

## HIGHER DERIVATIONS AND HOCHSCHILD HOMOLOGY

A. BANERJEE

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**ABSTRACT.** Let  $K$  be a commutative ring and let  $A$  be a  $K$ -algebra. A  $K$ -linear derivation  $D$  on  $A$  induces a morphism  $L_D$  on the Hochschild and cyclic homologies of  $A$ , which is the analogue of the Lie derivative in noncommutative geometry. In this paper, we extend this to higher (or Hasse–Schmidt) derivations on  $A$ .

**RÉSUMÉ.** Soit  $K$  un anneau commutatif et soit  $A$  une  $K$ -algèbre. Une dérivation  $K$ -linéaire  $D$  de  $A$  induit un endomorphisme  $L_D$  de l'homologie de Hochschild et de l'homologie cyclique de  $A$ , qui est l'analogue en géométrie non-commutative de la dérivée de Lie. Dans cet article, nous généralisons cette construction aux dérivations d'ordre supérieur (ou dérivations de Hasse–Schmidt) de  $A$ .

**1. Introduction.** Let  $K$  be a commutative ring and let  $A$  be a (not necessarily commutative)  $K$ -algebra. Then the cyclic homology  $HC_*(A)$  of  $A$ , as introduced by Connes [2] plays the role of de Rham cohomology in noncommutative geometry. If  $D$  is a derivation on  $A$ , Goodwillie [4] (see also Rinehart [8]) showed that  $D$  induces morphisms  $L_D$  of degree 0,

$$(1.1) \quad L_D: HC_*(A) \longrightarrow HC_*(A) \quad L_D: HH_*(A) \longrightarrow HH_*(A),$$

on the Hochschild homologies  $HH_*(A)$  and cyclic homologies  $HC_*(A)$  of the algebra  $A$ , which play the role of the Lie derivative in noncommutative geometry.

In this paper, we will extend the above mentioned result to higher derivations. We recall that a sequence  $D = \{d_n\}_{n \geq 0}$  of  $K$ -linear morphisms  $d_n: A \longrightarrow A$  with  $d_0 = \text{id}$  is said to be a *higher derivation* if, for each  $n \geq 1$ , we have:

$$d_n(ab) = \sum_{i=0}^n d_i(a)d_{n-i}(b) \quad \forall a, b \in A.$$

Higher derivations, first introduced by Hasse and Schmidt [5], have also been referred to as Hasse–Schmidt derivations in the literature. The study of higher derivations on an algebra  $A$ , particularly when the ring  $K$  is a field of positive characteristic, has been dealt with extensively in the literature (see Sweedler [10], Weisfeld [11]) and has interesting consequences in field extensions and Galois

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theory (see [3], [12]). More recently, Mirzavaziri [7] has obtained a simple characterization of higher derivations when  $K$  is a field of characteristic zero. For other applications and generalizations of higher derivations, see [1] and [13].

Given a higher derivation  $D = \{d_n\}_{n \geq 0}$  on  $A$ , for each  $n \geq 0$ , we construct morphisms

$$(1.2) \quad L_D^n: HC_*(A) \longrightarrow HC_*(A), \quad L_D^n: HH_*(A) \longrightarrow HH_*(A),$$

of degree 0 that generalize the Lie derivative in (1.1). Thereafter, given two  $K$ -algebras  $A$  and  $B$  carrying higher derivations  $D = \{d_n\}_{n \geq 0}$  and  $E = \{e_n\}_{n \geq 0}$ , we construct a higher derivation  $D \otimes E$  on  $A \otimes B$ . Then this is applied to show that when  $A$  is a commutative  $K$ -algebra, the sequence of maps  $\{L_D^n\}_{n \geq 0}$  defines a higher derivation on  $HH_*(A)$  with respect to the shuffle product.

Following this, we consider the Lie algebra  $\mathcal{L}$  of ordinary  $K$ -linear derivations on  $A$  and let  $\mathcal{H} = \mathcal{U}(\mathcal{L})$  denote the universal enveloping algebra of  $\mathcal{L}$ . Then in the particular case when  $K$  is a field of characteristic zero, we show, using the description of higher derivations in characteristic zero due to Mirzavaziri [7], that the operators  $L_D^n$  in (1.2) can be obtained from the coproduct in  $\mathcal{H}$ . Thereafter, we show that if  $u \in A$  is a given element, the sequence  $D_u = \{d_k\}_{k \geq 0}$  of  $K$ -linear maps on  $A$  given by  $d_0 = \text{id}$ ,  $d_k = \{(-1)^k u^{k-1} [u, \cdot]\}_{k \geq 1}$  defines a higher derivation on  $A$ . When  $K$  is a field of characteristic zero, we show that the maps  $L_{D_u}^n$ ,  $n \geq 1$  induced on  $HH_*(A)$  and  $HC_*(A)$  by the higher derivation  $D_u$  are zero. This generalizes the fact that the Lie derivative on  $HH_*(A)$  and  $HC_*(A)$  as in (1.1) induced by an inner derivation  $[u, \cdot]$  on  $A$  is zero.

