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EXPLICIT PROOF OF POINCARÉ INEQUALITY FOR DIFFERENTIAL FORMS ON MANIFOLDS

To the third anniversary of P. Milman's working seminar for his graduate students and postdocs

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Presented by Pierre Milman, FRSC

ABSTRACT. We prove a Poincaré type inequality for differential forms on compact manifolds by means of a constructive 'globalization' of a local Poincaré inequality on convex sets.

RÉSUMÉ. On prouve une inégalité de Poincaré pour les formes différentielles sur les variétés compactes à l'aide d'une 'globalisation' constructive d'une inégalité de Poincaré locale pour les ensembles convexes.

1. Introduction. In a recent paper V. Goldshtein and M. Troyanov proved Sobolev–Poincaré type inequality for differential forms on compact Riemannian manifolds [GoTr]. In this article we present a constructive alternative method of proof. The latter allows, in particular, to estimate the constants for the inequalities in geometric terms. Namely, we construct for any smooth r-form ω on a Riemannian manifold M a smooth r-form ξ on M such that $d\omega = d\xi$ and inequality

$$\|\xi\|_{L^p(M)} \leq C\|d\omega\|_{L^q(M)} \tag{1.1}$$

holds for p and q in a certain (standard) range with a positive constant C depending only on p, q, r and the manifold M (Theorems 2.2 and 3.1). The structure of the proof is first to show inequality (1.1) locally by means of adapting a proof of Lemma 3.11 from [BoMi] to our setting with differential forms, and then, globalizing it by means of a novel method that we present in Section 3. The rough idea of the latter construction was suggested to us by Pierre Milman.

We are mainly interested in Poincaré type inequalities due to the geometric information that they encode. Our primary goal is to study such inequalities on singular sets of algebraic nature, such as semialgebraic sets, in order to better understand the metric behavior of such sets. Constructive proofs of Poincaré type inequalities would, hopefully, allow to extend results of this type to a singular setting.

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Detailed proofs of all the results of Section 3 can be found in [S].

Throughout the paper we will use the following notations.

NOTATION 1.1.

- Suppose that X is a set and $f, g\colon X \to \mathbb{R}$ are two functions. We write $f \lesssim g$ if there exists a positive constant C such that $f \le Cg$.
- If A is a measurable subset of $\mathbb{R}^n$ we write $\mathrm{Vol}(A)$ to denote its n-dimensional volume.
- If $p > 1$ denote by p' its Hölder conjugate, that is, $1/p + 1/p' = 1$.
- If $x, y \in \mathbb{R}^n$, we write $d(x, y) := |x - y|$.

2. Local Poincaré inequality.

In this section we prove a local Poincaré type inequality for differential forms. That is, we prove inequality (1.1) with M being a convex set. This inequality is well known and was studied, *e.g.*, in [IwLu]. Our proof of local inequality (1.1) utilizes a slightly different approach from the one used in [IwLu]. We show that Poincaré inequality for differential forms (Theorem 2.2) is a simple consequence of a "universal" inequality (Proposition 2.1) that extends Lemma 3.11 from [BoMi] to differential forms.

Suppose that M is an orientable Riemannian manifold. We denote by $\Omega^\bullet(M)$ the algebra of smooth differentiable forms on M. Define an L^p norm of a form $\omega \in \Omega^r(M)$ by $\|\omega\|_{L^p} := (\int_M |\omega|^p \, d\,\mathrm{Vol})^{1/p}$, where $|\omega|$ denotes the pointwise norm of ω and $d\,\mathrm{Vol}$ denotes the volume form on M.

2.1. Poincaré inequality on a convex set in $\mathbb{R}^n$. Let $D \subset \mathbb{R}^n$ be a convex set. For each $y \in D$ define a homotopy operator

$$K_y\colon \Omega^r(D) \to \Omega^{r-1}(D)$$

by formula

$$K_y\omega := \int_0^1 \psi_y^*\omega \, dt,$$

where $\psi_y\colon D \times [0,1] \to D$, $\psi_y(x,t) := tx + (1-t)y$. It is easy to verify that $dK_y\omega + K_y d\omega = \omega$.

The next proposition is an extension of Lemma 3.11 from [BoMi] to differential forms with nearly the same proof, *i.e.*, by interchanging the order of integrations on the left hand side of the inequality.

