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Exact sequences for equivariantly formal spaces

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Abstract. Let T be a torus. We present an exact sequence relating the relative equivariant cohomologies of the skeletons of an equivariantly formal T-space. This sequence, which goes back to Atiyah and Bredon, generalizes the so-called Chang–Skjelbred lemma. As coefficients, we allow prime fields and subrings of the rationals, including the integers. We extend to the same coefficients a generalization of this "Atiyah–Bredon sequence" for actions without fixed points which has recently been obtained by Goertsches and Töben.

Résumé. Soit T un tore et soit X un T-espace dont la cohomologie équivariante est libre sur $H^*(BT)$. Nous construisons une suite exacte liant les cohomologies relatives equivariantes des squelettes de X, et dont les coefficients sont à valeurs dans un corps premier ou dans un sous-anneau des nombres rationnels, y compris l'anneau des entiers. Cette suite, qui remonte à Atiyah et Bredon, généralise le lemme de Chang–Skjelbred. Goertsches et Töben ont récemment démontré qu'une modification de cette suite "d'Atiyah–Bredon" à coefficients réels est exacte dans le cas plus général d'une action sans points fixes. Nous montrons que ceci reste vrai pour les coefficients mentionnés ci-dessus.

1. Introduction.

Let $T = (S^1)^n$ be a torus of dimension n and X a "nice" T-space. Suppose that X is *cohomologically equivariantly formal*[1] (or *CEF*), which means that its equivariant cohomology $H_T^*(X)$ (with rational coefficients) is free over the polynomial ring $H^*(BT)$. Then the so-called Chang–Skjelbred lemma [CS, (2.3)] asserts that the sequence

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X^T) \longrightarrow H_T^{*+1}(X_1, X^T) \tag{1.1}$$

is exact, where $X^T \subset X$ denotes the fixed point set and X_1 the union of all orbits of dimension at most 1. In other words, $H_T^*(X)$ coincides, as subalgebra of $H_T^*(X^T)$, with the image of $H_T^*(X_1) \to H_T^*(X^T)$. Usually X^T and X_1 are much simpler than X, so that (1.1) gives an easy method to compute $H_T^*(X)$. This can be used for instance for a short proof of Jurkiewicz's theorem about

[1] It seems that the terminology 'equivariantly formal' was coined by [GKM]. We add 'cohomologically' to avoid confusion with notions in rational homotopy theory.

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the cohomology of smooth projective toric varieties, because they are known to be equivariantly formal, see Remark 2.3. There are many other applications of the Chang–Skjelbred lemma in algebraic and symplectic geometry, often with implications for combinatorics.

Roughly at the same time as Chang and Skjelbred, Atiyah proved a much more general theorem in the context of equivariant K-theory [A, Chapter 7]. Bredon [Br] then observed that it applies equally to cohomology. In the context of toric varieties an Atiyah-like theorem was proven by Barthel–Brasselet–Fieseler–Kaup for cohomology and intersection homology [BBFK, Theorem 4.3]. Recently Goertsches and Töben [GT] made the very interesting observation that the Cohen–Macaulay property, which Atiyah used in an essential way, gives generalizations also for fixed-point-free actions.

The principal aim of this note is to present a generalized cohomological version of Atiyah's theorem. Our proof in Section 2 will be a modification of his, adapted to cohomology and with a special emphasis on coefficient rings R other than the rationals, namely on subrings of $\mathbb{Q}$ and on prime fields. As a corollary, we get an integral version of the Chang–Skjelbred lemma with certain restrictions on the isotropy groups. We also extend the Goertsches–Töben result to the same coefficient rings and to more general spaces than compact T-manifolds.

The present note is a revised version of our preprint [FP′]. It extends our results from finite T-CW complexes to a much larger class of T-spaces and takes into account the Goertsches–Töben result.

