

SUBGROUPS OF THE GROUP OF FORMAL POWER SERIES WITH THE BIG POWERS CONDITION

ALEXANDER BRUDNYI

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ABSTRACT. We study the structure of countable subgroups of the group $G[[r]]$ of complex formal power series under the operation of composition of series. In particular, we prove that every finitely generated fully residually free group is embeddable in $G[[r]]$.

RÉSUMÉ. Nous étudions la structure des sous-groupes dénombrables du groupe $G[[r]]$ des séries de puissance formelle sous l'opération de la composition des séries. En particulier, nous prouvons que chaque groupe qui est finement engendré et ω -résiduellement libre admet un plongement dans $G[[r]]$.

1. Main Result Let $G[[r]]$ be the pronipotent group of formal power series of the form $r + \sum_{i=1}^{\infty} c_i r^{i+1}$, $c_i \in \mathbb{C}$, $i \in \mathbb{N}$, under the operation \circ of composition of series. In the paper we study the problem of the structure of countable subgroups of $G[[r]]$. The problem is of importance, in particular, in connection with the classification of local analytic foliations and the holonomy of local differential equations (see, e.g., [C], [CL], [EV], [IP], [L], [NY] and references therein). The deep results of [EV] show that in contrast to free pronipotent groups (see [LM, Cor. 4.7]) the group $G[[r]]$ contains two-generator countable subgroups which are neither abelian nor free (see also [NY] for further results in this direction). In turn, in [Br, Problem 4.15] we asked with regard to the center problem for families of Abel differential equations whether the fundamental groups of orientable compact Riemann surfaces are embeddable in $G[[r]]$. In this paper we answer this question affirmatively. Our approach is purely group-theoretical and can be applied to a wide class of pronipotent groups.

To formulate the main result of the paper we introduce several definitions.

Let G be a group and $u = (u_1, \dots, u_k)$, $k \in \mathbb{N}$, be a tuple of non-trivial elements of G . We say that u is *commutation-free* if $[u_i, u_{i+1}] := u_i u_{i+1} u_i^{-1} u_{i+1}^{-1} \neq 1$ for all $1 \leq i \leq k-1$. In turn, u is called *independent* if there exists an integer $n = n(u) \in \mathbb{N}$ such that $u_1^{\alpha_1} \cdots u_k^{\alpha_k} \neq 1$ for any integers $\alpha_1, \dots, \alpha_k \geq n$.

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DEFINITION 1.1. *The group G satisfies the big powers condition if every commutation-free tuple in G is independent.*

The groups subject to the definition are referred to as *BP-groups*. The class of *BP-groups* contains torsion-free abelian groups, free groups and torsion-free hyperbolic groups. Also, subgroups and direct and inverse limits of *BP-groups* are *BP* as well. On the other hand, e.g., nonabelian torsion-free nilpotent groups are not *BP* (see [KMS, Thm. 1]). We recommend the paper [KMS] for the corresponding references and other examples and properties of *BP-groups* and their applications in group theory.

Let δ be an ordinal of cardinality $\leq \mathfrak{c}$ and

$$(1.1) \quad G_0 \leq G_1 \leq \cdots \leq G_\alpha \leq G_{\alpha+1} \leq \cdots \leq G_\delta$$

be a chain of subgroups such that for each limit ordinal λ

$$G_\lambda := \bigcup_{\alpha < \lambda} G_\alpha.$$

Suppose that for each successor ordinal $\alpha + 1 \leq \delta$ one of the following conditions holds:

- (i) $G_{\alpha+1} = G_\alpha *_{C_\alpha} F_\alpha$, where F_α is a nontrivial subgroup of G_α , and either $C_\alpha = \{1\}$ or $C_\alpha = C_{G_\alpha}(u) = C_{F_\alpha}(u)^1$ for some nontrivial $u \in F_\alpha$;
- (ii) $G_{\alpha+1}$ is an extension of a centralizer of G_α .

Recall that an *extension of a centralizer* of a group G is the group $\langle G, t \mid [c, t] = 1, c \in C_G(u) \rangle$ for some nontrivial $u \in G$.

THEOREM 1.2. *G_δ is a BP-group embeddable in $G[[r]]$ if and only if G_0 is.*

EXAMPLE 1.3. (1) Let $G_0 (\cong \mathbb{C})$ be a one-parametric subgroup of $G[[r]]$ and $G_{\alpha+1} = G_\alpha * G_0$ for all successor ordinals $\alpha + 1 \leq \delta$, where δ is of the cardinality of the continuum \mathfrak{c} . Then G_δ is isomorphic to the free product of \mathfrak{c} copies of \mathbb{C} and due to Theorem 1.2 it is a *BP-group* embeddable in $G[[r]]$.

(2) A group G is called *fully residually free* if for any finite subset X of G there exists a homomorphism from G to a free group that is injective on X . The notion was introduced in [B2] and since then extensively studied in connection with important problems of group theory and logic. Deep results of [MR] and [KM] assert that a *finitely generated fully residually free group* is embeddable in a finite sequence of extensions of centralizers of the free group of rank two. Hence, due to Theorem 1.2(b) and part (1) of the example a finitely generated fully residually free group is a *BP-group* embeddable in $G[[r]]$. Since all non-exceptional fundamental groups of compact Riemann surfaces (i.e., distinct from the fundamental groups of non-orientable surfaces of Euler characteristic 1, 0 or

¹ $C_G(u) \leq G$ stands for the centralizer of an element u of a group G .

