

CONVEXITY OF THE KREIN SPACE
TRACIAL NUMERICAL RANGE AND MORSE THEORY

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ABSTRACT. In this paper we present a Krein space convexity theorem on the tracial-numerical range of a matrix. This theorem is the analogue of Westwick's theorem. The proof is an application of Morse theory.

RÉSUMÉ. Nous présentons un théorème de convexité pour une gamme numérique généralisée des opérateurs sur un espace de Krein. Ce théorème est un analogue du théorème de Westwick. La preuve est une application de la théorie de Morse.

1. Introduction. Let J denote a Hermitian involutive matrix, that is, $J^* = J$, $J^2 = I_n$, with *signature* $(r, n - r)$, $0 < r \leq n$ (i.e., with r positive and $n - r$ negative eigenvalues). We consider \mathbf{C}^n endowed with the indefinite inner product induced by J :

$$[x, y] = y^* J x, \quad x, y \in \mathbf{C}^n.$$

Let M_n be the algebra of $n \times n$ complex matrices. A matrix $H \in M_n$ is J -Hermitian if $H^\# = H$, where $H^\# = JH^*J$ is the J -adjoint of H . A matrix $U \in M_n$ is J -unitary if $U^\#U = UU^\# = I_n$, the identity matrix of size n . The J -unitary matrices form a locally compact connected group, denoted by $G = \mathcal{U}(r, n - r)$ and called the J -unitary group. For $C, T \in M_n$, the J -tracial numerical range of T is denoted and defined as

$$W_C^J(T) = \{\text{Tr}(CUTU^{-1}) : U \in \mathcal{U}(r, n - r)\}.$$

This set is a connected set in the Gaussian plane \mathbf{C} and satisfies the symmetry property $W_C^J(T) = W_T^J(C)$ (see [1], [2], [4] for other properties). In the case $r = n$, the matrix J reduces to the identity matrix and the group $\mathcal{U}(n, 0) = \mathcal{U}(n)$ is the *unitary group*, which is a compact group. In this case $W_C^J(T)$ is simply denoted by $W_C(T)$ and called the C -numerical range of T . Hausdorff proved the convexity of $W_C(T)$ in the case C is a rank one orthogonal projection. Using

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Morse theory, Westwick [8] proved the convexity of $W_C(T)$ for C Hermitian and in [5] Poon obtained an alternative proof by majorization techniques. In [7], Tam generalized Westwick's result for certain compact Lie groups. In this paper we obtain a non-compact group analogue of Westwick's convexity theorem applying Morse theory (*cf.* Lemma 4.1). The eigenvalues of a J -unitarily diagonalizable J -Hermitian matrix T are real and will be denoted as

$$\begin{aligned}\sigma_+(T) &= \{\lambda \in \mathbf{C} : T\xi = \lambda\xi \text{ for some } \xi \in \mathbf{C}^n, [\xi, \xi] > 0\}, \\ \sigma_-(T) &= \{\lambda \in \mathbf{C} : T\xi = \lambda\xi \text{ for some } \xi \in \mathbf{C}^n, [\xi, \xi] < 0\}.\end{aligned}$$

The eigenvalues are said to be *noninterlacing* if

$$(1.1) \quad \max \sigma_-(T) < \min \sigma_+(T) \quad \text{or} \quad \max \sigma_+(T) < \min \sigma_-(T).$$

Replacing $<$ in (1.1) by \leq , the eigenvalues are said to be *weakly noninterlacing*. The noninterlacing property plays a key role in the investigation of numerical ranges associated with non-compact groups.

In the sequel we consider $J = I_r \oplus (-I_{n-r})$. Our main result is the following.