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2. The Atiyah–Bredon sequence. We use Alexander–Spanier cohomology, *cf.* [M, Chapter 8]. Coefficients are taken in a subring R of $\mathbb{Q}$ or in the finite field $R = \mathbb{Z}_p$. Here the letter p denotes a prime, as it does in the rest of the paper. All topological spaces will be assumed to be paracompact and Hausdorff. Recall that on locally contractible spaces Alexander–Spanier cohomology and singular cohomology coincide.

Let $T = (S^1)^n$ be a torus and X a T-space. (Note that its Borel construction X_T is again paracompact Hausdorff.) We assume X to be finite-dimensional, with only finitely many orbit types and such that $H^*(X^K)$ is finitely generated over R for all closed subgroups K of T.[2] For example, X may be a finite T-CW complex, a compact topological T-manifold or a complex algebraic variety with an algebraic action of $(\mathbb{C}^\times)^n \supset T$.

Denote by X_i, $-1 \le i \le n$, the equivariant i-skeleton of X, *i.e.*, the union of

[2]These assumptions can be weakened, *cf.* [AP, Section 3.2]. For example, X can be finitistic instead of finite-dimensional, and, for $R = \mathbb{Q}$, must only have finitely many *connected* orbit types.

all orbits of dimension $\le i$. In particular, $X_{-1} = \varnothing$, $X_0 = X^T$ and $X_n = X$. Since X is Hausdorff with only finitely many orbit types, each X_i is closed in X. The maximal p-torus $T_p \subset T$ is the subgroup of all elements of order p (together with $1 \in T$). It is isomorphic to $\mathbb{Z}_p^n$. The T_p-equivariant i-skeleton $X_{p,i}$ of X is the set of all T_p-orbits consisting of at most p^i points. One clearly has $X_i \subset X_{p,i}$.

The inclusion of pairs $(X_i, X_{i-1}) \hookrightarrow (X, X_{i-1})$ gives rise to a long exact sequence

(2.1)
$$\cdots \to H_T^*(X, X_i) \to H_T^*(X, X_{i-1}) \to H_T^*(X_i, X_{i-1}) \xrightarrow{\delta} H_T^{*+1}(X, X_i) \to \cdots,$$

and likewise $(X_{i+1}, X_{i-1}) \hookrightarrow (X_{i+1}, X_i)$ gives maps

$$H_T^*(X_i, X_{i-1}) \to H_T^{*+1}(X_{i+1}, X_i). \tag{2.2}$$

Theorem 2.1. *Let X be a T-space such that $H_T^*(X)$ is a free $H^*(BT)$-module. Suppose in addition that for all i*

$$X_{p,i} = X_i \tag{2.3a}$$

if $R = \mathbb{Z}_p$, or, in case $R \subset \mathbb{Q}$,

$$X_{p,i-1} \subset X_i \tag{2.3b}$$

for all p not invertible in R. Then the following “Atiyah–Bredon sequence” is exact:

$$\begin{aligned} 0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X_0) \longrightarrow H_T^{*+1}(X_1, X_0) \longrightarrow \cdots \\ \cdots \longrightarrow H_T^{*+n-1}(X_{n-1}, X_{n-2}) \longrightarrow H_T^{*+n}(X_n, X_{n-1}) \longrightarrow 0. \end{aligned} \tag{2.4}$$

More generally, if condition (2.3a) *or* (2.3b) *holds for all $i \le k$, then one still has exactness for*

$$\begin{aligned} 0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X_0) \longrightarrow H_T^{*+1}(X_1, X_0) \longrightarrow \cdots \\ \cdots \longrightarrow H_T^{*+k-1}(X_{k-1}, X_{k-2}) \longrightarrow H_T^{*+k}(X_k, X_{k-1}). \end{aligned} \tag{2.5}$$

The content of the Chang–Skjelbred lemma is just exactness of the Atiyah–Bredon sequence at the first terms. Rephrasing Theorem 2.1 for $k \in \{0, 1\}$ and integer coefficients, we get:

Corollary 2.2. *Assume $R = \mathbb{Z}$ and let X be a T-space such that $H_T^*(X)$ is free over $H^*(BT)$. Then*

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X_0)$$

*is exact. If in addition the isotropy group of each $x \notin X_1$ is contained in a proper subtorus of T (*i.e.*, if condition* (2.3b) *holds for $i \le 1$), then the following sequence is exact:*

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X_0) \longrightarrow H_T^{*+1}(X_1, X_0).$$

The injectivity of the map $H_T^*(X) \to H_T^*(X_0)$ is an immediate consequence of the localization theorem in equivariant cohomology, see Lemma 4.4 below.

