$, and assuming as we may, since all non-zero functions are strictly positive, that the sequence $(g_n)$ is increasing in the strict pointwise order, we have $f \leq g_n \leq g_{n+1} \leq g$ for some $n$, but also, for some $\varepsilon > 0$, $(1+\varepsilon)g_n \leq g_{n+1}$—so $f \leq g_n \leq (1+\varepsilon)g_n \leq g$, as desired.)

**Theorem.** *Let $K_1$ and $K_2$ be Choquet simplices. Denote by $K_1 \otimes K_2$ the tensor product simplex in the sense of [19], [9], [15], and [17] (see Proposition 2.10 of [17]). Then, with respect to the natural map*

$$\mathrm{LAff}_+(K_1) \times \mathrm{LAff}_+(K_2) \to \mathrm{LAff}_+(K_1 \otimes K_2),$$

$\mathrm{LAff}_+(K_1 \otimes K_2)$ *is the tensor product*

$$\mathrm{LAff}_+(K_1) \otimes_{\mathrm{Cu}} \mathrm{LAff}_+(K_2)$$

*of* $\mathrm{LAff}_+(K_1)$ *and* $\mathrm{LAff}_+(K_2)$ *in the Cuntz category defined (in terms of a universal property, and also shown to exist in general) in [2].*

*Proof.* First, let us consider the case that $K_1$ and $K_2$ are metrizable. In this case, by the Corollary to Theorem 5.2 of [16], $K_1$ and $K_2$ are the projective limits of sequences

$$K_1^1 \leftarrow K_1^2 \leftarrow \cdots \text{ and } K_2^1 \leftarrow K_2^2 \leftarrow \cdots$$

of finite-dimensional simplices with surjective maps:

$$K_1 = \varprojlim K_1^n \text{ and } K_2 = \varprojlim K_2^n.$$

Hence by Lemma 3, below,

$$K_1 \otimes K_2 = \varprojlim (K_1^n \otimes K_2^n).$$

By Lemma 4, below, it follows that

$$\mathrm{LAff}_+(K_1 \otimes K_2) = \varinjlim \mathrm{LAff}_+(K_1^n \otimes K_2^n)$$

(limit in the Cuntz category—see [6]). Of course, also by Lemma 4,

$$\mathrm{LAff}_+(K_1) = \varinjlim \mathrm{LAff}_+(K_1^n) \text{ and } \mathrm{LAff}_+(K_2) = \varinjlim \mathrm{LAff}_+(K_2^n).$$

Since the simplices $K_1^n$ and $K_2^n$ are finite-dimensional for each $n$, and therefore also the tensor product simplex is finite-dimensional (e.g. by Theorem 2.3 of [17]), an elementary computation yields that, for each $n$,

$$\mathrm{LAff}_+(K_1^n \otimes K_2^n) = \mathrm{LAff}_+(K_1^n) \otimes_{\mathrm{Cu}} \mathrm{LAff}_+(K_2^n),$$

where the tensor product on the right is the universal Cuntz category tensor product of [2]. (For any finite-dimensional simplex $K$, $\mathrm{LAff}_+(K)$ is naturally identified with the strictly positive extended positive real-valued functions on the (finite) set of extreme points of $K$, and by Theorem 2.3 of [17], the set of extreme points of $K_1^n \otimes K_2^n$ is naturally identified with the Cartesian product of the sets of extreme points of $K_1$ and $K_2$.)

It remains to apply Lemma 5, below, to the right hand side equation (natural isomorphism) above, for Cuntz category objects. One obtains the equalities (natural isomorphisms)

$$\begin{aligned}
\mathrm{LAff}_+(K_1 \otimes K_2) &= \varinjlim \mathrm{LAff}_+(K_1^n \otimes K_2^n) \\
&= \varinjlim (\mathrm{LAff}_+(K_1^n) \otimes_{\mathrm{Cu}} \mathrm{LAff}_+(K_2^n)) \\
&= \varinjlim \mathrm{LAff}_+(K_1^n) \otimes_{\mathrm{Cu}} \varinjlim \mathrm{LAff}_+(K_2^n) \\
&= \mathrm{LAff}_+(K_1) \otimes_{\mathrm{Cu}} \mathrm{LAff}_+(K_2),
\end{aligned}$$

as desired.

Now let us consider the general case. The main step is to show that every countable subset of $\mathrm{Aff}(K_1 \otimes K_2)$ is contained in a separable subspace $\mathrm{Aff}_+(K_1' \otimes K_2')$ corresponding to a map $K_1 \otimes K_2 \to K_1' \otimes K_2'$ consisting of the tensor product of maps $K_1 \to K_1'$ and $K_2 \to K_2'$, and such that the map $\mathrm{Aff}(K_1' \otimes K_2') \to \mathrm{Aff}(K_1 \otimes K_2)$ resulting in the first instance is injective, and in fact isometric—which requires that the map $K_1 \otimes K_2 \to K_1' \otimes K_2'$ be surjective. Separability of $\mathrm{Aff}(K_1' \otimes K_2')$ is of course equivalent to metrizability of $K_1' \otimes K_2'$, equivalently, of $K_1'$ and $K_2'$. We shall explain this reduction to the separable case below.