THEOREM 1.1. *Let C be a J -unitarily diagonalizable J -Hermitian matrix with noninterlacing eigenvalues and let $T \in M_n$ be a matrix such that for some $\theta \in \mathbf{R}$*

$$T_\theta = \cos \theta (T + T^\#) / 2 + \sin \theta (T - T^\#) / (2i)$$

is a J -unitarily diagonalizable J -Hermitian matrix with noninterlacing eigenvalues:

$$\begin{aligned}\sigma_+(C) &= \{c_1 \geq \dots \geq c_r\}, & \sigma_-(C) &= \{c_{r+1} \geq \dots \geq c_n\}, \\ \sigma_+(T_\theta) &= \{\lambda_1 \geq \dots \geq \lambda_r\}, & \sigma_-(T_\theta) &= \{\lambda_{r+1} \geq \dots \geq \lambda_n\}, \\ c_n &> c_1 \text{ or } c_r > c_{r+1}, & \lambda_n &> \lambda_1 \text{ or } \lambda_r > \lambda_{r+1}.\end{aligned}$$

Then $W_C^J(T)$ is a closed convex set in \mathbf{C} contained in a closed cone $\{z_0 + r e^{i\eta} : r \geq 0, \theta_1 \leq \eta \leq \theta_2\}$ with $0 \leq \theta_2 - \theta_1 < \pi$.

Denote by $C^\infty(G)$ the commutative algebra of all C^∞ -differentiable complex-valued functions on $G = \mathcal{U}(r, n-r)$. We consider the even-dimensional coset space $M = G/D$, where

$$D = \{\text{diag}(d_1, \dots, d_n) : d_1, \dots, d_n \in \mathbf{C}, |d_1| = \dots = |d_n| = 1\}.$$

Let the real valued function f_0 on M be defined by

$$(1.2) \quad f_0(g) = \sum_{h,j=1}^n c_h \lambda_j |u_{hj}|^2 \epsilon_h \epsilon_j = \text{Tr}(CgT_\theta g^{-1}),$$

where $\epsilon_h = 1$ for $1 \leq h \leq r$ and $\epsilon_h = -1$ for $r+1 \leq h \leq n$.

This paper is organized as follows. The critical points and the Morse indices of the function f_0 are investigated in Section 2. Global properties of f_0 and of $W_C^J(T)$ are studied in Section 3. The proof of Theorem 1.1 is presented in Section 4 using the results in Sections 2 and 3.

2. Local properties of the function f_0 . To treat local properties of the function f_0 , we introduce some notation and prerequisites. The Lie algebra of $G = \mathcal{U}(r, n-r)$ is given as

$$\mathfrak{G} = \{X \in M_n : X^\# = -X\}.$$

For $1 \leq k_0 < l_0 \leq r$ or $r+1 \leq k_0 < l_0 \leq n$, let

$$X_{k_0, l_0} = (\delta_{kk_0} \delta_{ll_0} - \delta_{kl_0} \delta_{lk_0}), \quad Y_{k_0, l_0} = i(\delta_{kk_0} \delta_{ll_0} + \delta_{kl_0} \delta_{lk_0}),$$

where δ_{kl} is the Kronecker symbol. For $1 \leq k_0 \leq r$, $r+1 \leq l_0 \leq n$, let

$$V_{k_0, l_0} = (\delta_{kk_0} \delta_{ll_0} + \delta_{kl_0} \delta_{lk_0}), \quad W_{k_0, l_0} = i(\delta_{kk_0} \delta_{ll_0} - \delta_{kl_0} \delta_{lk_0}).$$

Throughout, we consider the basis of the Lie algebra \mathfrak{G} constituted by the matrices X_{k_0, l_0} , Y_{k_0, l_0} , V_{k_0, l_0} , W_{k_0, l_0} and the diagonal matrices

$$D_1 = \text{diag}(i, 0, \dots, 0), \dots, D_n = \text{diag}(0, \dots, 0, i),$$

which generate the Lie algebra \mathfrak{D} . We shall use a canonical coordinate system of the second kind,

$$\exp(t_1 X_1) \exp(t_2 X_2) \cdots \exp(t_n X_n),$$

in a neighborhood of the identity I_n , where $\{X_{n^2-n+1}, \dots, X_{n^2}\}$ is the above basis of \mathfrak{D} , and $\{X_1, X_2, \dots, X_{n^2-n}\}$ is the remaining system of vectors of the basis of \mathfrak{G} .