**2. Lie derivative for higher derivations.** In this paper, unless otherwise mentioned, we will always let  $K$  be a commutative ring and let  $A$  be a  $K$ -algebra that is not necessarily commutative. For any  $n \geq 0$ , we let  $C_n(A) = A \otimes A^{\otimes n}$ , the tensor products being taken over the ring  $K$ . For the sake of convenience, an element  $(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \in C_n(A)$  will often be denoted by  $(a_0, a_1, \dots, a_n)$ . For any  $n \geq 0$ , we will let  $HH_n(A)$  and  $HC_n(A)$  respectively denote the Hochschild and cyclic homologies of  $A$ . We also set  $HH_*(A) = \bigoplus_{n=0}^{\infty} HH_n(A)$  and  $HC_*(A) = \bigoplus_{n=0}^{\infty} HC_n(A)$ . For details on Hochschild and cyclic homology, we refer the reader to Loday [6].

**DEFINITION 2.1.** Let  $K$  be a commutative ring and let  $A$  be a  $K$ -algebra. A sequence  $D = \{d_n\}_{n \geq 0}$  of  $K$ -linear maps  $d_n: A \longrightarrow A$  with  $d_0 = \text{id}$  is said to be a higher (or Hasse–Schmidt) derivation on  $A$ , if, for each  $n \geq 0$ , we have

$$d_n(ab) = \sum_{i=0}^n d_i(a) d_{n-i}(b) \quad \forall a, b \in A.$$

For instance, if  $d$  is an ordinary derivation and  $K$  is a field of characteristic zero, the sequence  $d_n := \{\frac{d^n}{n!}\}$ ,  $n \geq 0$  is a higher derivation on  $A$ . However, this is hardly the only example of a higher derivation in the literature. For several

other natural examples and applications of higher derivations, see [3], [9], [10], and [12].

It is well known (see [6]) that an ordinary derivation  $d$  on the algebra  $A$  induces natural operators of degree 0 on  $HH_*(A)$  and  $HC_*(A)$ . By abuse of notation, we denote both of these operators by  $L_d$ :

$$L_d: HH_*(A) \longrightarrow HH_*(A), \quad L_d: HC_*(A) \longrightarrow HC_*(A).$$

The operator  $L_d$  plays the role of a Lie derivative in noncommutative geometry. Our objective is to extend the operators  $L_d$  on  $HH_*(A)$  and  $HC_*(A)$  to higher derivations  $D = \{d_n\}_{n \geq 0}$  on the algebra  $A$ . For some  $k \geq 0$ , we consider  $(a_0, a_1, \dots, a_k) \in C_k(A)$  and define, for each  $n \geq 1$ :

$$(2.1) \quad L_D^n(a_0, a_1, \dots, a_k) := \sum_{\substack{(j_0, j_1, \dots, j_k) \\ j_0 + j_1 + \dots + j_k = n}} (d_{j_0}(a_0), d_{j_1}(a_1), \dots, d_{j_k}(a_k)).$$

For the sake of convenience, unless otherwise mentioned, we will often denote a sum as in (2.1), taken over all ordered tuples  $(j_0, \dots, j_k)$  of nonnegative integers  $j_i$ ,  $0 \leq i \leq k$ , such that  $j_0 + \dots + j_k = n$  simply by

$$\sum_{j_0 + j_1 + \dots + j_k = n} (d_{j_0}(a_0), d_{j_1}(a_1), \dots, d_{j_k}(a_k)).$$

Moreover, we let  $L_D^0 = \text{id}$  be the identity map on  $C_*(A)$ . We will now show that the maps  $L_D^n: C_*(A) \longrightarrow C_*(A)$  induce morphisms on the Hochschild and cyclic homology of  $A$ .

**PROPOSITION 2.2.** *Let  $K$  be a commutative ring and let  $A$  be a  $K$ -algebra. Let  $D = \{d_n\}_{n \geq 0}$  be a sequence of  $K$ -linear morphisms on  $A$  defining a higher derivation on  $A$ . Then for each  $n \geq 1$  the morphisms  $L_D^n$  defined in (2.1) induce morphisms of degree 0 on the Hochschild and cyclic homologies of  $A$ :*

$$L_D^n: HH_*(A) \longrightarrow HH_*(A), \quad L_D^n: HC_*(A) \longrightarrow HC_*(A).$$

**PROOF.** We choose  $n \geq 1$ . Then for each  $k \geq 1$  let  $b_k: C_k(A) \longrightarrow C_{k-1}(A)$  denote the standard Hochschild differential. Then by definition, for each  $k \geq 1$ ,  $b_k = \sum_{i=0}^k (-1)^i b_k^i$  where  $b_k^i(a_0, a_1, \dots, a_k) = (a_0, \dots, a_i, a_{i+1}, \dots, a_k)$  for all  $0 \leq i \leq k-1$  and  $b_k^k(a_0, a_1, \dots, a_k) = (a_k, a_0, a_1, \dots, a_{k-1})$ . Then we have, for