–1) are fully residually free (see [B1]), they are embeddable in $G[[r]]$ ². This answers [Br, Problem 4.15].

(3) Let $G^{\mathbb{Z}[t]}$ be the Lyndon's completion of a finitely generated fully residually free group G . The notion was introduced in [L] in order to describe the solutions of equations in a single variable with coefficients in a free group. The recent result of [MR] asserts that $G^{\mathbb{Z}[t]}$ is the direct limit of a countable chain of extensions of centralizers $G \leq G_1 \leq G_2 \leq \dots$. Hence, Theorem 1.2(b) and part (2) imply that $G^{\mathbb{Z}[t]}$ is a *BP*-group embeddable in $G[[r]]$.

REMARK 1.4. (1) Let $\mathbb{F} \subset \mathbb{C}$ be a subfield and $G_{\mathbb{F}}[[r]] < G[[r]]$ be the subgroup of series with coefficients in \mathbb{F} . A minor modification of the proof of Theorem 1.2 (see Section 4) leads to the following result.

THEOREM 1.5. *Suppose the cardinality of G_0 is less than \mathfrak{c} . Then G_{δ} is a *BP*-group embeddable in $G_{\mathbb{R}}[[r]]$ if and only if G_0 is.*

In particular, the Lyndon's completion $G^{\mathbb{Z}[t]}$, where G is a finitely generated fully residually free group, is embeddable in $G_{\mathbb{R}}[[r]]$.

(2) In view of our main result the following questions seem plausible.

PROBLEM. (a) *Is $G[[r]]$ a *BP*-group?*

(b) *Suppose groups G_1, G_2 are embeddable in $G[[r]]$. Is $G_1 * G_2$ embeddable in $G[[r]]$?*

(c) *Let $\bar{\mathbb{Q}}$ be the algebraic closure of the field of rational numbers \mathbb{Q} . Is a finitely generated fully residually free group embeddable in $G_{\bar{\mathbb{Q}}}[[r]]$?*

(d) *Is a finitely generated fully residually free group embeddable in $G[[r]] \cap \text{Diff}(\mathbb{R}, 0)$?*

(Note that the proof of Theorem 1.2 uses the fact that the transcendence degree of \mathbb{C} is \mathfrak{c} .)

In a forthcoming paper we present some applications of Theorems 1.2 and 1.5 to the center problem for ordinary differential equations.

2. Auxiliary Results

2.1. In our proofs we use the following notion equivalent to the *BP* condition.

We say that a group G satisfies the *separation condition* if for any positive integer k and any tuples $u = (u_1, \dots, u_k)$ and $g = (g_1, \dots, g_{k+1})$ of elements from G such that

$$[g_{i+1}^{-1} u_i g_{i+1}, u_{i+1}] \neq 1 \quad \text{for } i = 1, \dots, k-1,$$

there exists an integer $n = n(u, g)$ such that

$$g_1 u_1^{\alpha_1} g_2 u_2^{\alpha_2} \cdots g_k u_k^{\alpha_k} g_{k+1} \neq 1$$

²They are also embeddable in the group $\text{Diff}(\mathbb{R}, 0)$ of germs of real analytic diffeomorphisms at the origin $0 \in \mathbb{R}$, see [CCGS].

for any integers $\alpha_1, \dots, \alpha_k \geq n$.

It was proved in [KMS, Prop. 1] that a group G satisfies the big powers condition if and only if it satisfies the separation condition.

2.2. We also use some known facts about the pronipotent group $G[[r]]$.

The Lie algebra \mathfrak{g} of $G[[r]]$ consists of formal vector fields of the form $\sum_{j=1}^{\infty} c_j e_j$, $c_j \in \mathbb{C}$, where $e_j := -x^{j+1} \frac{d}{dx}$. Here the Lie bracket satisfies the identities $[e_i, e_j] = (i-j)e_{i+j}$ for all $i, j \in \mathbb{N}$. Moreover, if v_r is the formal solution of the initial value problem

$$\frac{dv}{dx} = \sum_{j=1}^{\infty} c_j v^{j+1}, \quad v(0) = r,$$

then the exponential map $\exp : \mathfrak{g} \rightarrow G[[r]]$ sends the element $\sum_{j=1}^{\infty} c_j e_j$ to $v_r(1)$, where

$$(2.1) \quad v_r(1) = r + \sum_{i=1}^{\infty} \left(\sum_{i_1 + \dots + i_k = i} \frac{(i_1 + 1)(i_1 + i_2 + 1) \cdots (i - i_k + 1) c_{i_1} \cdots c_{i_k}}{k!} \right) r^{i+1}.$$

The map \exp is bijective. We denote its inverse by $\log : G[[r]] \rightarrow \mathfrak{g}$. Then for $h = r + \sum_{i=1}^{\infty} h_i r^{i+1}$, $h_i \in \mathbb{C}$,

$$(2.2) \quad \log h = \sum_{i=1}^{\infty} P_i(h_1, \dots, h_i) e_i,$$

where $P_i \in \mathbb{Q}[x_1, \dots, x_i]$, $i \in \mathbb{N}$.