For $f \in C^\infty(G)$, consider a representation α of \mathfrak{G} on $C^\infty(G)$ given by

$$\alpha(X)(f)(g) = -\lim_{t \rightarrow 0} \frac{f(e^{tX} g) - f(g)}{t} = \lim_{t \rightarrow 0} \frac{f(e^{-tX} g) - f(g)}{t}.$$

The function space $C^\infty(G/D)$ is identified with the space

$$\{f \in C^\infty(G) : f(gd) = f(g) \text{ for } d \in D\}.$$

For S_n the symmetric group of degree n , let

$$S_{r, n-r} = \{\sigma_1 \sigma_2 : \sigma_1(j) = j \ (r+1 \leq j \leq n), \sigma_2(j) = j \ (1 \leq j \leq r)\}.$$

PROPOSITION 2.1. *Let $H = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $C = \text{diag}(c_1, \dots, c_n)$ be such that $c_h \neq c_j$, $\lambda_h \neq \lambda_j$ for $1 \leq h \neq j \leq n$. A point $g = (u_{hj})$ of G is a critical point of f_0 if and only if (u_{hj}) is a permutation matrix associated with a permutation in $S_{r, n-r}$.*

PROOF. We take an arbitrary critical point g . A straightforward computation shows that

$$-\alpha(X)(f_0)(g) = \text{Tr}(X[gHg^{-1}, C]) = 0,$$

where $[X, Y] = XY - YX$ stands for the commutator. Thus, gHg^{-1} commutes with the diagonal matrix C , whose diagonal entries are pairwise distinct, and so

$$gHg^{-1} = \text{diag}(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}),$$

for some $\sigma \in S_n$. For $g = (u_{hj})$, we have $u_{hj}\lambda_j = \lambda_{\sigma(h)}u_{hj}$ and so $u_{hj} = 0$ unless $\lambda_j = \lambda_{\sigma(h)}$. Therefore, $u_{hj} = \eta_j \delta_{j\sigma(h)}$, where $|\eta_j| = 1$. We notice that $gHg^{-1} = dgHg^{-1}d^{-1}$ for any $d \in D$. Moreover, σ belongs to $S_{r, n-r}$ because $g^{-1} = Jg^*J$. The converse is clear. \square

PROPOSITION 2.2. *Let $H = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $C = \text{diag}(c_1, \dots, c_n)$ be such that $c_h \neq c_j$, $\lambda_h \neq \lambda_j$ for $1 \leq h \neq j \leq n$. The numbers of positive and negative eigenvalues of the Hessian matrix of f_0 at each critical point are even.*

PROOF. We analyze the Hessian matrix of the function f_0 at a critical point. For $f \in C^\infty(G)$, the representation α of \mathfrak{G} satisfies

$$\alpha(X)\alpha(Y)(f)(g) = \left. \frac{d}{dt} \frac{d}{ds} f(e^{-sY} e^{-tX} g) \right|_{s,t=0}.$$

For $X = -X^\#$, consider the expansion of $f(e^{-tX} g)$ in powers of t

$$\begin{aligned} f_0(e^{-tX} g) &= \text{Tr}(CgHg^{-1}) - t \text{Tr}(C[X, gHg^{-1}]) \\ &\quad + \frac{t^2}{2} \text{Tr}(C[X, [X, gHg^{-1}]]) + \mathcal{O}(t^3). \end{aligned}$$

Assuming g is critical, we get

$$f_0(e^{-tX} g) = \text{Tr}(CgHg^{-1}) + \frac{t^2}{2} \text{Tr}(C[X, [X, gHg^{-1}]]) + \mathcal{O}(t^3).$$