$0 \leq i \leq k-1$ :

$$\begin{aligned}
(2.2) \quad L_D^n b_k^i(a_0, \dots, a_k) &= L_D^n(a_0, \dots, a_i a_{i+1}, \dots, a_k) \\
&= \sum_{j_0 + \dots + j_{k-1} = n} (d_{j_0}(a_0), \dots, d_{j_i}(a_i a_{i+1}), \dots, d_{j_{k-1}}(a_k)) \\
&= \sum_{j_0 + \dots + j_{k-1} = n} \sum_{j+j'=j_i} (d_{j_0}(a_0), \dots, d_j(a_i) d_{j'}(a_{i+1}), \dots, d_{j_{k-1}}(a_k)) \\
&= \sum_{j'_0 + \dots + j'_k = n} (d_{j'_0}(a_0), \dots, d_{j'_i}(a_i) d_{j'_{i+1}}(a_{i+1}), \dots, d_{j'_k}(a_k)) \\
&= b_k^i L_D^n(a_0, \dots, a_k).
\end{aligned}$$

As in (2.2), we can check that  $L_D^n b_k^i = b_k^i L_D^n$ . Hence,  $L_D^n b = \sum_{i=0}^k (-1)^i L_D^n b_k^i = \sum_{i=0}^k (-1)^i b_k^i L_D^n = b L_D^n$ . Hence, the higher derivation  $D$  induces morphisms  $L_D^n: HH_*(A) \longrightarrow HH_*(A)$  of degree 0 for each  $n \geq 1$ . Furthermore, if

$$t_k: C_k(A) \longrightarrow C_k(A), \quad t_k(a_0, \dots, a_k) := (-1)^k (a_k, a_0, \dots, a_{k-1})$$

denotes the cyclic operator on the cyclic set  $\{C_k(A)\}_{k \geq 0}$  defining the cyclic homology of  $A$  (see [6, Section 2.1]), it is clear that  $L_D^n t_k = t_k L_D^n$ . Hence the operators  $L_D^n$  also induce morphisms  $L_D^n: HC_*(A) \longrightarrow HC_*(A)$  of degree 0 on the cyclic homology of  $A$ .  $\square$

LEMMA 2.3. *Let  $A$  and  $B$  be two given  $K$ -algebras. Let  $D = \{d_n\}_{n \geq 0}$  and  $E = \{e_n\}_{n \geq 0}$  be two higher derivations on  $A$  and  $B$  respectively. For each  $n \geq 1$ , set*

$$(2.3) \quad (D \otimes E)_n: A \otimes B \longrightarrow A \otimes B \quad a \otimes b \mapsto \sum_{i=0}^n d_i(a) \otimes e_{n-i}(b)$$

and set  $(D \otimes E)_0 = \text{id}: A \otimes B \longrightarrow A \otimes B$ . Then the sequence  $D \otimes E := \{(D \otimes E)_n\}_{n \geq 0}$  is a higher derivation on  $A \otimes B$ .

PROOF. We choose elements  $a, a' \in A$  and  $b, b' \in B$ . Since the morphisms  $\{d_n\}_{n \geq 0}$  and  $\{e_n\}_{n \geq 0}$  are  $K$ -linear, it follows that the morphisms  $(D \otimes E)_n$

in (2.3) are well defined. Then we have

$$\begin{aligned}
(D \otimes E)_n(aa' \otimes bb') &= \sum_{i+j=n} d_i(aa') \otimes e_j(bb') \\
&= \sum_{i+j=n} \left( \sum_{i'+i''=i} d_{i'}(a)d_{i''}(a') \right) \otimes \left( \sum_{j'+j''=j} e_{j'}(b)e_{j''}(b') \right) \\
&= \sum_{i'+i''+j'+j''=n} (d_{i'}(a) \otimes e_{j'}(b)) \cdot (d_{i''}(a') \otimes e_{j''}(b')) \\
&= \sum_{n_1+n_2=n} (D \otimes E)_{n_1}(a \otimes b) \cdot (D \otimes E)_{n_2}(a' \otimes b').
\end{aligned}$$

□

Let  $A$  and  $B$  be two  $K$ -algebras. For any  $k \geq 0$  and given  $(a_0, a_1, \dots, a_k) \in C_k(A)$ , a permutation  $\sigma \in S_k$  acts on  $(a_0, a_1, \dots, a_k)$  as follows (see [6, 4.2.1.1]):

$$\sigma \cdot (a_0, a_1, \dots, a_k) := (a_0, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(k)}).$$