PROPOSITION 2.1. *Let D be a convex compact set in $\mathbb{R}^n$, $r \in \mathbb{N} \cup \{0\}$ and $p, q \ge 1$ such that*
(i) $p \ge q$ *and* $\frac{1}{q} - \frac{1}{p} < \frac{1}{n}$, *or*
(ii) $p < q$.

Then,

$$\left\| \frac{1}{\mathrm{Vol}(D)^{1/p}} \left\| \frac{K_y d\omega(x)}{d(x,y)} \right\|_{L^q(dy)} \right\|_{L^p(dx)} \le C(p,q,r,n) \|d\omega\|_{L^q(dx)}$$

for any r-form ω, where

$$C(p,q,r,n) := \begin{cases} \int_0^1 \min\big(t^{n/p},(1-t)^{n/p}\big)t^{r-n/p}(1-t)^{-n/q}\,dt & \textit{in case } \text{(i)} \\ \int_0^1 \min\big(t^{n/q},(1-t)^{n/q}\big)t^{r-n/q}(1-t)^{-n/q}\,dt & \textit{in case } \text{(ii)}. \end{cases}$$

Next, we prove the local Poincaré inequality.

THEOREM 2.2 (Local Poincaré inequality). *Suppose that $p,q \geq 1$ are as in Proposition* 2.1. *Let ω be a smooth r-form on a convex set $D \subset \mathbb{R}^n$ with diameter R. There exists an r-form ξ on D such that $d\omega = d\xi$ and*

$$\|\xi\|_{L^p(D)} \leq c\|d\omega\|_{L^q(D)},$$

where $c := \mathrm{Vol}(D)^{1/p-1/q}C(p,q,r,n)R$, with $C(p,q,r,n)$ from Proposition 2.1.

PROOF. Define an average homotopy operator A by the formula

$$A\omega := \frac{1}{\mathrm{Vol}(D)}\int_D K_y\omega\,dy.$$

Set $\xi = Ad\omega$. Note that $dA\omega + Ad\omega = \omega$ and therefore $d\xi = d\omega$. Using the Hölder inequality and Proposition 2.1 we obtain the following estimate.

$$\begin{aligned} \|Ad\omega\|_{L^p(D)} &= \left\|\frac{1}{\mathrm{Vol}(D)}\int_D K_y\,d\omega(x)\,dy\right\|_{L^p(D,dx)} \\ &= \frac{1}{\mathrm{Vol}(D)}\left\|\int_D \frac{K_y d\omega(x)}{d(x,y)}\,d(x,y)\,dy\right\|_{L^p(D,dx)} \\ \text{(Hölder inequality)} &\leq \frac{1}{\mathrm{Vol}(D)}\left\|\left\|\int_D \frac{K_y d\omega(x)}{d(x,y)}\right\|_{L^q(dy)}\|d(x,y)\|_{L^{q'}(dy)}\right\|_{L^p(D,dx)} \\ &\leq \frac{1}{\mathrm{Vol}(D)}\sup_{x\in D}\|d(x,y)\|_{L^{q'}(dy)}\left\|\left\|\int_D \frac{K_y d\omega(x)}{d(x,y)}\right\|_{L^q(dy)}\right\|_{L^p(D,dx)} \\ \text{(Proposition 2.1)} &\leq \mathrm{Vol}(D)^{1/p-1/q}c(p,q,r,n)R\|d\omega\|_{L^q(D)}. \end{aligned}$$

□

3. Globalization of Poincaré type inequality.

In this section we describe a constructive method of proof of Poincaré type inequality on a compact manifold. The idea of our construction was inspired from the construction of double Čech–De Rham complex (see [BT]).

The main Theorem of this section is the following.

THEOREM 3.1 (Global Poincaré inequality). *Let M be a compact Riemannian manifold and ω an exact r-form on it. Suppose that p and q are as in Proposition* 2.1. *There exists an $(r-1)$-form ξ on M such that*

$$d\xi = \omega \quad \textit{and} \quad \|\xi\|_{L^p(M)} \lesssim \|\omega\|_{L^q(M)}. \tag{3.1}$$

In what follows we describe the construction of the form ξ from the latter theorem. We begin with some basic definitions.