In turn, let $w(X_1, \dots, X_n)$ be a word in the free group with generators X_1, \dots, X_n . For some $a_1, \dots, a_n \in \mathfrak{g}$ we set

$$\tilde{w}(a_1, \dots, a_n) := w(\exp(a_1), \dots, \exp(a_n)).$$

Then the formula for the composition of series and (2.1) imply that

$$(2.3) \quad \tilde{w}(a_1, \dots, a_n) = r + \sum_{i=1}^{\infty} Q_i(a_1, \dots, a_n) r^{i+1},$$

where Q_i is a polynomial with rational coefficients of degree i whose set of variables is the union of the first i coefficients of the series expansions of a_k , $1 \leq k \leq n$.

We also use the following fact.

LEMMA 2.1. *Elements $\exp(a_1), \exp(a_2) \in G[[r]]$ with nonzero $a_1, a_2 \in \mathfrak{g}$ commute iff $a_1 = \lambda a_2$ for some $\lambda \in \mathbb{C}$.*

PROOF. If $\exp(-a_2)\exp(a_1)\exp(a_2) = \exp(a_1)$, then passing to the logarithm we get

$$\text{ad}(\exp(a_2))(a_1) = a_1,$$

where ad is the differential at 1 of the map $\text{Ad}(\exp(a_2))(g) := \exp(-a_2)g\exp(a_2)$, $g \in G[[r]]$. Multiplying both parts of the previous equation by $t \in \mathbb{C}$ and taking the exponents we obtain that $\exp(-a_2)\exp(ta_1)\exp(a_2) = \exp(ta_1)$ for all $x \in \mathbb{C}$. This implies

$$[a_1, a_2] := \lim_{t \rightarrow 0} \frac{1}{t} (\text{ad}(\exp(ta_1))(a_2) - a_2) = 0.$$

Further, if $a_k = \sum_{j=j_k}^{\infty} c_{jk}e_j$, where $c_{j_k k} \neq 0$, $k = 1, 2$, then

$$0 = [a_1, a_2] = \sum_{n=1}^{\infty} \sum_{i+j=n} c_{i1}c_{j2}[e_i, e_j] = \sum_{n=1}^{\infty} \left(\sum_{i+j=n} c_{i1}c_{j2}(j-i) \right) e_n.$$

Thus,

$$(2.4) \quad \sum_{i+j=n} c_{i1}c_{j2}(j-i) = 0 \quad \text{for all } n \geq 1.$$

In particular, $c_{j_1 1}c_{j_2 2}(j_2 - j_1) = 0$, i.e., $j_2 = j_1$ and there exists a nonzero $\lambda \in \mathbb{C}$ such that $c_{j_1 1} = \lambda c_{j_2 2}$.

Assume now that we have proved that $c_{j_1 1} = \lambda c_{j_2 2}$ for all $j_1 \leq j < n$. Let us prove that $c_{n1} = \lambda c_{n2}$ as well. Indeed, due to (2.4) and our hypothesis we obtain

$$\begin{aligned} 0 &= \sum_{i+j=n+j_1} c_{i1}c_{j2}(j-i) = c_{n1}c_{j_1 2}(j_1 - n) + \lambda c_{j_1 2}c_{n2}(n - j_1) \\ &\quad + \sum_{i+j=n+j_1, i>j_1} \lambda c_{i2}c_{j2}(j-i) \\ &= c_{n1}c_{j_1 2}(j_1 - n) + \lambda c_{j_1 2}c_{n2}(n - j_1). \end{aligned}$$

Since $c_{j_1 2}(j_1 - n)$ is not equal to zero, this gives the desired statement. Hence, we obtain by induction that $a_1 = \lambda a_2$.

The converse statement is obvious. \square

A subgroup H of a group G is called *malnormal* if $H \cap g^{-1}Hg = \{1\}$, $g \in G$ implies $g \in H$. A group is called *CSA* if every maximal abelian subgroup is malnormal.

As a corollary of Lemma 2.1 we obtain:

PROPOSITION 2.2. *Any subgroup of $G[[r]]$ is CSA.*

PROOF. Let $H \subset G[[r]]$ and $A \subset H$ be a maximal abelian subgroup of H . Without loss of generality we may assume that H is nontrivial. Then A contains a centralizer $C_H(h)$ of a nontrivial element $h \in H$. Due to Lemma 2.1, each

$g \in H$ such that $[g, h] = 1$ is of the form $\exp(\lambda \log(h))$ for some nonzero $\lambda \in \mathbb{C}$. Then $A = \langle \exp(\lambda \log(h)) : \lambda \in \mathbb{C} \rangle \cap H = C_{G[[r]]}(h) \cap H := C_H(h)$.

Further, suppose $(g^{-1}Ag) \cap A \neq \{1\}$ for some nontrivial $g \in H$. Let us show that $g \in A$.