Then, if  $p, q \geq 0$  are integers and  $S_{p,q} \subseteq S_{p+q}$  denotes the set of  $(p, q)$  shuffles, *i.e.*, the collection of all  $\sigma \in S_{p+q}$  such that  $\sigma(1) < \sigma(2) < \dots < \sigma(p)$  and  $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$ , we know that the shuffle product (see [6, 4.2.1.2])

$$\begin{aligned}
&\text{sh}_{p,q}((a_0, \dots, a_p) \otimes (b_0, \dots, b_q)) \\
&= \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) \sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \dots, a_p \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_q)
\end{aligned}$$

induces a product  $\text{sh}_{p,q}: HH_p(A) \otimes HH_q(B) \longrightarrow HH_{p+q}(A \otimes B)$  on Hochschild homology groups. In particular, when  $A$  is a commutative  $K$ -algebra, the multiplication  $\mu: A \otimes A \longrightarrow A$  is a morphism of  $K$ -algebras and hence the shuffle product composed with the map induced by  $\mu$  gives an inner shuffle product map  $\text{sh}_{p,q}^\mu: HH_p(A) \otimes HH_q(A) \longrightarrow HH_{p+q}(A)$ . The shuffle products  $\text{sh}_{p,q}^\mu: HH_p(A) \otimes HH_q(A) \longrightarrow HH_{p+q}(A)$ ,  $p, q \geq 0$  together induce a multiplication  $\text{sh}^\mu: HH_*(A) \otimes HH_*(A) \longrightarrow HH_*(A)$  on the Hochschild homology ring  $HH_*(A)$ .

**PROPOSITION 2.4.** *Let  $A$  and  $B$  be two given  $K$ -algebras. Let  $D = \{d_n\}_{n \geq 0}$  and  $E = \{e_n\}_{n \geq 0}$  be two higher derivations on  $A$  and  $B$  respectively. Then for any  $p, q \geq 0$ , the sequence of morphisms  $\{L_{(D \otimes E)}^n\}_{n \geq 0}$  on  $HH_*(A \otimes B)$  satisfies*

$$L_{(D \otimes E)}^n(\text{sh}_{p,q}(x \otimes y)) = \sum_{i=0}^n \text{sh}_{p,q}(L_D^i(x) \otimes L_E^{n-i}(y)),$$

where  $x \in HH_p(A)$  and  $y \in HH_q(B)$ . In particular, if  $A$  is a commutative  $K$ -algebra, the sequence of morphisms  $\{L_D^n\}_{n \geq 0}$  defines a higher derivation on the Hochschild homology ring  $HH_*(A)$  with respect to the shuffle product.

PROOF. For any  $n \geq 1$  and  $a, a' \in A$ , we have  $d_n(aa') = \sum_{i=0}^n d_i(a)d_{n-i}(a')$

and hence we note that  $d_n(1) = 0$ . Then from the definition of  $(D \otimes E)$  in (2.3), it follows that  $(D \otimes E)_n(a \otimes 1) = d_n(a) \otimes 1$  and  $(D \otimes E)_n(1 \otimes b) = 1 \otimes e_n(b)$  for any  $a \in A, b \in B$ .

We fix integers  $p, q \geq 0$  and some  $(p, q)$ -shuffle  $\sigma \in S_{p,q}$ . If we choose any  $x = (a_0, a_1, \dots, a_p) \in C_p(A)$  and  $y = (b_0, b_1, \dots, b_q) \in C_q(B)$ , we can verify that

$$\begin{aligned} & \sigma \cdot L_{(D \otimes E)}^n(a_0 \otimes b_0, a_1 \otimes 1, \dots, a_p \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_q) \\ &= \sigma \cdot \sum_{j_0 + \dots + j_{p+q} = n} ((D \otimes E)_{j_0}(a_0 \otimes b_0), (D \otimes E)_{j_1}(a_1 \otimes 1), \dots \\ & \quad \dots, (D \otimes E)_{j_p}(a_p \otimes 1), (D \otimes E)_{j_{p+1}}(1 \otimes b_1), \dots \\ & \quad \dots, (D \otimes E)_{j_{p+q}}(1 \otimes b_{p+q})) \\ &= \sigma \cdot \sum_{n_1 + n_2 = n} \sum_{\substack{j'_0 + \dots + j'_p = n_1 \\ j''_0 + \dots + j''_q = n_2}} (d_{j'_0}(a_0) \otimes e_{j''_0}(b_0), d_{j'_1}(a_1) \otimes 1, \dots, d_{j'_p}(a_p) \otimes 1, \\ & \quad 1 \otimes e_{j''_1}(b_1), \dots, 1 \otimes e_{j''_q}(b_q)). \end{aligned}$$