Let, now, $S$ be an object in the Cuntz category of [4], and let

$$\mathrm{LAff}_+(K_1) \times \mathrm{LAff}_+(K_2) \to S$$

be a map as in [2], biadditive, order preserving, and supremum preserving in each variable separately (and therefore jointly), and preserving compact containment jointly in both variables, and let us show that there exists a unique Cuntz category map $\mathrm{LAff}_+(K_1 \otimes K_2) \to S$ through which the given map factorizes (with respect to the canonical map $\mathrm{LAff}_+(K_1) \times \mathrm{LAff}_+(K_2) \to \mathrm{LAff}_+(K_1 \otimes K_2)$).

Uniqueness is in fact an immediate consequence of uniqueness in the setting of metrizable images $K_1'$ and $K_2'$ of $K_1$ and $K_2$ described above (since the corresponding subspace $\mathrm{Aff}(K_1' \otimes K_2') \subseteq \mathrm{Aff}(K_1 \otimes K_2)$ contains an arbitrary countable subset, and every element of $\mathrm{LAff}_+(K_1 \otimes K_2)$ is (by definition) the supremum of an increasing sequence of strictly positive elements of $\mathrm{Aff}(K_1 \otimes K_2)$. (Recall that a Cuntz category map preserves suprema of increasing sequences.)

Existence in fact also follows easily from the metrizable approximation described above. A given element of $\mathrm{LAff}_+(K_1 \otimes K_2)$ is, again, the supremum of an increasing sequence of strictly positive elements of a subspace $\mathrm{Aff}(K_1' \otimes K_2') \subseteq \mathrm{Aff}(K_1 \otimes K_2)$ as above, and existence of a factorization of the restricted map $\mathrm{LAff}_+(K_1') \times \mathrm{LAff}_+(K_2') \to S$ through $\mathrm{LAff}_+(K_1' \otimes K_2')$ yields an image in $S$ of this increasing sequence—and in fact also of its supremum—which since determined pointwise is the same on $K_1' \otimes K_2'$ and on its preimage $K_1 \otimes K_2$. Appropriate alternative choices of $K_1'$ and $K_2'$ show that this assignment (of an element of $S$ to a given element of $\mathrm{LAff}_+(K_1 \otimes K_2)$) is unique, is additive, preserves increasing sequential suprema, and preserves compact containment. (For this last, note that if $f_1 << f_2$ in $\mathrm{LAff}_+(K_1 \otimes K_2)$, and $f_1, f_2 \in \mathrm{LAff}_+(K_1' \otimes K_2') \subseteq \mathrm{LAff}_+(K_1 \otimes K_2)$ with $K_1'$ and $K_2'$ as above, then, since as pointed out at the beginning of this section, compact containment in $\mathrm{LAff}_+$ of a compact convex set is determined by a certain pointwise relation, it is the same for functions on a given compact convex set and the pullbacks of these functions to another one mapping onto it.)

Now, let $X$ be a countable subset of $\mathrm{Aff}(K_1 \otimes K_2)$, and let us construct surjections $K_1 \to K_1'$ and $K_2 \to K_2'$ such that $K_1'$ and $K_2'$ are metrizable simplices and $X \subseteq \mathrm{Aff}(K_1' \otimes K_2') \subseteq \mathrm{Aff}(K_1 \otimes K_2)$. To construct $K_1'$ and $K_2'$ and surjections $K_1 \to K_1'$ and $K_2 \to K_2'$, by the Krein extension theorem for order-unit ordered vector spaces it is equivalent to construct closed subspaces $E_1$ and $E_2$ of $\mathrm{Aff}(K_1)$ and $\mathrm{Aff}(K_2)$ containing the canonical order unit, and with the Riesz decomposition property (so that when they are written as $\mathrm{Aff}(K_i')$, $K_i'$ is a simplex), and we wish to do this in such a way that the countable set $X$ is contained in the closure, $E$, of $E_1 \otimes_{\mathrm{alg}} E_2$ in $\mathrm{Aff}(K_1 \otimes K_2)$. Of course, we also want $E$ to be equal to $\mathrm{Aff}(K_1' \otimes K_2')$, in the natural sense.

We follow the method of reducing to separable substructures introduced by Blackadar in [3]. We desire a separable subspace $E$ of $\mathrm{Aff}(K_1 \otimes K_2)$, containing the countable set $X$, and containing the canonical order unit 1, so that it will be $\mathrm{Aff}(K)$ for some (metrizable) compact convex set $K$ (a continuous affine image of $K_1 \otimes K_2$) and with the Riesz interpolation property, so that $K$ will be a simplex, and such that furthermore $E$ is the closure of $\mathrm{Aff}(K_1') \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2') \subseteq \mathrm{Aff}(K_1 \otimes K_2)$—also containing the canonical order units and with the Riesz interpolation property, so that the continuous affine images $K_1'$ and $K_2'$ of $K_1$ and $K_2$ are simplices.