The above "proper subtorus condition" is violated in the counterexample given by Tolman–Weitsman [TW, Section 4] because there all orbits of dimension 2 have isotropy group $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset S^1 \times S^1$. The full condition (2.3b) can be stated analogously: For $R = \mathbb{Z}$ the isotropy group T_x of any $x \in X$ must be contained in a subtorus of dimension $\le \dim T_x + 1$.

REMARK 2.3. In some cases general criteria guarantee the freeness of $H_T^*(X)$ over $H^*(BT)$, for instance if $H^*(X)$ is free over R and concentrated in even degrees (Leray–Hirsch). Assume that $H^*(X)$ is free over $R \subset \mathbb{Q}$. Then $H_T^*(X)$ is free over $H^*(BT)$ if X is a compact Kähler manifold with $X^T \neq \varnothing$ (for example, a smooth projective $(\mathbb{C}^\times)^n$-variety) [Bl, Theorem II.1.2] or if X is a compact Hamiltonian T-manifold [F]. If all isotropy groups of X are connected, then the exactness of the Atiyah–Bredon sequence with integer coefficients can be characterized by a homological condition on $H_T^*(X)$, see [FP].

3. Isotropy groups.

Before proving Theorem 2.1, we want to reformulate the conditions (2.3a) and (2.3b) on the equivariant skeletons in algebraic terms. In the sequel, we will refer to (2.3a) and (2.3b) together as condition (2.3). Notice that this condition is always satisfied for $R = \mathbb{Q}$.

For any closed subgroup $T' \subset T$ there is an isomorphism $T \cong (S^1)^n$ and a divisor chain $m_q \mid m_{q-1} \mid \cdots \mid m_1$ such that T' corresponds to the subgroup

$$\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_q} \times (S^1)^r \subset (S^1)^n. \tag{3.1}$$

(To see this, consider the inverse image of T' under the exponential map $\exp\colon \mathfrak{t} \to T$ and compare it with the lattice $\exp^{-1}(1)$.)

Recall that the dimension of a module M over a ring S is the Krull dimension of $S/\operatorname{Ann} M$, that is, the length of a maximal chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots$ of prime ideals in S, all containing the annihilator $\operatorname{Ann} M$ of M. We denote the dimension of R over itself by d. Hence, $d = 0$ if R is a field, and $d = 1$ if $R \subsetneq \mathbb{Q}$. Since R is a PID, the dimension of a polynomial ring $R[t_1, \ldots, t_s]$ is $d + s$ [S, Proposition III.13].

LEMMA 3.1. *Let $T' \subset T$ be a closed subgroup and let s be the number of m_j's in the decomposition* (3.1) *which are not invertible in R. Then*

$$\dim_{H^*(BT)} H^*(BT') = \begin{cases} r + s & \text{if } R \text{ is a field or } s > 0, \\ r + 1 & \text{if } R \subsetneq \mathbb{Q} \text{ and } s = 0. \end{cases}$$

Proof. The $H^*(BT)$-action on $H^*(BT')$ comes from the algebra map induced by the inclusion $T' \hookrightarrow T$. Hence, the dimension of the $H^*(BT)$-module $H^*(BT')$ is equal to the Krull dimension of the (evenly graded) image A of $H^*(BT)$ in $H^*(BT')$.