Now, adding over all  $\sigma \in S_{p,q}$  (with signs), we have

$$\begin{aligned} & L_{(D \otimes E)}^n(\text{sh}_{p,q}(x \otimes y)) \\ &= L_{(D \otimes E)}^n(\text{sh}_{p,q}((a_0, a_1, \dots, a_p) \otimes (b_0, b_1, \dots, b_q))) \\ &= \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) \sigma \cdot L_{(D \otimes E)}^n(a_0 \otimes b_0, a_1 \otimes 1, \dots, a_p \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_q) \\ &= \sum_{n_1 + n_2 = n} \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) \sigma \sum_{\substack{j'_0 + \dots + j'_p = n_1 \\ j''_0 + \dots + j''_q = n_2}} (d_{j'_0}(a_0) \otimes e_{j''_0}(b_0), \dots, d_{j'_p}(a_p) \otimes 1, \\ & \quad 1 \otimes e_{j''_1}(b_1), \dots, 1 \otimes e_{j''_q}(b_q)) \\ &= \sum_{n_1 + n_2 = n} \text{sh}_{p,q}(L_D^{n_1}(a_0, \dots, a_p) \otimes L_E^{n_2}(b_0, \dots, b_q)) \\ &= \sum_{n_1 + n_2 = n} \text{sh}_{p,q}(L_D^{n_1}(x) \otimes L_E^{n_2}(y)). \end{aligned}$$

In particular, suppose that  $A = B$  and  $D = E$ . Then if  $A$  is a commutative  $K$ -algebra, the multiplication  $\mu: A \otimes A \rightarrow A$  is a morphism of  $K$ -algebras and for any  $k \geq 0$ , we denote by  $C_k(\mu)$  the induced morphism  $C_k(\mu): C_k(A \otimes A) \rightarrow C_k(A)$ . We note that, for any  $n \geq 1$ , we have  $L_D^n \circ C_k(\mu) = C_k(\mu) \circ L_{(D \otimes D)}^n$ . Then for any fixed integers  $p, q \geq 0$  and  $x = (a_0, a_1, \dots, a_p) \in C_p(A)$ ,  $y = (b_0, b_1, \dots, b_q) \in C_q(B)$ , we have

$$\begin{aligned} L_D^n(\text{sh}^\mu(x \otimes y)) &= L_D^n(C_{p+q}(\mu) \circ \text{sh}_{p,q}(x \otimes y)) \\ &= C_{p+q}(\mu) \circ L_{(D \otimes D)}^n(\text{sh}_{p,q}(x \otimes y)) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n_1+n_2=n} C_{p+q}(\mu) \circ \text{sh}_{p,q}(L_D^{n_1}(x) \otimes L_D^{n_2}(y)) \\
 &= \sum_{n_1+n_2=n} \text{sh}^\mu(L_D^{n_1}(x) \otimes L_D^{n_2}(y)). \quad \square
 \end{aligned}$$

Let  $D$  be a  $K$ -linear derivation on  $A$  in the ordinary sense. Then, as mentioned before,  $D$  induces a morphism  $L_D: C_*(A) \rightarrow C_*(A)$  of degree 0 on the Hochschild complex  $(C_*(A), b)$  of  $A$ . Further, we note that given derivations  $D_1$  and  $D_2$  on  $A$ , we have  $L_{[D_1, D_2]} = L_{D_1} \circ L_{D_2} - L_{D_2} \circ L_{D_1}$ . Let  $\mathcal{L}$  denote the Lie algebra of all  $K$ -linear derivations on  $A$  and let  $\mathcal{H} = \mathcal{U}(\mathcal{L})$  denote the universal enveloping algebra of  $\mathcal{L}$ . Then  $\mathcal{L}$  has a Lie algebra action on  $(C_*(A), b)$ . Hence, for each element  $h \in \mathcal{H} = \mathcal{U}(\mathcal{L})$ , there exists a morphism  $L_h: C_*(A) \rightarrow C_*(A)$  of degree 0 on the Hochschild complex of  $A$ . If  $\Delta$  denotes the coproduct on  $\mathcal{H}$  and for any  $l \geq 0$  and any  $h \in \mathcal{H}$ , we denote  $\Delta^l(h) = \sum h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(l+1)}$ , we note that the equation

$$(2.4) \quad L_h(a_0, \dots, a_l) := \sum (h_{(1)}(a_0), \dots, h_{(l+1)}(a_l)) \quad \forall (a_0, \dots, a_l) \in C_l(A)$$

holds for all  $h \in \mathcal{L} \subseteq \mathcal{H}$  and hence for all  $h \in \mathcal{H} = \mathcal{U}(\mathcal{L})$ . The morphisms  $L_h$  on the Hochschild complex descend to morphisms  $L_h: HH_*(A) \rightarrow HH_*(A)$  of degree 0 on the Hochschild homology groups of  $A$ .