To construct $E$ as above, note first that the algebraic tensor product $\mathrm{Aff}(K_1)\otimes_{\mathrm{alg}}\mathrm{Aff}(K_2)$ is contained in $\mathrm{Aff}(K_1\otimes K_2)$ in the natural way, isometrically and order isomorphically, since by Theorem 2.3 of [17] (together with Proposition 2.10) the extreme points of $K_1 \otimes K_2$ are contained in $K_1 \times K_2$—here we are considering the elements of the algebraic tensor product as (biaffine) functions on $K_1 \times K_2$—, and furthermore is dense in $\mathrm{Aff}(K_1 \otimes K_2)$ (as by Proposition 2.10 of [17] the states on $\mathrm{Aff}(K_1 \otimes K_2)$, i.e., the points of $K_1 \otimes K_2$, are the same as the states on the order-unit space $\mathrm{Aff}(K_1) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2)$ with the pointwise order on $K_1 \times K_2$—and so the continuous real-valued linear functionals on $\mathrm{Aff}(K_1 \otimes K_2)$, which are real linear combinations of (two) states are determined by their values on the subspace).

It follows that $X$ is contained in the closure of $E_1 \otimes E_2$ for separable closed subspaces $E_1$ and $E_2$ of $\mathrm{Aff}(K_1)$ and $\mathrm{Aff}(K_2)$, which we may suppose on enlargement contain the canonical order unit, so that they are equal to $\mathrm{Aff}(K_1')$ and $\mathrm{Aff}(K_2')$ for surjective, continuous affine maps $K_1 \to K_1'$ and $K_2 \to K_2'$, and which we may also suppose have the Riesz interpolation property (the reformulation by Birkhoff of Riesz's decomposition property), so that $K_1'$ and $K_2'$ are simplices. This last is slightly tricky—we must note that it is enough to have the (on the face of it, weaker) Riesz interpolation property for the strict pointwise order (this results in the same cone of positive functionals, the lattice property of which is characteristic) to have it for a dense subset—and then enlarging $E_1$ (and $E_2$) in successive operations to include interpolants for a countable dense subset of the subspace at each preceding stage, one has (much as in the analogous setting of [3]) in the limit a new subspace $E_1$ (or $E_2$) with a (countable) dense subset with interpolants, and thus a subspace with the Riesz property, as desired. It remains to check that the subspace $E = (E_1 \otimes_{\mathrm{alg}} E_2)^-$ is $\mathrm{Aff}(K_1' \otimes K_2')$, but this follows as above for the closure of $\mathrm{Aff}(K_1) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2)$ and $\mathrm{Aff}(K_1 \otimes K_2)$.

**3. Lemma.** *Tensor product of simplices commutes with (sequential) projective limit.*

*Proof.* Let

$$K_1^1 \leftarrow K_1^2 \leftarrow \cdots \text{ and } K_2^1 \leftarrow K_2^2 \leftarrow \cdots$$

be sequences of simplices (or just compact convex sets) with projective limits $K_1$ and $K_2$, and let us show that, in the natural sense,

$$K_1 \otimes K_2 = \lim_{\leftarrow}(K_1^n \otimes K_2^n),$$

where the tensor product is in the sense of [19].

By definition (Section 7 of [19]), together with Corollary 2.6 of [17], $K_1 \otimes K_2$ is the state space of the order unit vector space $\mathrm{Aff}(K_1) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2)$ with the pointwise order on $K_1 \times K_2$, equivalently, of the completion, which is then therefore equal to

$$\mathrm{Aff}(K_1 \otimes K_2).$$

What we must show, then, is that, in the natural sense (universal property for inductive limits of order unit Banach spaces),

$$\mathrm{Aff}(K_1 \otimes K_2) = \lim_{\rightarrow} \mathrm{Aff}(K_1^n \otimes K_2^n).$$

(We refer to the well-known Kadison duality between order unit Banach spaces and compact convex sets, converting inductive limits of the first into projective limits of the second.)

So, let $\mathrm{Aff}(K)$ be any order unit Banach space ($K$ the (arbitrary) compact convex set of states), and let compatible positive unital maps into it from the sequence

$$\mathrm{Aff}(K_1^1 \otimes K_2^1) \to \mathrm{Aff}(K_2^2 \otimes K_2^2) \to \cdots$$

be given. Since all $K$'s with indices are simplices, by Corollary 2.6 of [17] the order unit spaces $\mathrm{Aff}(K_1^n \otimes K_2^n)$ and $\mathrm{Aff}(K_1 \otimes K_2)$ are the supremum norm completions of the algebraic tensor products $\mathrm{Aff}(K_1^n) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2^n)$ and $\mathrm{Aff}(K_1) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2)$, respectively, with these spaces considered as spaces of functions on $K_1^n \times K_2^n$ and $K_1 \times K_2$, respectively. Since the union of the images of the spaces $\mathrm{Aff}(K_1^n) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2^n)$ is dense in $\mathrm{Aff}(K_1 \otimes K_2)$, and hence in $\mathrm{Aff}(K_1 \otimes K_2)$, and all the maps are contractions in the sup norm, the canonically specified map from this union to $\mathrm{Aff}(K)$ extends uniquely by continuity to a (positive, unital) map $\mathrm{Aff}(K_1 \otimes K_2) \to \mathrm{Aff}(K)$, as desired.