We have $g^{-1}hg = \exp(\mu \log(h))$ for some $\mu \in \mathbb{C}$. Let $h = r + \sum_{j=p}^{\infty} h_j r^{j+1}$ with $h_p \neq 0$. Let $G_{p+1} < G[[r]]$ be the normal subgroup of series of the form $r + \sum_{j=p+1}^{\infty} c_j r^{j+1}$, $c_j \in \mathbb{N}$, and $\varphi_{p+1} : G[[r]] \rightarrow G[[r]]/G_{p+1}$ be the quotient homomorphism. Then $\varphi_{p+1}(C_{G[[r]]}(h))$ belongs to the central subgroup and is isomorphic to \mathbb{C} , where the isomorphism sends $\varphi_{p+1}(\exp(\lambda \log(h)))$ to λh_p , $\lambda \in \mathbb{C}$. Hence,

$$\varphi_{p+1}(g^{-1}hg) = \varphi_{p+1}(h) = \varphi_{p+1}(\exp(\mu \log(h)))$$

which implies that $\mu = 1$. Thus $[g, h] = 1$ and by Lemma 2.1 $g \in C_{G[[r]]}(h) \cap H := C_H(h)$.

This completes the proof of the proposition. \square

3. Proof of Theorem 1.2

3.1. First, we prove the particular case of the theorem for the ordinal δ of cardinality 2, i.e., the following result.

THEOREM 3.1. (a) *Let H_1 and H_2 be nontrivial subgroups of a BP-group $H_0 \subset G[[r]]$. Then the group $H_1 *_C H_2$, where either $C = \{1\}$ or $H_1 \cap H_2 \neq \{1\}$ and there is a nontrivial $u \in H_1 \cap H_2$ such that $C = C_{H_1}(u) = C_{H_2}(u)$, is a BP-group embeddable in $G[[r]]$.*

(b) *An extension of a centralizer of a BP-subgroup of $G[[r]]$ is a BP-group embeddable in $G[[r]]$.*

PROOF. (a) Let $S \subset \mathbb{R}$ be the transcendence basis of \mathbb{C} over \mathbb{Q} . It is known that S is of the cardinality of the continuum. We write $S = S_0 \sqcup S_0^c$, where S_0 and S_0^c are of the cardinality of the continuum, and choose some $s, t \in S_0^c$. Then a bijection $S \rightarrow S_0$ extends to an embedding $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ such that s and t are algebraically independent over $\sigma(\mathbb{C})$. The isomorphism $\mathbb{C} \cong \sigma(\mathbb{C})$ induces an isomorphism $G[[r]] \cong G_{\sigma(\mathbb{C})}[[r]]$, where the latter is the subgroup of $G[[r]]$ of series with coefficients in $\sigma(\mathbb{C})$. Thus without loss of generality we may assume that $H_0 \leq G_{\sigma(\mathbb{C})}[[r]]$.

Let $C \leq H_1 \cap H_2$ be as in the statement of the theorem. First, we consider the case $C \neq \{1\}$. Then $C \leq C_{G[[r]]}(c) := \langle c^\alpha : \alpha \in \mathbb{C} \rangle$ for a fixed $c \in C \setminus \{1\}$; here we set for brevity $c^\alpha := \exp(\alpha \log(c))$.

LEMMA 3.2. *The group $\bar{H}_2 := c^{-s} H_2 c^s$ satisfies $\bar{H}_2 \cap H_1 = C$.*

PROOF. Since $C \leq C_{G[[r]]}(c)$, $C \leq \bar{H}_2 \cap H_1$. Suppose that there exists some $u \in (\bar{H}_2 \cap H_1) \setminus C$. Then $u = c^{-s} v c^s$ for some $v \in H_2 \setminus C$. Since s is algebraically independent over $\sigma(\mathbb{C})$ and the coefficients of the series expansion of u belong to

$\sigma(\mathbb{C})$, the latter identity implies that $u = c^{-\alpha}vc^\alpha$ for all $\alpha \in \mathbb{C}$ (see (2.2),(2.3)). Thus for $\alpha = 0$ we have $u = v$ and from here for $\alpha = 1$ we obtain that $[u, c] = 1$. Then Lemma 2.1 implies that $v = u \in C_{G[[r]]}(c) \cap H_2 = C$, a contradiction that proves the lemma. \square

Let $\tilde{H} \leq G[[r]]$ be a subgroup generated by \tilde{H}_2 and H_1 . Consider the epimorphism $\varphi : H_1 * H_2 \rightarrow \tilde{H}$ such that $f(h_1) := h_1$, $h_1 \in H_1$, and $f(h_2) := c^{-s}h_2c^s \in \tilde{H}_2$, $h_2 \in H_2$. Since $c^{-s}Cc^s = C$, φ descends to an epimorphism $\tilde{\varphi} : H_1 *_C H_2 \rightarrow \tilde{H}$.