Let  $D = \{d_k\}_{k \geq 0}$  be a higher derivation on  $A$ . For the rest of this paper, we will assume that the ground ring  $K$  is actually a field of characteristic zero. In this case, Mirzavaziri [7] has recently shown that each  $d_k$ ,  $k \geq 1$  actually lies in the Hopf algebra  $\mathcal{H}$  generated by  $K$ -linear derivations on  $A$ . More precisely, Mirzavaziri [7] has shown that, corresponding to  $D$ , there exists a sequence  $\{\delta_n\}_{n \geq 1}$  of ordinary derivations on  $A$  such that

$$(2.5) \quad d_k = \sum_{i=1}^k \left( \sum_{\sum_{j=1}^i r_j = k} \left( \prod_{j=1}^i \frac{1}{r_j + \cdots + r_i} \right) \delta_{r_1} \cdots \delta_{r_i} \right)$$

where the sum in (2.5) is taken over all positive integers  $r_j$  with  $\sum_{j=1}^i r_j = k$ . Hence, it follows from (2.4) that for each  $k \geq 0$ ,  $d_k \in \mathcal{H}$  induces a morphism  $L_{d_k}$  on  $HH_*(A)$ . We will now show that  $L_{d_k} = L_D^k$ , where  $L_D^k: HH_*(A) \rightarrow HH_*(A)$  is the morphism induced by (2.1). Since  $K$  is a field of characteristic zero, we will often speak of  $\mathbb{Z}$  and  $\mathbb{Q}$  as subrings of  $K$ .

**PROPOSITION 2.5.** *Let  $K$  be a field of characteristic zero and let  $A$  be a  $K$ -algebra. Let  $D = \{d_k\}_{k \geq 0}$  denote a higher derivation on  $A$ . For each  $k \geq 1$ , let  $L_{d_k}: HH_*(A) \rightarrow HH_*(A)$  be the morphism induced by (2.4) and let  $L_D^k: HH_*(A) \rightarrow HH_*(A)$  be the morphism induced by (2.1). Then for  $k \geq 1$  we have  $L_{d_k} = L_D^k$ .*

PROOF. Since  $K$  has characteristic zero, we know from (2.5) that each  $d_k$ ,  $k \geq 0$  lies in  $\mathcal{H}$  and that there exists an infinite sequence  $\{\delta_n\}_{n \geq 1}$  of derivations on  $A$  determining  $D = \{d_k\}_{k \geq 0}$ . Hence, from the definitions of  $L_{d_k}$  and  $L_D^k$  in (2.4) and (2.1) respectively, it suffices to show that for any  $l \geq 0$ , we have

$$(2.6) \quad \Delta^l(d_k) = \sum_{\sum_{i=0}^l j_i = k} d_{j_0} \otimes d_{j_1} \otimes \cdots \otimes d_{j_l}.$$

It is clear that the equation (2.6) holds for  $k = 0$  and  $k = 1$ . Given  $l$ , we suppose that (2.6) holds for all  $0 \leq k \leq K$ . From [7, Proposition 2.1], we know that

$$(2.7) \quad (M+1)d_{M+1} = \sum_{m=0}^M \delta_{m+1} d_{M-m} \quad \forall M \geq 0.$$

Hence  $\Delta^l(d_{K+1}) = \frac{1}{K+1} \sum_{m=0}^K \Delta^l(\delta_{m+1}) \Delta^l(d_{K-m})$ , from which it follows that

$$(2.8) \quad \Delta^l(d_{K+1}) = \frac{1}{K+1} \sum_{m=0}^K \left( \sum_{p=0}^l \delta_{m+1}^p \right) \left( \sum_{\sum_{i=0}^l j_i = K-m} d_{j_0} \otimes d_{j_1} \otimes \cdots \otimes d_{j_l} \right),$$

where  $\Delta^l(\delta_{m+1}) = \sum_{p=0}^l \delta_{m+1}^p$  and  $\delta_{m+1}^p$  refers to the term  $1 \otimes 1 \otimes \cdots \otimes \delta_{m+1} \otimes \cdots \otimes 1$  ( $\delta_{m+1}$  in the  $p$ -th position) appearing in the expression for  $\Delta^l(\delta_{m+1})$ . Let  $(j'_0, j'_1, \dots, j'_l)$  be a tuple of nonnegative integers with  $j'_0 + j'_1 + \cdots + j'_l = K+1$  and  $p$  be a chosen position such that  $j'_p \geq 1$ . Then we note that

$$\begin{aligned} & \sum_{m=0}^{j'_p-1} \delta_{m+1}^p \cdot (d_{j'_0} \otimes \cdots \otimes d_{j'_p-m-1} \otimes \cdots \otimes d_{j'_l}) \\ &= \sum_{m=0}^{j'_p-1} (d_{j'_0} \otimes \cdots \otimes \delta_{m+1} d_{j'_p-m-1} \otimes \cdots \otimes d_{j'_l}) \\ &= j'_p \cdot (d_{j'_0} \otimes \cdots \otimes d_{j'_p} \otimes \cdots \otimes d_{j'_l}), \end{aligned}$$

where the second equality above follows by applying (2.7) with  $M = j'_p - 1$ . As  $p$  varies from 0 to  $l$  and  $(j'_0, j'_1, \dots, j'_l)$  varies over all tuples of nonnegative integers satisfying  $j'_0 + \cdots + j'_l = K+1$ , we note that

$$\begin{aligned} & \sum_{m=0}^K \left( \sum_{p=0}^l \delta_{m+1}^p \right) \left( \sum_{\sum_{i=0}^l j_i = K-m} d_{j_0} \otimes d_{j_1} \otimes \cdots \otimes d_{j_l} \right) \\ &= \sum_{\sum_{i=0}^l j'_i = K+1} \sum_{p=0, j'_p \geq 1}^l \sum_{m=0}^{j'_p-1} \delta_{m+1}^p \cdot (d_{j'_0} \otimes \cdots \otimes d_{j'_p-m-1} \otimes \cdots \otimes d_{j'_l}) \\ &= \sum_{\sum_{i=0}^l j'_i = K+1} \sum_{p=0, j'_p \geq 1}^l j'_p \cdot (d_{j'_0} \otimes \cdots \otimes d_{j'_p} \otimes \cdots \otimes d_{j'_l}). \end{aligned}$$