DEFINITIONS 3.2. Let M be a Riemannian manifold. A subset $D \subset M$ is called *convex* if for every two points $p, q \in D$ there exists a unique geodesic that connects p with q and lies entirely in D. Let $\mathcal{U} = \{U_i\}$ be a cover of M. Denote by U_I the set $U_{i_0} \cap \cdots \cap U_{i_s}$ where $I = (i_0, \ldots, i_s)$. The cover $\mathcal{U}$ is called a *good cover* if every $U \in \mathcal{U}$ is convex. The *nerve complex* of $\mathcal{U}$ is a simplicial complex $\big(C_j(\mathcal{U}), \partial\big)$ generated by $\{[I] : U_I \neq \varnothing, I = (i_0, \ldots, i_j)\}$ where $\partial \colon C_{j+1}(\mathcal{U}) \to C_j(\mathcal{U})$ is defined by

$$\partial[(i_0, \ldots, i_{j+1})] := \sum_k (-1)^k [(i_0, \ldots, \hat{i_k} \ldots, i_{j+1})].$$

The *Čech complex* associated with the cover $\mathcal{U}$ is denoted by $\big(C^j(\mathcal{U}), \delta\big)$ where $C^j(\mathcal{U}) := \operatorname{Hom}\big(C_j(\mathcal{U}), \mathbb{R}\big)$ and $\delta := \partial^* \colon C^j(\mathcal{U}) \to C^{j+1}(\mathcal{U})$ is the dual operator to ∂. The k-th cohomology group of $C^\bullet(\mathcal{U})$ is denoted by $H^k\big(C^\bullet(\mathcal{U})\big)$.

REMARK 3.3. It is well known that sufficiently small balls in a Riemannian manifold M are convex (see [D, Proposition 4.2]). Therefore, there exists a good cover $\mathcal{U}$ for M.

From here on, we will assume that we are in the setting of Theorem 3.1. Let $\mathcal{U} := \{U_i\}$, $i = 1, \ldots, N$ be a good cover of M. In the definition below we define the Čech complex associated with the sheaf of smooth r-forms on M.

DEFINITION 3.4. Set

$$K^{r,0} := \Omega^r(M), \quad K^{r,s} := \bigoplus_{i_0 < \cdots < i_{s-1}} \Omega^r(U_I).$$

Let $\alpha \in K^{r,s}$. Denote by α_I, $I = (i_0, \ldots, i_{s-1})$ the components of α. Define $\delta \colon K^{r,s} \to K^{r,s+1}$, $s \geq 0$,

$$(\delta\alpha)_J := \Big(\sum_{t=0}^{s} (-1)^t \alpha_{j_0 \cdots \hat{j}_t \cdots j_s}\Big)\Big|_{U_J}, \quad J = (j_0, \ldots, j_s).$$

Define an L^p norm on $K^{r,s}$ as follows.

$$\|\alpha\|_{L^p(K^{r,s})} := \sum_{i_0 < \cdots < i_{s-1}} \|\alpha_I\|_{L^p(U_I)}.$$

We will use the following convention.

CONVENTION 3.5. If $\alpha \in K^{r,s}$ with components α_I, $I = (i_0, \ldots, i_{s-1})$, $i_0 < \cdots < i_{s-1}$ and τ is a permutation of $\{0, \ldots, s-1\}$ then $\alpha_I = \alpha_{\tau(I)} \operatorname{sign}(\tau)$.

Table 1: Construction of ξ^k

	2	$\omega \rightarrow$	ω_{i_0}		
	1		$\uparrow$ $\xi^0_{i_0} \rightarrow$	$(\delta\xi^0)_{i_0,i_1}$	
$d \uparrow$	0			$\uparrow$ $\xi^1_{i_0,i_1} \rightarrow$	$(\delta\xi^1)_{i_0,i_1,i_2}$
		$\Omega^\bullet(M)$	$\bigoplus_{i_0} \Omega^\bullet(U_{i_0})$	$\bigoplus_{i_0,i_1} \Omega^\bullet(U_{i_0,i_1})$	$\bigoplus_{i_0,i_1,i_2} \Omega^\bullet(U_{i_0,i_1,i_2})$

$\delta \quad \rightarrow$

In the next proposition we list fundamental properties of the complex $(K^{r,\bullet}, \delta)$.