LEMMA 3.3. *$\tilde{\varphi}$ is an isomorphism.*

PROOF. Let $h \in H_1 *_C H_2$ be such that $\tilde{\varphi}(h) = 1$. Then there exist h_1, \dots, h_{2k} , where $h_{2i-1} \in H_1$, $h_{2i} \in H_2$, $1 \leq i \leq k$, such that $h = h_1 * \dots * h_{2k}$ (here $*$ stands for the product on $H_1 *_C H_2$). Thus we have

$$\tilde{\varphi}(h) = h_1c^{-s}h_2c^s \dots h_{2k-1}c^{-s}h_{2k}c^s = 1.$$

Since s is algebraically independent over $\sigma(\mathbb{C})$ the latter implies a similar identity with an arbitrary $\alpha \in \mathbb{C}$ instead of s (see (2.2),(2.3)). In particular, for all $n \in \mathbb{Z}$,

$$(3.1) \quad h_1c^{-n}h_2c^n \dots h_{2k-1}c^{-n}h_{2k}c^n h_{2k+1} = 1, \quad h_{2k+1} := 1.$$

Since the element on the right belongs to the BP -group H_0 , by the separation condition (see Section 2.1) there exists $1 \leq j \leq 2k - 1$ such that

$$[h_{j+1}^{-1}c^{(-1)^j}h_{j+1}, c^{(-1)^{j+1}}] = 1.$$

Now Lemma 2.1 implies that $h_{j+1}^{-1}ch_{j+1} \in C := C_{G_s}(c)$, $s = 1, 2$. Hence, due to Proposition 2.2, $h_{j+1} \in C$. If $k = 1$, this and (3.1) imply that $\tilde{\varphi}(h) = h_1h_2 = 1$, $h_2 \in C$, and so $h_1 \in C$ as well. In particular, $h = h_1 * h_2 \in C \leq H_1 *_C H_2$. Since $\tilde{\varphi}|_C$ is identity, $h = 1$ in this case.

If $k > 1$, then

$$h_jc^{(-1)^j}h_{j+1}c^{(-1)^{j+1}}h_{j+2} = h_jh_{j+1}h_{j+2} \in G_{\frac{s}{2}}, \quad s = 3 + (-1)^j.$$

Therefore

$$h = \tilde{h}_1 * \dots * \tilde{h}_{2k-2}, \quad \text{where } \tilde{h}_i = h_i \text{ if } i \neq j, \text{ and } \tilde{h}_j := h_j * h_{j+1} * h_{j+2}.$$

Here $\tilde{h}_{2i-1} \in H_1$, $\tilde{h}_{2i} \in H_2$, $1 \leq i \leq k - 1$.

Applying such reductions $k-1$ times and using at the end the above considered case of $k = 1$ we obtain that $h = 1$.

This proves that $\tilde{\varphi}$ is a monomorphism and, hence, it is an isomorphism (as $\tilde{\varphi}$ is an epimorphism by definition). \square

Thus we have proved that \tilde{H} is a subgroup of $G[[r]]$ isomorphic to $H_1 *_C H_2$ for $C \neq \{1\}$.

Now suppose that $C = \{1\}$. Let us take $c := \exp(se_1 + s^2e_2) \in G[[r]] \setminus G_{\sigma(\mathbb{C})}[[r]]$ and set

$$\bar{H}_2 := c^{-t}H_2c^t.$$

Then similarly to Lemma 3.2 we get the following statement.

LEMMA 3.4. $\bar{H}_2 \cap H_1 = \{1\}$.

PROOF. If there exists some nontrivial $u \in \bar{H}_2 \cap H_1$, then $u = c^{-t}vc^t$ for some $v \in H_2$. As in the proof of Lemma 3.2 this implies $u = v \in H_2 \cap H_1$. If $H_2 \cap H_1 = \{1\}$, then we obtain a contradiction. For otherwise, as in the proof above the separation condition and Proposition 2.2 imply that $u = c^\alpha$ for some nonzero $\alpha \in \mathbb{C}$. Hence, $\log(u) = \alpha se_1 + \alpha s^2e_2$. Since the coefficients of the series expansion of $\log u$ belong to $\sigma(\mathbb{C})$, the latter yields $\alpha s, \alpha s^2 \in \sigma(\mathbb{C})$; hence $s = \frac{\alpha s^2}{\alpha s} \in \sigma(\mathbb{C})$. This contradicts the algebraic independence of s over $\sigma(\mathbb{C})$ and completes the proof of the lemma. \square

Let $\tilde{H} \leq G[[r]]$ be the subgroup generated by H_1 and \bar{H}_2 . Consider the surjective homomorphism $\varphi : H_1 * H_2 \rightarrow \tilde{H}$ such that $\varphi(h_1) = gh_1$, $h_1 \in H_1$, and $\varphi(h_2) = c^{-t}h_2c^t$, $h_2 \in H_2$.

LEMMA 3.5. φ is an isomorphism.