We choose isomorphisms $H^{2*}(BS^1) \cong R[t]$ and $H^{2*}(B\mathbb{Z}_m) \cong R[t]/mt$ such that the map $H^{2*}(BS^1) \to H^{2*}(B\mathbb{Z}_m)$ corresponds to the canonical quotient map $R[t] \to R[t]/mt$. By the Künneth theorem for cohomology, we find

$$\begin{aligned} A &\cong H^{2*}(B\mathbb{Z}_{m_1}) \otimes \cdots \otimes H^{2*}(B\mathbb{Z}_q) \otimes H^{2*}(BS^1)^{\otimes r} \\ &\cong R[t_1, \ldots, t_q, t'_1, \ldots, t'_r]/(m_1 t_1, \ldots, m_q t_q) \\ &= R[t_1, \ldots, t_s, t'_1, \ldots, t'_r]/(m_1 t_1, \ldots, m_s t_s) \end{aligned}$$

since $m_{s+1}, \ldots, m_q$ are invertible in R.

R, a field or $s = 0$: then $A \cong R[t_1, \ldots, t_s, t'_1, \ldots, t'_r]$, so the assertion is clear.

$R \subsetneq \mathbb{Q}$ and $s > 0$: An ascending chain of prime ideals in A of length k is the same as an ascending chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_k$ of prime ideals in $B = R[t_1, \ldots, t_s, t'_1, \ldots, t'_r]$ with $(m_1 t_1, \ldots, m_s t_s) \subset \mathfrak{p}_0$. Since one can clearly find such a chain of length $r + s$, we have to show that there are no longer chains. If $\mathfrak{p}_0$ contains some prime factor p of m_s, then $B/\mathfrak{p}_0$ is a quotient of

$$\mathbb{Z}_p[t_1, \ldots, t_s, t'_1, \ldots, t'_r],$$

which is of dimension $r + s$. Otherwise, $t_s \in \mathfrak{p}_0$ and $B/\mathfrak{p}_0$ is a quotient of $R[t_1, \ldots, t_{s-1}, t'_1, \ldots, t'_r]$, which by induction has again dimension $r + s$. In any case, we find that $k \leq r + s$. □

Proposition 3.2. *The conditions* (2.3) *for all $i \leq k$ are equivalent to the conditions*

$$\forall x \in X \quad \dim_{H^*(BT)} H^*(BT_x) \geq d + n - i \quad \Longrightarrow \quad x \in X_i \tag{3.2}$$

for all $i \leq k$.

Proof. $R = \mathbb{Q}$: Since the dimension of $H^*(BT_x)$ is just the rank of T_x, condition (3.2) holds for all spaces, as does (2.3).

$R = \mathbb{Z}_p$: By Lemma 3.1, condition (3.2) means that the rank of the maximal p-torus contained in T_x equals the dimension of T_x for $x \in X_k$, and differs by at most $i - k - 1$ for $x \in X_i \setminus X_{i-1}$, $i > k$. This is equivalent to $X_{p,i} = X_i$ for $i \leq k$.

$R \subsetneq \mathbb{Q}$: Here condition (3.2) translates into the following: for each non-invertible prime p the rank of the maximal p-torus in T_x differs by at most 1 from the dimension of T_x for $x \in X_k$, and by at most $i - k$ for $x \in X_i \setminus X_{i-1}$, $i > k$. This is the same as saying $X_{p,i-1} \subset X_i$ for all $i \leq k$. □

Note that conditions (3.2) are always true in K-theory (see [A, (7.1)]), essentially because the representation ring of a finite group is a finitely generated $\mathbb{Z}$-module.

4. **Proof of Theorem 2.1.** Before starting in earnest, we remark that our assumptions on X imply that all R-modules $H^*(X_j, X_i)$, $j \geq i$, are finitely generated, hence so are the $H^*(BT)$-modules $H_T^*(X_j, X_i)$. (Use the usual long exact sequences for the first claim. For equivariant cohomology, the proof given in [AP, (3.10.1)] for $R = \mathbb{Q}$ still applies.)