This proof in fact works with the $K$'s with indices arbitrary compact convex sets, except that one has to interpret the tensor products of compact convex sets in the sense considered, instead of in the universal sense considered in [19]. (In other words, one has to define the tensor products of $K_1^n$ with $K_2^n$ and of $K_1$ with $K_2$ in the state spaces of the algebraic tensor products $\mathrm{Aff}(K_1^n) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2^n)$ and $\mathrm{Aff}(K_1) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2)$ with the pointwise order, as considered in [17] (as the second definition of tensor product, on page 472). Equivalently, as the states have norm one in the sup norm, as the state spaces of the completions of these order-unit spaces—pointwise order and supremum, or order-unit, norm.)

**4. Lemma.** *The operation* $\mathrm{LAff}_+$ *on simplices, or arbitrary compact convex sets, transforms sequential projective limits of compact convex sets to (Cuntz category) inductive limits of Cuntz category objects.*

*Proof.* Given a sequence

$$K_1 \leftarrow K_2 \leftarrow \cdots$$

of compact convex sets, with projective limit $\lim\limits_{\leftarrow} K_n$ equal to $K$, i.e., such that $\mathrm{Aff}(K) = \lim\limits_{\rightarrow} \mathrm{Aff}(K_n)$ we must show that the inductive limit of the corresponding sequence

$$\mathrm{LAff}_+(K_1) \to \mathrm{LAff}_+(K_2) \to \cdots$$

in the Cuntz category **Cu** is equal to $\mathrm{LAff}_+(K)$. (It will be useful, in the proof, to consider this general case.)

Let the ordered semigroup $S$ be a Cuntz category object, as in [6], and let there be given a compatible sequence of maps

$$\mathrm{LAff}_+(K_1) \to S, \quad \mathrm{LAff}_+(K_2) \to S, \cdots .$$

Let us show first that the restrictions of these to $\mathrm{Aff}_+(K_1), \mathrm{Aff}_+(K_2), \cdots$ (the strictly positive functions in $\mathrm{Aff}(K_1), \mathrm{Aff}(K_2), \cdots$, together with zero) factor uniquely through $\mathrm{Aff}_+(K)$ as order-preserving semigroup maps. (Recall that the addition and order in $\mathrm{Aff}_+(K_n)$ coincide with those in $\mathrm{LAff}_+(K_n)$.)

If $f$ is a non-zero element of $\mathrm{LAff}_+(K)$, then $f$ is the pointwise limit of an increasing sequence of elements of $\mathrm{Aff}_+(K)$, and hence (in the natural sense) of a sequence of elements $f_n \in \mathrm{Aff}_+(K_n)$, which may be chosen to be increasing in the strict pointwise order (in the natural sense, i.e., when compared in $\mathrm{Aff}_+(K)$). The images of these elements in $S$ are increasing, and so we may, tentatively, map $f$ to the supremum. It is only necessary to note that this supremum is independent of the choice of the sequence $(f_n)$, which is clear since any two such sequences have intertwined subsequences. It follows immediately from this observation (independence of the choice) that the map is additive. Let us check that it preserves order, increasing sequential suprema, and compact containment (the relation of being "way below", as some would have it—more cautiously, perhaps, "well below", or "far below".)

Let, now, $f$ and $g$ be elements of $\mathrm{LAff}_+(K)$ with $f \leq g$ (pointwise), and let us show that the images of $f$ and $g$ in $S$ are also comparable in this way. As in the preceding paragraph, we have $f$ and $g$ the suprema of rapidly increasing sequences $f_n \in \mathrm{Aff}_+(K_n)$ and $g_n \in \mathrm{Aff}_+(K_n)$, respectively, and since $f_n << f \leq g = \sup g_k$, so that $f_n \leq g_k$ for some $k$, passing to subsequences we may suppose that $f_n \leq g_n$ for all $n$. This comparability then holds for the images of each $f_n$ and $g_n$ in $S$, and hence (as in any ordered set) for their suprema, i.e., for the images of $f$ and $g$ in $S$.

Let $f$ again be an element of $\mathrm{LAff}_+(K)$, and let $f$ be the supremum of an increasing sequence $(g_n)$ in $\mathrm{LAff}_+(K)$. Passing to a subsequence, we may suppose that $(g_n)$ is rapidly increasing, in the sense of [6], i.e., $g_n << g_{n+1}$ (compactly contained in) for all $n$, since by definition, any element of a Cuntz category object is the supremum of at least one rapidly increasing sequence. Again, since (see Section 2) in $\mathrm{LAff}_+(K)$ any compact containment is interspersed with, or engineered by, a compact containment of elements of $\mathrm{Aff}_+(K)$ (i.e., if $k_1 << k_2$