Hence, the total number of times  $(d_{j'_0} \otimes \cdots \otimes d_{j'_p} \otimes \cdots \otimes d_{j'_l})$  appears in the sum above is  $\sum_{p=0, j'_p \geq 1}^l j'_p = \sum_{p=0}^l j'_p = K + 1$ . Considering the factor of  $1/(K + 1)$  appearing in (2.8), it now follows that  $\Delta^l(d_{K+1}) = \sum_{\sum_{p=0}^l j'_p = K+1} d_{j'_0} \otimes d_{j'_1} \otimes \cdots \otimes d_{j'_l}$ .  $\square$

It is well known (see [6, Section 4.1]) that if  $u \in A$  is a given element, the Lie derivative on  $HH_*(A)$  and  $HC_*(A)$  induced by the inner derivation  $[u, \cdot]$  on  $A$  is zero. We will now generalize this to higher derivations.

LEMMA 2.6. *Let  $A$  be a  $K$ -algebra and let  $u \in A$ . Let  $d_0 = \text{id}: A \rightarrow A$  and let  $d_n: A \rightarrow A$  be defined by  $d_n(a) = (-1)^n u^{n-1}[u, a]$ ,  $\forall a \in A$ . Then the sequence  $D_u = \{d_n\}_{n \geq 0}$  forms a higher derivation on  $A$ .*

PROOF. We choose some  $a, b \in A$ . It is clear that the morphism  $d_1 = -[u, \cdot]: A \rightarrow A$  is a derivation on  $A$ . Further, for any  $n \geq 2$ , we have:

$$\begin{aligned} & \sum_{l=0}^n d_l(a)d_{n-l}(b) \\ &= (-1)^n u^{n-1}[u, a]b + \sum_{l=1}^{n-1} (-1)^n u^{n-l-1}[u, a]u^{l-1}[u, b] + (-1)^n au^{n-1}[u, b] \\ &= (-1)^n u^{n-1}[u, a]b + (-1)^n u^{n-1}aub - (-1)^n au^n b - (-1)^n u^{n-1}abu \\ & \quad + (-1)^n au^{n-1}bu + (-1)^n au^{n-1}[u, b] \\ &= (-1)^n u^n ab - (-1)^n au^n b - (-1)^n u^{n-1}abu + (-1)^n au^n b = d_n(ab). \quad \square \end{aligned}$$

PROPOSITION 2.7. *Let  $A$  be an algebra over a field  $K$  of characteristic zero and let  $u \in A$ . Let  $D_u = \{d_n\}_{n \geq 0}$  be the higher derivation on  $A$  defined by setting  $d_0 = \text{id}: A \rightarrow A$  and  $d_n = (-1)^n u^{n-1}[u, \cdot]: A \rightarrow A$ . Then for each  $n \geq 1$ , the morphism  $L_{D_u}^n$  is zero on Hochschild homology.*

PROOF. We may verify directly that  $u[u, \cdot] = (1/2) \cdot ([u, [u, \cdot]] + [u^2, \cdot])$ . Let  $k$  be a fixed integer and suppose that for all  $2 \leq l \leq k$ , we have:

(a) The operator  $u[u^{l-1}, \cdot]$  on  $A$  can be expressed as a  $\mathbb{Q}$ -linear combination of the operators of the form

$$\{[u^{s_m}, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]], s_i > 0, m \geq 1, \sum_{i=1}^m s_i = l - 1\} \text{ and } [u^l, \cdot].$$

(b) There exists some  $\xi_l \in \mathbb{Q} \subseteq K$ ,  $\xi_l \geq 0$  such that the operator  $u^{l-1}[u, \cdot] - \xi_l[u^l, \cdot]$  on  $A$  can be expressed as a  $\mathbb{Q}$ -linear combination of the operators of the form

$$(2.9) \quad \{[u^{s_m}, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]], s_i > 0, m \geq 1, \sum_{i=1}^m s_i = l - 1\}.$$

We will now show that this implies that (a) and (b) are also true for  $l = k + 1$ . We start by noting that

$$(2.10) \quad u^k[u, \cdot] + u[u^k, \cdot] = [u^{k+1}, \cdot] + [u^k, [u, \cdot]].$$