The sequences (2.4) and (2.5) are exact if and only if for all $i \geq 0$ (respectively $0 \leq i \leq k$) the sequence (2.1) splits into short exact sequences

$$0 \longrightarrow H_T^*(X, X_{i-1}) \longrightarrow H_T^*(X_i, X_{i-1}) \longrightarrow H_T^{*+1}(X, X_i) \longrightarrow 0, \tag{4.1}$$

i.e., if and only if the inclusion of pairs $(X, X_{i-1}) \hookrightarrow (X, X_i)$ induces the zero map in cohomology. (See [FP, Lemma 4.1].)

Let M be a graded $H^*(BT)$-module. Then any prime ideal associated with M is homogeneous, *i.e.*, generated by its homogeneous elements [BH, Lemma 1.5.6], hence contained in some maximal homogeneous ideal $\mathfrak{m}$. This implies $M = 0$ if and only if $M_\mathfrak{m} = 0$ for all maximal homogeneous ideals $\mathfrak{m} \subset H^*(BT)$. It suffices therefore to prove exactness after localizing at such an ideal $\mathfrak{m}$. Note that it always contains $H^{>0}(BT)$. From now on all localized modules will be over the regular local ring $A = H^*(BT)_\mathfrak{m}$, whose maximal ideal we denote by $\mathfrak{m}_A$.

By induction on $i \geq 0$ (resp. $0 \leq i \leq k$), we show:

CLAIM 4.1. *The sequence* (4.1) *is exact, and* $H_T^*(X, X_i)_\mathfrak{m}$ *is zero or a Cohen–Macaulay module of dimension* $d + n - i - 1$.

Recall that the *depth* of a finitely generated A-module M is the maximal length of an M-regular sequence in the maximal ideal $\mathfrak{m}_A$. An M-regular sequence is a sequence $a_1, \ldots, a_l \in \mathfrak{m}_A$ such that $(a_1, \ldots, a_l)M \neq M$ and such that a_j is not a zero divisor on $M/(a_1, \ldots, a_{j-1})M$ for all $j \leq l$. One always has

$$\operatorname{depth} M \leq \dim M; \tag{4.2}$$

if $M \neq 0$ and

$$\operatorname{depth} M = \dim M, \tag{4.3}$$

then M is called *Cohen–Macaulay*. (We use [S] and [BH] as references for commutative algebra. The reader might also find the summary of results in [AP, Appendix A] helpful. They were compiled with applications in equivariant cohomology in mind.)

We start the proof with several lemmas. The first two are standard.

LEMMA 4.2. *Let* $0 \to U \to M \to N \to 0$ *be an exact sequence of* A*-modules. Then*

$$\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}.$$

PROOF. Since
$$\operatorname{depth} M = \inf\{i : \operatorname{Ext}^i(K, M) \neq 0\},$$
where K denotes the residue field of A, the assertion follows from the long exact sequence
$$\cdots \longrightarrow \operatorname{Ext}^i(K, U) \longrightarrow \operatorname{Ext}^i(K, M) \longrightarrow \operatorname{Ext}^i(K, N) \longrightarrow \operatorname{Ext}^{i+1}(K, U) \longrightarrow \cdots .$$
□

LEMMA 4.3. *Let M be a Cohen–Macaulay A-module. If $U \subset M$ is a non-zero submodule, then* $\dim U = \dim M$.

PROOF. Choose a prime ideal $\mathfrak{p}$ associated to U, hence also to M. Since M is Cohen–Macaulay, we have $\dim A/\mathfrak{p} = \dim M$ by [S, Proposition IV.13]. On the other hand, $\dim A/\mathfrak{p} \leq \dim U \leq \dim M$. □

LEMMA 4.4. *For all $i \leq k$ one has* $\dim H_T^*(X, X_i) \leq d + n - i - 1$.

PROOF. This follows from the localization theorem in equivariant cohomology: For a prime ideal $\mathfrak{p} \subset H^*(BT)$, define $X^{\mathfrak{p}} = \{x \in X : \mathfrak{p} \supset \operatorname{Ann} H^*(BT_x)\}$. The localization theorem asserts that the inclusion $X^{\mathfrak{p}} \hookrightarrow X$ induces an isomorphism
$$H_T^*(X)_{\mathfrak{p}} = H_T^*(X^{\mathfrak{p}})_{\mathfrak{p}},$$
see for example [AP, Section 3.2]. An analogous formula holds for relative cohomology.