in $\mathrm{LAff}_+(K)$ then $k_1 \leq k_1' << k_2' \leq k_2$ with $k_1', k_2' \in \mathrm{Aff}_+(K)$, and $k_1' << k_2'$ just means $k_1'$ majorized by $k_2'$ in the strict pointwise order), we may replace the sequence $(g_n)$ by an intertwined one (so still with supremum $f$) in $\mathrm{Aff}_+(K)$. Again, since $\mathrm{Aff}(K) = \varinjlim(K_n)$ (as order-unit Banach spaces), passing again to a subsequence we may suppose that, in the natural sense, $g_n$ belongs to $\mathrm{Aff}(K_n)$ (since the maps $\mathrm{Aff}K_n \to \mathrm{Aff}K$ are not isometric in general some argument is needed here, for the simplest solution is just to note that we may clearly reduce the problem for this case, since $K_n$ and $K$ are just any compact convex sets, so $K_n$ might as well be the image of $K$.) Then, by construction, the image of $f$ (the supremum of $(g_n)$) in $S$ is the supremum of the given images of $g_n \in \mathrm{Aff}(K_n)$ in $S$, which are, of course also, by construction, the images of $g_n \in \mathrm{Aff}(K_n)$ in $S$, as desired.

Finally, let $f << g$ be a compact containment relation in $\mathrm{LAff}_+(K)$. As recalled in the preceding paragraph, this just means that $f \leq k \leq (1+\varepsilon)k \leq g$ for some $k \in \mathrm{Aff}_+K)$ and $\varepsilon > 0$. Choosing $k$, as we may (slightly larger, with $\varepsilon$ slightly smaller), to belong to the image of $\mathrm{Aff}_+(K_n)$ for some $n$, we have then (in the natural sense), $k << (1+\varepsilon)k$ in $\mathrm{LAff}_+(K_n)$, so the image of $k$, hence of $f$, in $S$ is compactly contained in the image of $(1+\varepsilon)k$, hence of $g$, as desired.

**5. Lemma.** *Cuntz category tensor product commutes with (Cuntz category) (sequential) inductive limits.*

*Proof.* This is a simple exercise in universal properties. Let, in the Cuntz category, $R$ and $S$ be the inductive limits of sequences $R_1 \to R_2 \to \cdots$ and $S_1 \to S_2 \to \cdots$, and note that the natural maps $R_n \times S_n \to R_{n+1} \otimes S_{n+1}$ and $R_n \times S_n \to R \otimes S$ yield maps (by the universal property) $R_n \otimes S_n \to R_{n+1} \otimes S_{n+1}$ and $R_n \otimes S_n \to R \otimes S$. To prove

$$R \otimes S = \varinjlim(R_n \otimes S_n),$$

we must show that any sequence of compatible maps $R_n \otimes S_n \to U$ in the Cuntz category factorizes uniquely through $R \otimes S$, with respect to the sequence of maps $R_n \otimes S_n \to R \otimes S$. By the tensor product universal property, it is sufficient to construct a suitable map $R \times S \to U$, through which the Cartesian product maps $R_n \times S_n \to R \times S$ factor. This can be achieved by considering the larger family of maps $R_n \times S_m \to R \times S$, $n, m = 1, 2, \ldots$, and first, for fixed $m$, factoring the maps $R_n \times S_m \to U$ through $R \times S_m$, and then factoring the maps $R \times S_m \to U$ through $R \times S$, in both steps using the actual construction of the Cuntz category inductive limit (i.e., $\varinjlim R_n$ and $\varinjlim S_m$) given in [6].

**6.** Recall that by Proposition 3.4 of [17], if $A$ is a C*-algebra and $a \in A^+$ is a full element of the Pedersen ideal of $A$, then the set of densely defined lower semicontinuous traces on $A$ equal to 1 on $a$ is a Choquet simplex (in the topology of pointwise convergence on the Pedersen ideal). Let us denote this by $T_a$.

**Theorem.** *Let $A$ and $B$ be C*-algebras, and let full positive elements $a$ and $b$ in the Pedersen ideals of $A$ and $B$ be given. Then the simplex $T_{a \otimes b}$ of traces on*

*$A \otimes B$, normalized and in particular non-zero on $a \otimes b$, is the universal tensor product of the simplices $T_a$ and $T_b$ in the categorical sense of Section 7 of [19]. Here, $A \otimes B$ refers to any C*-algebra containing the algebraic tensor product as a dense sub-*-algebra, and mapping into the usual, minimal, tensor product.*

(Theorem 3.6 of [13] is a related statement, but with the tensor product of trace spaces a priori depending on the C*-algebra context.)

*Proof.* By Proposition 2.10 of [17], the universal tensor product of [19] for compact convex sets, realized as the state space of the order-unit space of continuous biaffine functions, is isomorphic in the case of simplices to the state space of the algebraic tensor product of the two affine function spaces considered with the positive cone consisting of those elements which are positive on the functionals consisting of the product of a state on each of the affine function spaces of the two given simplices.

In other words, the positive cone of the algebraic tensor product of the two affine function spaces which is to be considered is just those elements which are positive pointwise when considered as functions of two variables (on the Cartesian product of the two simplices), i.e., the subset of the positive cone of the space of (continuous) biaffine functions consisting of elements arising from the algebraic tensor product. Thus, one is looking at the state space of the subspace of the space of biaffine functions consisting of the algebraic tensor product, with the relative (i.e., inherited) order, instead of the state space of the whole space of biaffine functions, but in the simplex case these state spaces are the same, i.e., every state of the subspace extends uniquely.