Proposition 3.6.

(i) $(K^{r,\bullet}, \delta)$ *is a complex,* i.e., $\delta^2 = 0$.

(ii) $(K^{r,\bullet}, \delta)$ *is an exact complex,* i.e., δ *cohomology of* $(K^{r,\bullet}, \delta)$ *are trivial and moreover, if* $\beta \in K^{r,s+1}$ *with* $\delta\beta = 0$ *then there exists* $\alpha \in K^{r,s}$ *such that* $\beta = \delta\alpha$ *and*

- $\|\alpha\|_{L^p(K^{r,s})} \lesssim \|\beta\|_{L^p(K^{r,s+1})}$,
- $\|d\alpha\|_{L^p(K^{r+1,s})} \lesssim \|\beta\|_{L^p(K^{r,s+1})} + \|d\beta\|_{L^p(K^{r+1,s+1})}$.

The proof of this proposition without estimates can be found in [BT, Propositions 8.3 and 8.5]. Part (i) follows from a direct computation of δ^2. For part (ii), suppose that $\beta \in K^{r,s+1}$, $\delta\beta = 0$. Let ρ_j be a partition of unity subordinate to the cover $\{U_i\}$. Set

$$\alpha_{i_0,\ldots,i_{s-1}} := \sum_j \rho_j \beta_{j,i_0,\ldots,i_{s-1}}. \tag{3.2}$$

Direct computation shows that $\delta\alpha = \beta$. The estimates of norms of α and $d\alpha$ follow trivially from (3.2) (by applying Hölder and triangle inequalities) to expression of norms of α and $d\alpha$.

Before we give the general construction of the form ξ that satisfies 3.1 we illustrate the construction on an example.

Example 3.7. Suppose that ω is a closed 2 form on M. Consider Table 1. An entry in the table represents the components of an element in the space indicated in the the same column at bottom row. The vertical arrows represent the exterior derivative d and the horizontal arrows represent the action of differential δ. Start off by placing ω in the first column of the table in the second row (corresponding to the degree of the form). Apply δ to ω to obtain an element $\bigoplus \omega_{i_0}$ in the second column of the table. Since $d\omega_{i_0} = 0$ and $\mathcal{U}$ is a good cover, we can apply the local Poincaré inequality to obtain an element $\xi^0 := \bigoplus \xi^0_{i_0}$ such that

$$d\xi^0_{i_0} = \omega_{i_0}$$

and

$$\|\xi^0_{i_0}\|_{L^p(U_{i_0})} \lesssim \|\omega\|_{L^q(U_{i_0})} \leq \|\omega\|_{L^q(M)} \quad \text{for all } i_0.$$

Next, we apply δ to ξ^0 to get an element $\delta\xi^0 := \bigoplus(\delta\xi^0)_{i_0,i_1}$. Observe that

$$d\delta\xi^0 = \delta d\xi^0 = \delta\delta^0\omega = 0.$$

Therefore, once again, we can apply local Poincaré inequality to $\delta\xi^0$ to obtain an element $\xi^1 := \bigoplus \xi^1_{i_0,i_1}$ such that

$$d\xi^1 = \delta\xi^0$$

and

$$\begin{aligned}\|\xi^1_{i_0,i_1}\|_{L^p(U_{i_0,i_1})} \lesssim \|(\delta\xi^0)_{i_0,i_1}\|_{L^p(U_{i_0,i_1})} &\leq \|\xi^0_{i_0}\|_{L^p(U_{i_0})} + \|\xi^0_{i_1}\|_{L^p(U_{i_1})} \\ &\leq 2\|\omega\|_{L^q(M)},\end{aligned}$$

for all i_0, i_1. Finally, note that

$$d\delta\xi^1 = \delta d\xi^1 = \delta\delta\xi^0 = 0.$$

Since the components $(\delta\xi^1)_{i_0,i_1,i_2}$ of $\delta\xi^1$ are functions with zero exterior derivatives it follows that they are constants. So far we have only used the fact that ω is closed. In order to find a global form ξ that satisfies (3.1) we have to assume that ω is exact. Therefore, in the next step of the construction we assume that ω is exact and find a global $(r-1)$ form ξ that satisfies (3.1). By Theorem 3.10 below, there exists an element $c \in \bigoplus \Omega^0(U_{i_0,i_1})$ with constant components c_{i_0,i_1} for all i_0, i_1 such that