Now assume that $\dim H_T^*(X, X_i) \geq d + n - i$. This means that there is a prime ideal $\mathfrak{p} \supset \operatorname{Ann} H_T^*(X, X_i)$ such that $\dim H^*(BT)/\mathfrak{p} \geq d + n - i$. Hence $\dim H^*(BT_x) \geq d + n - i$ for any $x \in X^{\mathfrak{p}}$. By Proposition 3.2, this implies $x \in X_i$, *i.e.*, $X^{\mathfrak{p}} \subset X_i$. The localization theorem now gives
$$H_T^*(X, X_i)_{\mathfrak{p}} = H_T^*(X^{\mathfrak{p}}, X_i^{\mathfrak{p}})_{\mathfrak{p}} = H_T^*(X^{\mathfrak{p}}, X^{\mathfrak{p}})_{\mathfrak{p}} = 0.$$
But this is impossible, because for any prime ideal $\mathfrak{p}$ one has $\mathfrak{p} \supset \operatorname{Ann} H_T^*(X, X_i)$ if and only if $H_T^*(X, X_i)_{\mathfrak{p}} \neq 0$ [S, Proposition I.3]. (Here we are using that $H_T^*(X, X_i)$ is finitely generated over $H^*(BT)$.) Hence $\dim H_T^*(X, X_i) \leq d + n - i - 1$. □

LEMMA 4.5. *For all $i \geq 0$ one has* $\operatorname{depth} H_T^*(X_i, X_{i-1})_{\mathfrak{m}} \geq n - i$.

PROOF. Each $x \in X_i \setminus X_{i-1}$ is fixed by exactly one $(n - i)$-dimensional torus $T_\alpha \subset T$. Because X has only finitely many orbit types, we can write $X_i \setminus X_{i-1}$ as the disjoint union of finitely many subsets $Y_\alpha = (X_i \setminus X_{i-1})^{T_\alpha}$. Since $\bar{Y}_\alpha \subset Y_\alpha \cup X_{i-1}$, a Mayer–Vietoris argument gives
$$H_T^*(X_i, X_{i-1}) = \bigoplus_\alpha H_T^*(\bar{Y}_\alpha, \bar{Y}_\alpha \cap X_{i-1}).$$

(Recall that with respect to Alexander–Spanier cohomology, the Mayer–Vietoris sequence is exact for all pairs of closed subspaces, *cf.* [M, Section 8.6].)

We have

$$M := H_T^*(\bar{Y}_\alpha, \bar{Y}_\alpha \cap X_{i-1}) = H_{T/T_\alpha}^*(\bar{Y}_\alpha, \bar{Y}_\alpha \cap X_{i-1}) \otimes_R H^*(BT_\alpha)$$

because $\bar{Y}_\alpha$ is fixed by T_α. A sequence of generators $x_1, \dots, x_{n-i} \in H^2(BT_\alpha) \subset \mathfrak{m}$ clearly is $M_\mathfrak{m}$-regular. Hence

$$\operatorname{depth} M_\mathfrak{m} \geq n - i.$$

The description of depth via Ext (*cf.* the proof of Lemma 4.2) shows that the depth of a direct sum is the minimum of the depths of the summands. This gives the result. □

We now prove Claim 4.1 by induction on i, assuming that $H_T^*(X, X_{i-1})_\mathfrak{m}$ is zero or a Cohen–Macaulay module of dimension $d + n - i$. For $i = 0$ this is true because we assume $H_T^*(X, X_{-1}) = H_T^*(X)$ to be free over the regular ring $H^*(BT)$.