PROOF. Let $h \in \text{Ker}(\varphi)$. Then $h = h_1 * \dots * h_{2k}$ for some $h_{2i-1} \in H_1$, $h_{2i} \in H_2$, $1 \leq i \leq k$ (here $*$ stands for the product on $H_1 * H_2$). Thus we have

$$\varphi(h) = h_1c^{-t}h_2c^t \dots h_{2k-1}c^{-t}h_{2k}c^t = 1.$$

Since t is algebraically independent over $\sigma(\mathbb{C})$, arguing as in the proof of Lemma 3.3 we obtain that there exists $1 \leq j \leq 2k - 1$ such that

$$[h_{j+1}^{-1}c^{(-1)^j}h_{j+1}, c^{(-1)^{j+1}}] = 1.$$

Now Lemma 2.1 implies that $h_{j+1}^{-1}ch_{j+1} \in C_{G[[r]]}(c)$. Hence, due to Proposition 2.2, $h_{j+1} \in C_{G[[r]]}(c)$, i.e., $h_{j+1} = c^\alpha$ for some $\alpha \in \mathbb{C}$. Then arguing as in the proof of Lemma 3.4 we obtain that $\alpha = 0$. Hence, $h_{j+1} = 1$ and so $h = \tilde{h}_1 * \dots * \tilde{h}_{2k-2}$, where $\tilde{h}_i = h_i$ if $i \neq j$ and $\tilde{h}_j := h_j * h_{j+1} * h_{j+2}$. Here $\tilde{h}_{2i-1} \in H_1$, $\tilde{h}_{2i} \in H_2$, $1 \leq i \leq k - 1$.

Applying such reductions $k - 1$ times we obtain at the end that $h = 1$.

This completes the proof of the lemma. \square

Thus we have proved that in this case \tilde{H} is a subgroup of $G[[r]]$ isomorphic to $H_1 * H_2$.

Finally, in both cases groups \tilde{H} are *BP* by Theorem 4 and Corollary 6 of [KMS] whose conditions are satisfied due to [KMS, Prop. 5] and our Proposition 2.2.

This completes the proof of part (a) of the theorem.

(b) Let G be a BP -subgroup of $G[[r]]$ and $C = C_G(u)$ for a nontrivial $u \in G$. As in the proof of (a) we assume that $G \leq G_{\mathbb{F}}[[r]]$, where \mathbb{F} is a proper subfield of \mathbb{C} and $s \in \mathbb{C} \setminus \mathbb{F}$ is algebraically independent over \mathbb{F} . Consider a subgroup $\tilde{G} \leq G[[r]]$ generated by G and u^s .

LEMMA 3.6. *\tilde{G} is isomorphic to the group $G_t := \langle G, t \mid [c, t] = 1, c \in C_G(u) \rangle$.*

PROOF. Consider the epimorphism $\varphi : G * \mathbb{Z} \rightarrow \tilde{G}$ such that $\varphi(g) = g, g \in G$, and $\varphi(n) = u^{ns}, n \in \mathbb{Z}$. Since $[\varphi(1), c] = 1, c \in C_G(u)$, φ descends to an epimorphism $\tilde{\varphi} : G_t \rightarrow \tilde{G}$. Let us show that $\tilde{\varphi}$ is a monomorphism. This will complete the proof of the lemma.

Let $g \in \text{Ker}(\tilde{\varphi})$. Then $g = g_1 * t^{\alpha_1} * \dots * g_k * t^{\alpha_k}$, where $g_i \in G, \alpha_i \in \mathbb{Z}, 1 \leq i \leq k$ (here $*$ is the product on G_t). Thus we have

$$(3.2) \quad \tilde{\varphi}(g) = g_1 u^{\alpha_1 s} \dots g_k u^{\alpha_k s} = 1.$$

If $k = 1$, then we obtain that $g_1 = u^{-\alpha_1 s}$. Since s is algebraically independent over \mathbb{F} and the coefficients of the series expansion of g_1 belong to \mathbb{F} , this implies that $\alpha_1 = 0$, hence, $g_1 = 1$ and $g = g_1 * t^{\alpha_1} = 1$.

For otherwise, by the same reason (3.2) implies that

$$g_1 u^{\alpha_1 n} \dots g_k u^{\alpha_k n} = 1, \quad n \in \mathbb{Z}.$$

The expressions on the right belong to the BP -group G , hence, due to the separation condition (see Section 2.1) there exists $1 \leq i < k$ such that

$$[g_{i+1}^{-1} u^{\alpha_i} g_{i+1}, u^{\alpha_{i+1}}] = 1.$$

If both $\alpha_i, \alpha_{i+1} \neq 0$, then arguing as in the proof of part (a) we obtain that $g_{i+1} \in C_G(u)$. This reduces the length of the word representing g from k to $k - 1$. The same is true if $\alpha_i = \alpha_{i+1} = 0$ and $i + 1 < k$. Finally, if $\alpha_k = 0$, then the separation condition provides a similar commutativity relation with a new $i < k - 1$ which leads to the word reduction for g as well. Applying this reduction procedure $k - 1$ times and using the above considered case $k = 1$, we get that $g = 1$, i.e. $\tilde{\varphi}$ is an injection. \square

To complete the proof of part (b) note that G_t is a BP -group due to [KMS, Thm. 4]. \square

3.2. Proof of Theorem 1.2

PROOF. Let $S \subset \mathbb{C}$ be the transcendence basis of \mathbb{C} over \mathbb{Q} . We write $S = S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3$, where all S_i are of the cardinality of the continuum. Then a bijection $S \rightarrow S_0$ extends to an embedding $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ such that $S \setminus S_0$ is the transcendence basis of \mathbb{C} over $\sigma(\mathbb{C})$. The isomorphism $\mathbb{C} \cong \sigma(\mathbb{C})$ induces an

isomorphism $G[[r]] \cong G_{\sigma(\mathbb{C})}[[r]]$. Thus without loss of generality we may assume that $G_0 \leq G_{\sigma(\mathbb{C})}[[r]]$.