$$\delta\xi^1 - \delta c = \delta(\xi^1 - c) = 0.$$

Moreover, by Corollary 3.11 we have

$$\|c_{i_0,i_1}\|_{L^p(U_{i_0,i_1})} \lesssim \|\omega\|_{L^q(M)}.$$

We will construct (inductively) elements $x^1 \in \bigoplus \Omega^0(U_{i_0})$ and $x^0 \in \Omega^1(M)$ such that $\xi := x^0$ satisfies (3.1), see Table 2.

Note that each row r of the latter table is the complex $(K^{r,\bullet}, \delta)$. By Proposition 3.6 each such row is exact. Therefore, by the same proposition, there exists an element x^1 such that $\delta x^1 = \xi^1 - c$ and the following estimates hold

$$\|x^1\|_{L^p(K^{0,1})} \lesssim \|\xi^1 - c\|_{L^p(K^{0,2})}$$

and

$$\|dx^1\|_{L^p(K^{1,1})} \lesssim \|\xi^1 - c\|_{L^p(K^{0,2})} + \|d(\xi^1 - c)\|_{L^p(K^{1,2})}.$$

Table 2: Construction of x^k

	2	$\omega \to$	ω_{i_0}		
	1	$\uparrow$ $x^0 \to$	$\uparrow$ $\xi^0_{i_0} - dx^1_{i_0} \to$	$0, \quad (\delta\xi^0)_{i_0,i_1}$	
$d \uparrow$	0		$x^1_{i_0} \to$	$\uparrow$ $\xi^1_{i_0,i_1} - c_{i_0,i_1} \to$	0
		$\Omega^\bullet(M)$	$\bigoplus_{i_0} \Omega^\bullet(U_{i_0})$	$\bigoplus_{i_0,i_1} \Omega^\bullet(U_{i_0,i_1})$	$\bigoplus_{i_0,i_1,i_2} \Omega^\bullet(U_{i_0,i_1,i_2})$

$\delta \quad \to$

Note that

$$\delta(\xi^0 - dx^1) = d\xi^1 - d\delta x^1 = d(\xi^1 - \xi^1 + c) = 0.$$

Hence, by exactness of the second row there exists an element x^0 such that $\delta x^0 = \xi^0 - dx^1$ and we have

$$\|x^0\|_{L^p(M)} \lesssim \|\xi^0 - dx^1\|_{L^p(K^{1,1})}$$

and

$$\|dx^0\|_{L^p(M)} \lesssim \|\xi^0 - dx^1\|_{L^p(K^{1,1})} + \|d(\xi^0 - dx^1)\|_{L^p(K^{2,1})}.$$

Set $\xi := x^0$. We claim that $d\xi = \omega$. Indeed,

$$\delta(\omega - dx^0) = \delta\omega - d\delta x^0 = \delta\omega - d(\xi^0 - dx^1) = \delta\omega - d\xi^0 = 0.$$

It follows that $(\omega - dx^0)|_{U_{i_0}} = 0$ for all i_0 and therefore $\omega = dx^0$ on M. Moreover, combining all the estimates from above we obtain

$$\begin{aligned}
\|\xi\|_{L^p(M)} &\lesssim \|\xi^0 - dx^1\|_{L^p(K^{1,1})} \\
&\lesssim \|\xi^0\|_{L^p(K^{1,1})} + \|dx^1\|_{L^p(K^{1,1})} \\
&\lesssim \|\omega\|_{L^q(M)} + \|\xi^1 - c\|_{L^p(K^{0,2})} + \|d\xi^1\|_{L^p(K^{1,2})} \\
&\lesssim \|\omega\|_{L^q(M)} + \|\xi^1\|_{L^p(K^{0,2})} + \|c\|_{L^p(K^{0,2})} + \|\delta\xi^0\|_{L^p(K^{1,2})} \\
&\lesssim \|\omega\|_{L^q(M)}.
\end{aligned}$$

This concludes the example.