In the case $H_T^*(X, X_{i-1})_\mathfrak{m} = 0$ the sequence (2.1) splits into short exact sequences for trivial reasons. Suppose therefore that $H_T^*(X, X_{i-1})_\mathfrak{m}$ is non-zero and Cohen–Macaulay of dimension $d + n - i$. Since

$$\dim H_T^*(X, X_i)_\mathfrak{m} \leq \dim H_T^*(X, X_i) \leq d + n - i - 1 \tag{4.4}$$

by Lemma 4.4 and $\dim H_T^*(X, X_{i-1})_\mathfrak{m} = d + n - i$, the image of $H_T^*(X, X_i)_\mathfrak{m}$ in $H_T^*(X, X_{i-1})_\mathfrak{m}$ has lower dimension than the target module, hence is zero by Lemma 4.3. In other words, (4.1) is exact in this case as well.

Lemma 4.2 now gives

$$\operatorname{depth} H_T^*(X, X_i)_\mathfrak{m} \geq d + n - i - 1 \tag{4.5}$$

since $\operatorname{depth} H_T^*(X_i, X_{i-1})_\mathfrak{m} \geq n - i$ by Lemma 4.5 and $\operatorname{depth} H_T^*(X, X_{i-1})_\mathfrak{m} = d + n - i$ (or infinity if $H_T^*(X, X_{i-1})_\mathfrak{m} = 0$).

Comparing (4.4) and (4.5) with the inequality (4.2) finishes the proof.

5. **Actions without fixed points.** The Atiyah–Bredon sequence (2.4) can never be exact if T acts without fixed points (unless $X = \varnothing$). Yet there are interesting fixed-point-free actions. In the context of compact differentiable T-manifolds and real coefficients, Goertsches and Töben [GT] observed that one can modify (2.4) to accommodate for this case. We briefly comment on such a generalization in our setting.

PROPOSITION 5.1. *Let k be the minimal dimension of the T-orbits in X. If X satisfies condition* (2.3)*, then* $\dim H_T^*(X) = d + n - k$.

Proof. Since $X_{k-1} = \varnothing$, we have $\dim H_T^*(X) \le d + n - k$ by Lemma 4.4.

For the reverse inequality, pick an $x \in X_k$, say with isotropy group T'. The map $H^*(BT) \to H^*(BT')$ factors through the map $H_T^*(X) \to H_T^*(Tx) = H^*(BT')$ induced by the inclusion $Tx \hookrightarrow X$. This implies $\operatorname{Ann} H_T^*(X) \subset \operatorname{Ann} H^*(BT')$, hence $\dim H_T^*(X) \ge \dim H^*(BT')$. But, under condition (2.3), Lemma 3.1 gives $\dim H^*(BT') = d + n - k$. □

A finitely generated graded module M over $H^*(BT)$ is called *Cohen–Macaulay* if $M_{\mathfrak{m}}$ is Cohen–Macaulay over $H^*(BT)_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \in \operatorname{Supp} M$. If M is free over $H^*(BT)$, then it is Cohen–Macaulay of maximal dimension $d+n$ (*cf.* the proof of Claim 4.1). We remark in passing that the converse holds as well, even if R is not a field: The Auslander–Buchsbaum formula [BH, Theorem 1.3.3] together with [S, Corollary IV.C.2] implies that M is projective; freeness then follows from the (comparatively easy) analogue of the Quillen–Suslin theorem for finitely generated *graded* modules over polynomial rings with coefficients in a PID [L, Corollary 4.7].

Theorem 5.2. *Let k be the minimal dimension of the T-orbits in X. If $H_T^*(X)$ is Cohen–Macaulay over $H^*(BT)$ and X satisfies condition* (2.3)*, then the following sequence is exact:*

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(X_k) \longrightarrow H_T^{*+1}(X_{k+1}, X_k) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_T^{*+n-1}(X_{n-1}, X_{n-2}) \longrightarrow H_T^{*+n}(X_n, X_{n-1}) \longrightarrow 0.$$

The only change to the proof of Theorem 2.1 is that the induction now starts at $i = k$, which is the dimension of $H_T^*(X)$ by Proposition 5.1.

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