Further, since the ordinal δ is of cardinality $\leq \mathfrak{c}$, there exist injections $\tau_i : \delta \rightarrow S_i$, $1 \leq i \leq 3$.

To prove the result we use the transfinite induction based on Theorem 3.1.

Specifically, we prove that for each $\lambda \leq \delta$, G_λ is a *BP*-group and there is a monomorphism $\varphi_\lambda : G_\lambda \rightarrow G_{\mathbb{F}_\lambda}[[r]]$, where $\mathbb{F}_\lambda \subset \mathbb{C}$ is the minimal subfield containing $\sigma(\mathbb{C})$ and all $\tau_i(\gamma)$, $\gamma \leq \lambda$, $i = 1, 2, 3$, such that $\varphi_\lambda|_{G_\alpha} = \varphi_\alpha$ for all $\alpha < \lambda$.

For $\lambda = 0$ the result holds trivially with $\varphi_0 = \text{id}$. Assuming that the result holds for all ordinals $< \lambda$ let us prove it for λ .

First, assume that λ is a limit ordinal. By the definition,

$$G_\lambda := \bigcup_{\alpha < \lambda} G_\alpha.$$

Since all G_α , $\alpha < \lambda$, are *BP*-groups by the induction hypothesis, their union G_λ is a *BP*-group as well.

Now, we set

$$\varphi_\lambda(g) := \varphi_\alpha(g), \quad g \in G_\alpha, \quad \alpha < \lambda.$$

Then due to the induction hypothesis, φ_λ is a well-defined monomorphism of G_λ to $G[[r]]$. Moreover, the coefficients of the series expansions of elements of $\varphi_\lambda(G_\lambda)$ belong to $\bigcup_{\alpha < \delta} \mathbb{F}_\alpha$. Clearly, the latter is a subfield of \mathbb{F}_λ which proves the required statement in this case.

Next, assume that λ is a successor ordinal, i.e., $\lambda = \alpha + 1$ for an ordinal $\alpha < \lambda$. We apply Theorem 3.1 as follows.

If $G_{\alpha+1} = G_\alpha *_{C_\alpha} F_\alpha$, where F_α is a nontrivial subgroup of G_α , and either $C_\alpha = \{1\}$ or $C_\alpha = C_{G_\alpha}(u) = C_{F_\alpha}(u)$ for some nontrivial $u \in F_\alpha$, then we choose in Theorem 3.1(a) $H_0 = H_1 = \varphi_\alpha(G_\alpha)$, $H_2 = \varphi_\alpha(F_\alpha)$ and $s = \tau_1(\alpha + 1) \in S_1$, $t = \tau_2(\alpha + 1) \in S_2$. Then the proof of the theorem implies that $G_{\alpha+1}$ is embeddable in $G[[r]]$ and the corresponding monomorphism of Lemma 3.3 $\tilde{\varphi}$ denoted in our case by $\varphi_{\alpha+1}$ extends φ_α and is such that the coefficients of series expansions of elements of $\varphi_\alpha(G_{\alpha+1})$ belong to the minimal subfield of \mathbb{C} containing \mathbb{F}_α and $\tau_1(\alpha + 1)$, $\tau_2(\alpha + 1)$ which is clearly a subfield of $\mathbb{F}_{\alpha+1}$.

If $G_{\alpha+1}$ is an extension of a centralizer of G_α , then we set in the proof of Theorem 3.1(b), $G = \varphi_\alpha(G_\alpha)$ and $s = \tau_3(\alpha + 1)$. Due to the theorem, $G_{\alpha+1}$ is embeddable in $G_{\mathbb{C}}[[r]]$ and the corresponding monomorphism of Lemma 3.6 $\tilde{\varphi}$ denoted now by $\varphi_{\alpha+1}$ extends φ_α and is such that the coefficients of series expansions of elements of $\varphi_\alpha(G_{\alpha+1})$ belong to the minimal subfield of \mathbb{C} containing \mathbb{F}_α and $\tau_3(\alpha + 1)$ which is a subfield of $\mathbb{F}_{\alpha+1}$. Moreover, in both cases $G_{\alpha+1}$ is a *BP*-group. This completes the proof of the inductive step and, hence, of Theorem 1.2. \square

4. Proof of Theorem 1.5 Repeating word-for-word the proof of Proposition 2.2 one obtains that any subgroup of the group $G_{\mathbb{R}}[[r]]$ is *CSA* and, moreover, maximal abelian subgroups of a nontrivial $H \leq G_{\mathbb{R}}[[r]]$ have the form $C_H(u) = C_{G_{\mathbb{R}}[[r]]}(u) \cap H = \langle \exp(\lambda \log(u)) : \lambda \in \mathbb{R} \rangle \cap H$ for nontrivial $u \in H$. One uses this to prove the following version of Theorem 3.1.

Let $\mathbb{F} \subset \mathbb{R}$ be a subfield such that the transcendence degree of \mathbb{R} over \mathbb{F} is at least two.