In what follows we give the general construction of the forms ξ^s and x^s as in the example above.

3.1. Construction of elements $\xi^s \in K^{r-s-1,s+1}$.

DEFINITION 3.8. Set $\xi^{-1} := \omega$ and define ξ^s by setting the I-th component, ξ^s_I, to be a solution of the equation

$$d\xi^s_I = (\delta\xi^{s-1})_I \tag{3.3}$$

in U_I, $I = (i_0, \ldots, i_s)$ such that

$$\|\xi^s_I\|_{L^p(U_I)} \lesssim \|(\delta\xi^{s-1})_I\|_{L^p(U_I)}, \tag{3.4}$$

for $0 \le s \le r-1$.

We remark that equation (3.3) can be solved with an estimate (3.4) by means of local Poincaré inequality since U_I is convex and $d\delta\xi^{s-1} = 0$ (*cf.* Example 3.7). We have the following estimate of ξ^s in terms of the norm of ω:

PROPOSITION 3.9. *Let $I = (i_0, \ldots, i_s)$. Then,*

$$\|\xi^s_I\|_{L^p(U_I)} \lesssim \|\omega\|_{L^q(M)}. \tag{3.5}$$

This proposition can be easily proven by induction on s.

Note that ξ^{r-1} is a collection of 0-forms that satisfy $d\delta\xi^{r-1} = 0$. It means that $(\delta\xi^{r-1})_I$ are constants on each U_I, $I = (i_0, \ldots, i_r)$. (We use the same notation to denote the extension of $(\delta\xi^{r-1})_I$ to a globally defined constant function on M.)

THEOREM 3.10. *There exists an element $c \in K^{0,r}$ with constant components c_I, $I = (i_0, \ldots, i_{r-1})$ such that*

$$(\delta c)_I = \sum_{t=0}^{r} (-1)^t c_{i_0,\ldots,\hat{i_t},\ldots,i_r} = (\delta\xi^{r-1})_I, \quad \textit{for all } I = (i_0, \ldots, i_r). \tag{3.6}$$

Moreover, there exist $b_{I,L} \in \mathbb{R}$, $I = (i_0, \ldots, i_{r-1})$, $L = (l_1, \ldots, l_r)$ such that

$$c_I = \sum_L b_{I,L}(\delta\xi^{r-1})_L$$

where $b_{I,L}$ depend only on the cover $\mathcal{U}$.

We prove this theorem in the last subsection of this article. As a consequence of Theorem 3.10, we obtain the following corollary.

COROLLARY 3.11. *The constants c_I from Theorem 3.10 admit the following estimate:*

$$\|c_I\|_{L^p(U_I)} \le \|\omega\|_{L^q(M)}.$$

PROOF. By Theorem 3.10 we may represent each c_I as

$$c_I = \sum_L b_{I,L}(\delta\xi^{r-1})_L.$$

By the triangle inequality we have

$$\|c_I\|_{L^p(U_I)} \le \sum_L |b_{I,L}|\, \|(\delta\xi^{r-1})_L\|_{L^p(U_I)}. \tag{3.7}$$

Observe that $(\delta\xi^{r-1})_L$ is a globally defined constant function and therefore, similarly to the proof of Proposition 3.9 we have

$$\|(\delta\xi^{r-1})_L\|_{L^p(U_I)} = \|(\delta\xi^{r-1})_L\|_{L^p(U_L)}\Big(\frac{\mathrm{Vol}(U_I)}{\mathrm{Vol}(U_L)}\Big)^{1/p} \lesssim \|\omega\|_{L^q(M)}. \tag{3.8}$$

Now, from (3.7) and (3.8) we obtain the desired estimate. □

3.2. Construction of elements $x^s \in K^{r-s-1,s}$. The final step of the construction is to glue all the forms ξ^s, $s = 0, \dots, r-1$ to a global solution ξ that satisfies (3.1). We construct inductively forms $x^s \in K^{r-s-1,s}$ such that $\xi := x^0$ is the desired global form (*cf.* Example 3.7). Set $\tilde{\xi}_I^{r-1} = \xi_I^{r-1} - c_I$ where c_I is given by Theorem 3.10 and $I = (i_0, \dots, i_{r-1})$. Note that $d\tilde{\xi}_I^{r-1} = d\xi_I^{r-1}$ and $\delta\tilde{\xi}^{r-1} = 0$. It follows from Proposition 3.6 (ii) that there exists a form $x^{r-1} \in K^{0,r-1}$ such that $\delta x^{r-1} = \tilde{\xi}^{r-1}$ and