Consider first the case that the algebras $A$ and $B$ are unital, with $a$ and $b$ the units.

We wish to show that every state of the last space above, in the case of the simplices $\mathrm{T}_a$ and $\mathrm{T}_b$, arises in the natural way from a state, indeed, tracial state, on the given tensor product C*-algebra. It is enough to show it is norm decreasing with respect to the quotient norm on the algebraic tensor product of the affine function spaces viewed as the image of the algebraic tensor product of the two C*-algebras. (This implies that it arises from a self-adjoint linear functional on the given C*-algebra completion of the algebraic tensor product of $A$ and $B$, of norm at most one and equal to 1 on the unit, and therefore a state—and a trace for the obvious reason.)

For this purpose, it is enough to show that if $x \in \mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(\mathrm{T}_b)$ and $x$ is the image of an element of $A \otimes_{\mathrm{alg}} B$ of norm at most $1+\varepsilon$ then $-(1+\varepsilon) \leq x \leq 1+\varepsilon$. (Then, if $x$ has norm at most one in the quotient norm, then $-1 \leq x \leq 1$, and so for any state $f$ of $\mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(\mathrm{T}_b)$ in the order under consideration (pointwise on $\mathrm{T}_a \times \mathrm{T}_b$), $-1 \leq f(x) \leq 1$; this shows that $\|f\| \leq 1$.) But if $x_\varepsilon = x_\varepsilon^* \in A \otimes_{\mathrm{alg}} B$ and $\|x_\varepsilon\| \leq 1+\varepsilon$, with respect to the given C*-algebra norm, then for any $\tau_1 \in \mathrm{T}_a$ and $\tau_2 \in \mathrm{T}_b$, $\tau_1 \otimes \tau_2$ is a tracial state on the given C*-algebra completion $A \otimes B$—since as is well known this is true for $A \otimes_{\min} B$—and so $|(\tau_1 \otimes \tau_2)(x_\varepsilon)| \leq 1$; in other words, $-(1+\varepsilon) \leq x \leq 1+\varepsilon$.

We must still consider the general case. In this case, it is still true that the

Pedersen ideal of $A$ maps onto $\mathrm{Aff}(\mathrm{T}_a)$, and similarly for $B$. (But we shall not use this, except when $A = \mathrm{Ped}A$ and $B = \mathrm{Ped}B$—see below.) What we must show is that any element of $\mathrm{T}_a \otimes \mathrm{T}_b$, i.e., any state of the order-unit ordered vector space $\mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(\mathrm{T}_b)$ (with the pointwise order considered as functions on $\mathrm{T}_a \times \mathrm{T}_b$) arises in the natural way from a unique element of $\mathrm{T}_{a\otimes b}$. This correspondence will then be the desired isomorphism.

Clearly, an element of $\mathrm{T}_{a\otimes b}$ (or any lower semicontinuous trace on $A \otimes B$) is determined by its restrictions to the hereditary sub-C*-algebras of $A \otimes B$ generated by an increasing approximate unit. Indeed, such a trace is defined by any compatible family of lower semicontinuous traces on such a family of hereditary subalgebras (assumed to be upward directed, with dense union). So, to get an element of $\mathrm{T}_{a\otimes b}$ corresponding to a given element of $\mathrm{T}_a \otimes \mathrm{T}_b$, it is sufficient to replace each of $A$ and $B$ by a hereditary sub-C*-algebra contained in the Pedersen ideal and containing $a$, respectively $b$, and which we may assume is singly generated ($\sigma$-unital). In particular now $A$ and $B$ are equal to their Pedersen ideals (see [21]) and so every trace is bounded. Such a modification of $A$ and $B$ will in fact not change $\mathrm{T}_a$ and $\mathrm{T}_b$, as in general, any lower semicontinuous trace on $A$ or $B$, is determined by its restriction to a full hereditary sub-C*-algebra of the Pedersen ideal. By [8], $A$ and $B$ (or, rather, the self-adjoint parts of these algebras) map onto $\mathrm{Aff}(\mathrm{T}_a)$ and $\mathrm{Aff}(\mathrm{T}_b)$, respectively, as these are just the Banach space duals of the spaces of self-adjoint bounded traces on $A$ and $B$. So, as stated above, it is sufficient to show that any element of $\mathrm{T}_a \otimes \mathrm{T}_b$, i.e., any state of the ordered vector space $\mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(\mathrm{T}_b)$, with the pointwise order on $\mathrm{T}_a \times \mathrm{T}_b$, arises from an element of $\mathrm{T}_{a\otimes b}$, i.e., a bounded trace on $A \otimes B$ equal to 1 on $a \otimes b$. This can now be proved in much the same way as above in the case $a = b = 1$. Namely, such a state of $\mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(\mathrm{T}_b)$ must be proved to be bounded with respect to the quotient norm arising from $A \otimes_{\mathrm{alg}} B$, with the given tensor product norm—as before, it is enough to consider $A \otimes_{\min} B$.