Applying (b) with  $l = k$ , it follows that  $u^{k-1}[u, \cdot] - \xi_k[u^k, \cdot]$  can be expressed as a  $\mathbb{Q}$ -linear combination of operators of the form

$$(2.11) \quad \{[u^{s_m}, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]], s_i > 0, m \geq 1, \sum_{i=1}^m s_i = k - 1\}.$$

Since each  $s_i > 0$  in (2.11), we have  $2 \leq s_m + 1 \leq k$ . Applying (a) with  $l = s_m + 1$ , it follows that for each  $[u^{s_m}, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]]$  appearing in (2.11),  $u[u^{s_m}, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]]$  can be expressed as a  $\mathbb{Q}$ -linear combination of operators of the form

$$\{[u^{s'_m}, [\dots u^{s'_1}, [u, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]] \dots]], s'_i > 0, m' \geq 1, \sum_{i=1}^{m'} s'_i = s_m\}$$

and  $[u^{s_m+1}, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]]$ .

From the above, it follows that there exist elements  $a_{t_1 t_2 \dots t_m} \in \mathbb{Q}$  such that  $u^k[u, \cdot] - \xi_k u[u^k, \cdot] = u(u^{k-1}[u, \cdot] - \xi_k[u^k, \cdot])$  may be expressed as:

$$(2.12) \quad u^k[u, \cdot] - \xi_k u[u^k, \cdot] = \sum a_{t_1 t_2 \dots t_m} [u^{t_m}, [u^{t_{m-1}}, [\dots u^{t_1}, [u, \cdot] \dots]]]$$

where the sum in (2.12) varies over all tuples of positive integers  $(t_1, \dots, t_m)$  with  $\sum_{i=1}^m t_i = k$  and  $1 \leq m \leq k$ . Since  $\xi_k \geq 0$  by assumption (b), it follows that  $\xi_k \neq -1$ . Then, solving the equations (2.10) and (2.12) for  $u[u^k, \cdot]$  and  $u^k[u, \cdot]$ , we can prove (a) for  $l = k + 1$  and show that there exists  $\xi_{k+1} \in \mathbb{Q}$  such that  $u^k[u, \cdot] - \xi_{k+1}[u^{k+1}, \cdot]$  can be expressed as a  $\mathbb{Q}$ -linear combination of the operators in (2.9) with  $l = k + 1$ .

It remains to show that  $\xi_{k+1} \geq 0$ . On inspecting these equations further, we note that  $\xi_{k+1} = \xi_k / (\xi_k + 1)$ . Since  $\xi_k \geq 0$ , we have  $\xi_{k+1} \geq 0$ . This proves (b) for  $l = k + 1$ .

From Lemma 2.6, we know that  $D_u = \{d_k\}_{k \geq 0}$  is a higher derivation on  $A$ . Since  $K$  is a field of characteristic zero, it follows that each  $d_k \in \mathcal{H}$ , the Hopf algebra generated by derivations on  $A$ . By induction, we have shown that any  $d_k = (-1)^k u^{k-1}[u, \cdot] \in \mathcal{H}$  may be expressed as a  $\mathbb{Q}$ -linear combination of operators of the form

$$(2.13) \quad \{[u^{s_m}, [u^{s_{m-1}}, [\dots u^{s_1}, [u, \cdot] \dots]]], s_i > 0, m \geq 1, \sum_{i=1}^m s_i = k - 1\} \text{ and } [u^k, \cdot].$$

Further, each of the terms in (2.13) also lies in the Hopf algebra  $\mathcal{H}$ . Then from Proposition 2.5 it follows that each  $L_{D_u}^k = L_{d_k}$  can be written as a  $\mathbb{Q}$ -linear combination of operators:

$$\{L_{u^{s_m}} \circ L_{u^{s_{m-1}}} \circ \dots \circ L_{u^{s_1}} \circ L_u, \sum_{i=1}^m s_i = k - 1, s_i > 0, m \geq 1\} \text{ and } L_{u^k}$$

where every  $L_{u^t}$ ,  $t \in \mathbb{Z}$ ,  $t > 0$ , denotes the morphism on  $HH_*(A)$  induced by the inner derivation  $[u^t, \cdot]$ . Since it is well known (see [6, Section 4.1]) that each  $L_{u^t}$  is zero, it follows that each  $L_{D_u}^k$  is also zero.  $\square$

REMARK 2.8. (1) It is clear that results similar to Proposition 2.5 and Proposition 2.7 hold with the Hochschild homology groups  $HH_*(A)$  replaced by the cyclic homology groups  $HC_*(A)$ .

(2) For given  $u \in A$ , the higher derivation  $D_u = \{d_k\}$  defined by  $d_0 = \text{id}$  and  $d_k = (-1)^k u^{k-1} [u, \cdot]$  is an example of an inner higher derivation in the sense of Roy and Sridharan [9].

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Dept. of Mathematics, Ohio State University, 231 W 18th Ave, Columbus, Ohio 43210, USA  
*e-mail:* abhishekbanerjee1313@gmail.com