Let $\mathcal{M}(X, Y) = \alpha(X)\alpha(Y)(f_0)(g) = \text{Tr}(C[Y, [X, gHg^{-1}]])$. Having in mind the Jacobi identity and performing some computations, we find

$$\alpha(X)\alpha(Y)(f_0)(g) = \text{Tr}(C[Y, [X, gHg^{-1}]]) = \text{Tr}([C, Y][X, gHg^{-1}]),$$

and so $\mathcal{M}(X, Y) = \mathcal{M}(Y, X)$. For $X = (x_{kl})$, $Y = (y_{kl})$, we have $[C, Y] = (y_{hj}(c_h - c_j))$ and $[X, gHg^{-1}] = (x_{hj}(\lambda_{\sigma(j)} - \lambda_{\sigma(h)}))$. By straightforward computations we obtain

$$\begin{aligned} \mathcal{M}(X, Y) &= \text{Tr}(C[Y, [X, gHg^{-1}]]) = \text{Tr}\left(\left(y_{hj}(c_h - c_j)\right)\left(x_{hj}(\lambda_{\sigma(j)} - \lambda_{\sigma(h)})\right)\right) \\ &= \sum_{k,l=1}^n y_{kl}x_{lk}(c_k - c_l)(\lambda_{\sigma(k)} - \lambda_{\sigma(l)}). \end{aligned}$$

The matrix representation of $\mathcal{M}(X, Y)$, relative to the fixed basis in $\mathfrak{G}/\mathfrak{D}$, is diagonal. In fact, for $1 \leq k_0 < l_0 \leq r$ or $r+1 \leq k_0 < l_0 \leq n$, an easy computation leads to

$$\mathcal{M}(X_{k_0 l_0}, X_{k'_0 l'_0}) = \mathcal{M}(Y_{k_0 l_0}, Y_{k'_0 l'_0}) = -2\delta_{k_0 k'_0} \delta_{l_0 l'_0} (c_{k_0} - c_{l_0})(\lambda_{\sigma(k_0)} - \lambda_{\sigma(l_0)}).$$

For $1 \leq k_0, k'_0 \leq r, r+1 \leq l_0, l'_0 \leq n$, we analogously have

$$\mathcal{M}(V_{k_0 l_0}, V_{k'_0 l'_0}) = \mathcal{M}(W_{k_0 l_0}, W_{k'_0 l'_0}) = 2\delta_{k_0 k'_0} \delta_{l_0 l'_0} (c_{k_0} - c_{l_0})(\lambda_{\sigma(k_0)} - \lambda_{\sigma(l_0)}).$$

The off-diagonal entries of $\mathcal{M}(X, Y)$ vanish. Thus, the $(n^2 - n) \times (n^2 - n)$ -Hessian matrix of the function f_0 at every critical point is nonsingular. Moreover, the numbers of positive and negative eigenvalues of the Hessian matrix are even. \square

We recall, in passing, that the number of negative eigenvalues of the Hessian is called the *Morse index*.

3. Global properties of the function f_0 and $W_C^J(H)$.

PROPOSITION 3.1. *Let C and H be J -unitarily diagonalizable J -Hermitian matrices with respective eigenvalues*

$$\begin{aligned} \sigma_+(C) &= \{c_1, \dots, c_r\}, & \sigma_-(C) &= \{c_{r+1}, \dots, c_n\}, \\ \sigma_+(H) &= \{\lambda_1, \dots, \lambda_r\}, & \sigma_-(H) &= \{\lambda_{r+1}, \dots, \lambda_n\} \end{aligned}$$

satisfying the conditions

$$\begin{aligned} c_{r+1} &\geq \dots \geq c_n > c_1 \geq \dots \geq c_r, \\ \lambda_1 &\geq \dots \geq \lambda_r > \lambda_{r+1} \geq \dots \geq \lambda_n. \end{aligned}$$

Then the superlevel set $\{U \in \mathcal{U}(r, n-r) : a \leq f_0(U)\}$ is compact for an arbitrary real number a .