$$\|x^{r-1}\|_{L^p(K^{0,r-1})} \lesssim \|\tilde{\xi}^{r-1}\|_{L^p(K^{0,r})},$$
$$\|dx^{r-1}\|_{L^p(K^{1,r-1})} \lesssim \|\tilde{\xi}^{r-1}\|_{L^p(K^{0,r})} + \|d\tilde{\xi}^{r-1}\|_{L^p(K^{1,r})}.$$

It follows from Corollary 3.11 and Proposition 3.9 that

$$\|x^{r-1}\|_{L^p(K^{0,r-1})} \lesssim \|\omega\|_{L^q(M)} \tag{3.9}$$

and

$$\begin{aligned} \|dx^{r-1}\|_{L^p(K^{1,r-1})} &\lesssim \|\omega\|_{L^q(M)} + \|\delta\xi^{r-2}\|_{L^p(K^{1,r})} \\ &\le \|\omega\|_{L^q(M)} + \|\xi^{r-2}\|_{L^p(K^{1,r-1})} \\ &\lesssim \|\omega\|_{L^q(M)}. \end{aligned} \tag{3.10}$$

Suppose that $x^{r-(t-1)}$ was constructed. By Proposition 3.6 (ii) there exists x^{r-t} such that

$$\delta x^{r-t} = \xi^{r-t} - dx^{r-t+1},$$

where

$$\|x^{r-t}\|_{L^P(K^{t-1,r-t})} \lesssim \|\xi^{r-t} - dx^{r-t+1}\|_{L^P(K^{t-1,r-t+1})}, \tag{3.11}$$

and

$$
\begin{aligned}
&\|dx^{r-t}\|_{L^p(K^{t,r-t})} \lesssim \|\xi^{r-t} - dx^{r-t+1}\|_{L^p(K^{t-1,r-t+1})} + \|d\xi^{r-t}\|_{L^p(K^{t,r-t+1})} \\
&\qquad \le \|\xi^{r-t}\|_{L^p(K^{t-1,r-t+1})} + \|dx^{r-t+1}\|_{L^p(K^{t-1,r-t+1})} \\
&\qquad\quad + \|\delta\xi^{r-t-1}\|_{L^p(K^{t,r-t+1})} \\
&\qquad \lesssim \|\omega\|_{L^q(M)} + \|dx^{r-t+1}\|_{L^p(K^{t-1,r-t+1})},
\end{aligned}
\tag{3.12}
$$

provided that $\delta(\xi^{r-t} - dx^{r-t+1}) = 0$. Let us verify this condition:

$$
\begin{aligned}
\delta(\xi^{r-t} - dx^{r-t+1}) &= \delta\xi^{r-t} - d\delta x^{r-t+1} \\
&= \delta\xi^{r-t} - d(\xi^{r-t+1} - dx^{r-t+2}) \\
&= \delta\xi^{r-t} - d\xi^{r-t+1} \\
&= 0.
\end{aligned}
$$

Using estimates (3.9), (3.10), (3.11) and (3.12), one can prove by induction the following proposition.

Proposition 3.12. *The forms x^s admit the following estimates:*
(1) $\|x^{r-t}\|_{L^p(K^{t-1,r-t})} \lesssim \|\omega\|_{L^q(M)}$.
(2) $\|dx^{r-t}\|_{L^p(K^{t,r-t})} \lesssim \|\omega\|_{L^q(M)}$.

Finally, set $\xi := x^0$. To see that $dx^0 = \omega$ observe that

$$\delta(\omega - dx^0) = \delta\omega - d\delta x^0 = \delta\omega - d(\xi^0 - dx^1) = \delta\omega - d\xi^0 = 0.$$

But $\delta(\omega - dx^0)_i = (\omega - dx^0)|_{U_i}$ and therefore $\omega = dx^0$ on M. The estimate of ξ follows from Proposition 3.12 for $t = r$.