Suppose, then, that $x \in \mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(\mathrm{T}_b)$ and $x$ is the image of an element of $A \otimes_{\mathrm{alg}} B$ of norm at most $1 + \varepsilon$. Note that since, now, $A = \mathrm{Ped}A$, by Theorem 4.7 and Lemma 4.4 of [10], there is a positive element $a'$ of $A \otimes M_n$ for some $n = 1, 2, \ldots$ acting as the unit on an embedding of $A$ as a full hereditary subalgebra, and it follows that since also $A \otimes M_n = \mathrm{Ped}(A \otimes M_n)$ and $a$ is full in $A \otimes M_n$, $a' = \sum_{i=1}^{m} x_i a x_i^*$ for some $m = 1, 2, \ldots$, necessarily every trace in $\mathrm{T}_a$ is of norm at most $m$ (on $A$). Similarly, the traces in $\mathrm{T}_b$ are uniformly bounded on $B$, say also by (a possibly larger) $m$. It follows that, for any $\tau_1 \in \mathrm{T}_a$, $\tau_2 \in \mathrm{T}_b$,

$$|(\tau_1 \otimes \tau_2)(x)| \leq m^2(1 + \varepsilon),$$

and since $\varepsilon$ is arbitrary, $|(\tau_1 \otimes \tau_2)(x)| \leq m^2$. In other words,

$$-m^2 \leq x \leq m^2$$

in $\mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(T_b)$ (i.e., pointwise on $\mathrm{T}_a \times \mathrm{T}_b$). This shows that any state of $\mathrm{Aff}(\mathrm{T}_a) \otimes_{\mathrm{alg}} \mathrm{Aff}(T_b)$ has norm at most $m^2$ in the quotient norm arising from $A \otimes B$, as desired.

**7.** In view of Theorem 6, our abstract Theorem 1 has the following consequence.

**Theorem.** *Let $A$ and $B$ be simple stable projectionless $C^*$-algebras that are also Jiang-Su stable, and suppose that the minimal $C^*$-algebra tensor product $C^*$-algebra, $A \otimes_{\min} B$, is also projectionless. Note that (by [15]), $A \otimes_{\min} B$ is also simple, and (see Section 1), $A \otimes_{\min} B$ is also Jiang-Su stable. Suppose that $A$, $B$, and $A \otimes_{\min} B$ (automatic, by [14]) are exact (e.g., nuclear, equivalently, amenable). Then, in the natural sense,*

$$\mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \cong \mathrm{Cu}(A \otimes_{\min} B).$$

*Proof.* By Theorems 6.6 and 5.7 of [11], in the setting of the present hypotheses, $\mathrm{Cu}(A)$, $\mathrm{Cu}(B)$ and $\mathrm{Cu}(A \otimes_{\min} B)$ are naturally isomorphic to the ordered semigroups $\mathrm{LAff}_+(\mathrm{T}(A))$, $\mathrm{LAff}_+(\mathrm{T}(B))$, and $\mathrm{LAff}_+(\mathrm{T}(A \otimes B))$, respectively, where $\mathrm{T}(A)$ denotes the topological cone of densely defined lower semicontinuous traces on $A$, and the same for $B$ and $A \otimes B$ (we write this simply for $A \otimes_{\min} B$). (We are using that (2-)quasitraces are traces—[12], [4]; see also [5].)

With $a$ and $b$ non-zero (and therefore full) positive elements of the Pedersen ideals of $A$ and $B$, note that $\mathrm{T}(A)$, $\mathrm{T}(B)$, and $\mathrm{T}(A \otimes B)$ are determined by the simplices $\mathrm{T}_a$, $\mathrm{T}_b$, and $\mathrm{T}_{a\otimes b}$ of Section 6 (as the cones they generate). (The topology of $\mathrm{T}(A)$ is also determined by that of the compact generating subset $\mathrm{T}_a$—a net $(\lambda_i \tau_i)$ in $\mathrm{T}(A)$ with $\tau_i \in \mathrm{T}_a$ converges to $\lambda\tau \in \mathrm{T}(A)$ with $\tau \in \tau_a$ if and only if $\lambda_i$ converges to $\lambda$ (evaluate at $a$) and, if $\lambda \neq 0$, then $\tau_i$ converges to $\tau$—so the definition of $\mathrm{LAff}_+(\mathrm{T}(A))$ can be just as the linear extensions of elements of $\mathrm{LAff}_+(\mathrm{T}_a)$.)

By Theorem 6, $\mathrm{T}_{a\otimes b} = \mathrm{T}_a \otimes \mathrm{T}_b$. By Theorem 2,

$$\mathrm{LAff}_+(\mathrm{T}_a \otimes \mathrm{T}_b) = \mathrm{LAff}_+(\mathrm{T}_a) \otimes_{\mathrm{Cu}} \mathrm{LAff}_+(\mathrm{T}_b).$$

In other words, $\mathrm{Cu}(A \otimes_{\min} B) = \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} (B)$.