THEOREM 4.1. (a) *Let H_1 and H_2 be nontrivial subgroups of a BP-group $H_0 \subset G_{\mathbb{F}}[[r]]$. Then the group $H_1 *_C H_2$, where either $C = \{1\}$ or $H_1 \cap H_2 \neq \{1\}$ and there is a nontrivial $u \in H_1 \cap H_2$ such that $C = C_{H_1}(u) = C_{H_2}(u)$, is a BP-group embeddable in $G_{\mathbb{R}}[[r]]$.*

(b) *An extension of a centralizer of a BP-subgroup of $G_{\mathbb{F}}[[r]]$ is a BP-group embeddable in $G_{\mathbb{R}}[[r]]$.*

PROOF. Suppose $S = S_0 \sqcup S_0^c \subset \mathbb{R}$ is the transcendence basis of \mathbb{R} over \mathbb{Q} , where S_0 is the transcendence basis of \mathbb{F} over \mathbb{Q} . By the definition of \mathbb{F} there exist some $s, t \in S_0^c$ algebraically independent over \mathbb{F} . Starting with these elements we repeat literally the proof of Theorem 3.1 replacing $\sigma(\mathbb{C})$ by \mathbb{F} , \mathbb{C} by \mathbb{R} and $G[[r]]$ by $G_{\mathbb{R}}[[r]]$ to get the required statement. \square

PROOF OF THEOREM 1.5. Since the cardinality of G_0 is less than \mathfrak{c} , the field $\mathbb{F} \subset \mathbb{R}$ generated by coefficients of series expansions of elements from G_0 has the cardinality less than \mathfrak{c} as well. Suppose $S = S_0 \sqcup S_0^c \subset \mathbb{R}$ is the transcendence basis of \mathbb{R} over \mathbb{Q} such that S_0 is the transcendence basis of \mathbb{F} over \mathbb{Q} . Since S is of the cardinality of the continuum, $S_0^c := S \setminus S_0$ is of the cardinality of the continuum as well. Hence, we can write $S_0^c = S_1 \sqcup S_2 \sqcup S_3 \subset \mathbb{R}$ where all S_i , $1 \leq i \leq 3$, are of the cardinality of the continuum. From now on the proof repeats literally that of Theorem 1.2 with $\sigma(\mathbb{C})$ replaced by \mathbb{F} , \mathbb{C} by \mathbb{R} and $G[[r]]$ by $G_{\mathbb{R}}[[r]]$. We leave the details to the reader. \square

REFERENCES

- B1.** G. Baumslag, On generalised free products, *Math. Z.* **78** (1962), 423–438.
- B2.** B. Baumslag, Residually free groups, *Proc. Lond. Math. Soc.* **17** (1967), 402–418.
- Br.** A. Brudnyi, Some algebraic aspects of the center problem for ordinary differential equations, *Qual. Theory Dyn. Syst.* **9** (2010), 9–28.
- C.** D. Cerveau, Une liste de problèmes, in *Ecuaciones Diferenciales*, Univ. Valladolid (1997), 455–460.
- CL.** D. Cerveau and F. Loray, Un théorème de Frobenius singulier via l’arithmétique élémentaire, *J. Number Theory* **68** (1998), 217–228.
- CCGS.** S. Cantat, D. Cerveau, V. Guirardel and J. Souto, Surface groups in the group of germs of analytic diffeomorphisms in one variable, Preprint, 2019.
- EV.** J. Ecalle, B. Vallet, Intertwined mappings, *Ann. Fac. Sci. Toulouse Math.* (6) **13** (2004), 291–376.
- IP.** Yu. S. Ilyashenko and A. S. Pyartli, The monodromy group at infinity of a generic polynomial vector field on the complex projective plane, *Russian J. Math. Phys.* **2** (1994), 275–315.
- KM.** O. Kharlampovich and A. Myasnikov, Irreducible affine varieties over a free group, II. Systems in triangular quasi-quadratic form and description of residually free groups, *J. Algebra* **200** (1998), 517–570.

- KMS.** A. V. Kvaschuk, A. G. Myasnikov and D. E. Serbin, Pregarps and the big powers condition, *Algebra and Logic* **48** (2009), 193–213.
- KS.** A. Karrass and D. Solitar, The subgroups of a free product of two groups with an amalgamated subgroup, *Trans. Amer. Math. Soc.* **150** (1970), 227–255.
- LM.** A. Lubotzky and A. R. Magid, Free prounipotent groups. *J. Algebra* **80** (1983), 323–349.
- L.** F. Loray, Formal invariants for nonsolvable subgroups of $\text{Diff}^\omega(\mathbb{C}, 0)$, *J. Algebra* **247** (2002), 95–103.
- MR.** A. Myasnikov and V. Remeslennikov, Exponential groups, II. Extensions of centralizers and tensor completion of CSA-groups, *Internat. J. Algebra Comput.* **6** (1996), 687–711.
- NY.** I. Nakai and K. Yanai, Relations of formal diffeomorphisms and the center problem, *Mosc. Math. J.* **10**, (2010), 415–468.

Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada
T2N 1N4
e-mail: abrudnyi@ucalgary.ca