3.3. Proof of Theorem 3.10. Note that the linear system of equations (3.6) has a solution if and only if $\sum_I (\delta\xi^{r-1})_I a_I = 0$ for every $a = \sum a_I[I] \in \ker\partial = \ker\delta^*$. Therefore, Theorem 3.10 is equivalent to the following proposition.

Proposition 3.13. *Let $a = \sum_I a_I[I]$ be an r-cycle then*

$$\sum_I a_I(\delta\xi^{r-1})_I = 0.$$

In the proof of this proposition we construct an explicit isomorphism from De Rham cohomology to Čech cohomology of a good cover from which Proposition 3.13 follows immediately. Another proof of the latter proposition is given in [S]. It is based on a direct computation of an integral of a closed form over a chain in M. It is shown there that if T is a triangulation of M and $\mathcal{U}$ is a cover consisting of open stars of vertices of T then the integral of the closed r-form ω over a cycle $a = \sum a_I[I]$ equals to $(-1)^{\lfloor \frac{r}{2} \rfloor} \sum a_I(\delta\xi^{r-1})_I$, where $[I]$ is identified here with the simplex of T with vertices $(i_0, \ldots, i_r)$.

In this note we prove:

PROPOSITION 3.14. *The map*

$$\mathrm{Int}\colon H^r\big(\Omega^\bullet(M)\big)\to H^r\big(C^\bullet(\mathcal{U})\big)$$

defined by

$$(\mathrm{Int}\,\omega)[I] := (-1)^{\lfloor \frac{r}{2}\rfloor}\delta\xi_I^{r-1}, \tag{3.13}$$

where ξ_I^{r-1} *is defined for the form* ω *as in Definition* 3.8 *is a well defined isomorphism.*

Before we prove this proposition, we show how Proposition 3.13 follows from it.

PROOF OF PROPOSITION 3.13. If ω is an exact r-form then $\mathrm{Int}\,\omega = \delta W$. If $a=\sum a_I[I]$ is a cycle, then

$$(\mathrm{Int}\,\omega)a = (-1)^{\lfloor \frac{r}{2}\rfloor}\sum a_I\delta\xi_I^{r-1} = (\delta W)a = W\partial a = 0. \qquad \square$$

Next, we prove Proposition 3.14.

PROOF. Consider an auxiliary complex $(K^\bullet, D)$ defined by

$$K^m := \bigoplus_{r+s=m} K^{r,s+1}$$

and

$$D := d + (-1)^s\delta \quad \text{on } K^{r,s+1}.$$

This complex is called Čech–De Rham complex, see [BT] for details. Denote by $H^r(K^\bullet)$ the r-th cohomology group of $K^\bullet$. The map $\delta\colon \Omega^r(M)\to K^r$ induces an isomorphism

$$h^0\colon H^r\big(\Omega^\bullet(M)\big)\to H^r(K^\bullet)$$

[BT, Proposition 8.8]. Similarly, the map $g\colon C^r(\mathcal{U})\to K^r$, defined by sending an element in $C^r(\mathcal{U})$ to the corresponding element in $K^{0,r+1}$ induces an isomorphism

$$h^1\colon H^r\big(C^\bullet(\mathcal{U})\big)\to H^r(K^\bullet).$$

We claim that $\mathrm{Int} = (h^1)^{-1}\circ h^0$. Indeed, let us compute the action of $(h^1)^{-1}\circ h^0$ on closed form ω. First applying h^0 to ω we get an element defined by the D cohomology class of $\delta\omega$. Note that

$$D\xi^{s+1} = d\xi^{s+1} + (-1)^s\delta\xi^{s+1},$$

A direct computation, using the latter formula and fact that $\delta\xi^s = d\xi^{s+1}$, shows that

$$\delta\omega - \sum_{j=0}^{k}(-1)^{\lfloor \frac{j}{2}\rfloor}D\xi^j = (-1)^{\lfloor \frac{k+1}{2}\rfloor}\delta\xi^k, \quad k\ge 0.$$

It follows from this that $(h^1)^{-1}\delta^0\omega$ is defined by the element that sends $[I]$ to $(-1)^{\lfloor \frac{r}{2}\rfloor}\delta\xi_I^{r-1}$ which is what was required to prove. $\square$

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