**8. Remark.** It would seem to be an interesting question whether Theorem 2 holds for arbitrary compact convex sets instead of just simplices, with the appropriate notion of tensor product—if not the universal one of Semadeni, [19], then perhaps one of the other two proposed by Namioka and Phelps in [17]—note that none of these gives the tensor product of state spaces of (unital) C*-algebras corresponding to the tensor product (any C*-algebra tensor product) of the two C*-algebras, i.e., the state space of the tensor product—except, of course, in the commutative case, when it is a question of (Bauer) simplices.

It is perhaps pertinent to point out that (as remarked there) the proof of Lemma 3 is valid for at least one tensor product of compact convex sets, namely, for which (by definition) the order-unit space $\mathrm{Aff}(K_1 \otimes K_2)$ is (in the natural way) the completion of the algebraic tensor product $\mathrm{Aff}(K_1) \otimes_{\mathrm{alg}} \mathrm{Aff}(K_2)$ in the supremum norm on $K_1 \times K_2$. Conceivably this can be parlayed into a proof that $\mathrm{LAff}_+(K_1) \otimes_{\mathrm{Cu}} \mathrm{Aff}_+(K_2)$ is equal to $\mathrm{LAff}_+(K_1 \otimes K_2)$. Also, Lemma 4 is valid in the general compact convex set context (and this is used, conceivably indispensably, in the proof), and Lemma 5 concerns the general Cuntz category (not only what arises from compact convex sets, although that is the only use we make of it).

(Thus, perhaps, altogether, plausible that Theorem 2 holds beyond the case of simplices—with the appropriate tensor product—as in the proof only the projective limit reduction to finite-dimensional simplices, together with the nature of these, is used.) (Of course, a defeatist approach might be just to define the tensor product this way, possibly no longer the universal one.)

## References


**1**. E. M. Alfsen, Convex Sets and Boundary Integrals, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

**2**. R. Antoine, F. Perera, and H. Thiel, Tensor products and regularity properties of Cuntz semigroups, Mem. Amer. Math. Soc. **251**, No. 1199 (2018). DOI: https://doi.org/10.1090/memo/1199.

**3**. B. E. Blackadar, Weak expectations and nuclear C*-algebras, Indiana Univ. Math. J. **27** (1078), 1021–1026.

**4**. E. Blanchard and E. Kirchberg, Non-simple purely infinite C*-algebras: the Hausdorff case, J. Funct. Anal. **207** (2004), 461–513.

**5**. N. P. Brown and W. Winter, Quasitraces are traces: a short proof of the finite-nuclear-dimension case, C. R. Math. Acad. Sci. Soc. R. Can. **33** (2011), 44–49.

**6**. K. T. Coward, G. A. Elliott, and C. Ivanescu, The Cuntz semigroup as an invariant for C*-algebras, J. Reine Angew. Math. **2008**, Issue 623 (2008), 161–193.

**7**. J. Cuntz, Dimension functions on simple C*-algebras, Math. Ann. **233** (1978), 145–153.

**8**. J. Cuntz and G. K. Pedersen, Equivalence and traces on C*-algebras, J. Funct. Anal. **33** (1979), 135–164.

**9**. E. B. Davies and G. F. Vincent-Smith, Tensor products, infinite products and projective limits of Choquet simplexes, Math. Scand. **22** (1968), 145–164.

**10**. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, Simple stably projectionless C*-algebras with generalized tracial rank one, preprint, 2017 (68 pages). arXiv:1711.01240.

**11**. G. A. Elliott, L. Robert, and L. Santiago, The cone of lower semicontinuous traces on a C*-algebra, Amer. J. Math. **133** (2011), 969–1005.

**12**. U. Haagerup, Quasitraces on exact C*-algebras are traces, C. R. Math. Acad. Sci. Soc. R. Can. **36** (2014), 67–92.

**13**. C. Ivanescu and D. Kucerovsky, Traces and Pedersen ideals of tensor products of nonunital C*-algebras, New York J. Math. **25** (2019), 423–450.

**14**. E. Kirchberg, UHF-algebras and functorial properties of exactness, J. Reine Angew. Math. **452** (1994), 39–77.

**15**. A. J. Lazar, Affine products of simplexes, Math. Scand. **22** (1968), 165–175.

**16**. A. J. Lazar and J. Linderstrauss, Banach spaces whose duals are $L_1$ spaces and their representing matrices, Acta Math. **126** (1971), 165–193.

**17**. I. Namioka and R. R. Phelps, Tensor products of compact convex sets, Pacific J. Math. **31** (1969), 469–480.

**18**. L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, Adv. Math. **231** (2012), 2802–2936.

**19**. Z. Semadeni, Categorical methods in convexity, Proc. Colloquium on Convexity (Copenhagen, 1965), 281–307. Københavns Univ. Mat. Inst., Copenhagen, 1967.

**20**. M. Takesaki, On the cross-norm of the direct product of C*-algebras, Tôhoku Math. J. (2) **16** (1964), 111–122.

**21**. A. P. Tikuisis and A. S. Toms, On the construction of Cuntz semigroups in (possibly) nonunital C*-algebras, Canad. Math. Bull. **58** (2015), 402–414.
**22**. W. Winter, Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras, Inv. Math. **187** (2012), 259–342.

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