PROOF. Analogously to (1.2) we define a function g_0 for the diagonal J -Hermitian matrices $C' = \text{diag}(c'_1, \dots, c'_n)$, $H' = \text{diag}(\lambda'_1, \dots, \lambda'_n)$ with eigenvalues $c'_1 = \dots = c'_r = c_1$, $c'_{r+1} = \dots = c'_n = c_n$ and $\lambda'_1 = \dots = \lambda'_r = \lambda_r$, $\lambda'_{r+1} = \dots = \lambda'_n = \lambda_{r+1}$. We show that the functions f_0, g_0 satisfy the inequality $f_0(g) \leq g_0(g)$ for $g \in G$. To prove this inequality, we may assume that $c_n > 0 > c_1$, $\lambda_r > 0 > \lambda_{r+1}$ by adding suitable scalar operators to C and H . Observing that $c_1 \lambda_r - c_h \lambda_j = (c_1 - c_h) \lambda_r - c_h (\lambda_j - \lambda_r)$ ($1 \leq h, j \leq r$), the function $g_0(U) - f_0(U)$ can be expressed as a linear combination of $|u_{h,j}|^2$ ($1 \leq h, j \leq n$) with nonnegative coefficients. Hence, the inclusion

$$(3.1) \quad \{[U] \in M : f_0([U]) \geq a\} \subset \{[U] \in M : g_0(U) \geq a\}$$

holds and the left-hand side of (3.1) is a closed subset of the set in the right-hand side. It can be easily checked that

$$g_0(U) = rc_1\lambda_r + (n-r)c_n\lambda_{r+1} - \frac{c_n - c_1}{4}(\lambda_r - \lambda_{r+1})\left(\sum_{h,j=1}^n |u_{h,j}|^2 - n\right),$$

so the right-hand side of (3.1) is compact for an arbitrary a and so is the left-hand side. \square

PROPOSITION 3.2. *Let C be a J -unitarily diagonalizable J -Hermitian matrix with noninterlacing eigenvalues $\sigma_+(C) = \{c_1 \geq \dots \geq c_r\}$, $\sigma_-(C) = \{c_{r+1} \geq \dots \geq c_n\}$, $c_n > c_1$. Let H be a J -Hermitian matrix such that $W_C^J(H) = (-\infty, a_0]$. Then the matrix H is J -unitarily diagonalizable and its eigenvalues are weakly noninterlacing.*

PROOF. We assume that the maximum value a_0 is attained as $\text{Tr}(CgHg^{-1})$. This assumption implies that $\text{Tr}(X[gHg^{-1}, C]) = 0$ for an arbitrary $X \in \mathfrak{G}$, so that $[gHg^{-1}, C] = 0$. The condition $c_n > c_1$ implies that $gHg^{-1} = H_1 \oplus H_2$, where H_1, H_2 are Hermitian block matrices of sizes r and $n-r$, respectively. Thus, gHg^{-1} is J -unitarily diagonalizable as $VgHg^{-1}V^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n)$, where V is a J -unitary matrix of the form $V = V_1 \oplus V_2$, the blocks V_1, V_2 being unitary of sizes r and $n-r$, respectively. The weak noninterlacing property of the eigenvalues follows from Theorem 1.1(iii) in [1]. \square

PROPOSITION 3.3. *Let $C = \text{diag}(c_1, \dots, c_n)$ with $\sigma_+(C) = \{c_1 \geq \dots \geq c_r\}$, $\sigma_-(C) = \{c_{r+1} \geq \dots \geq c_n\}$, $c_n > c_1$, and let $H = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $\sigma_+(H) = \{\lambda_1 \geq \dots \geq \lambda_r\}$, $\sigma_-(H) = \{\lambda_{r+1} \geq \dots \geq \lambda_n\}$, $\lambda_r \geq \lambda_{r+1}$, so that*

$$\{\text{Tr}(CgHg^{-1}) : g \in \mathcal{U}(r, n-r)\} = \left(-\infty, \sum_{j=1}^n c_j \lambda_j\right].$$

Then, for K J -Hermitian, the set

$$\mathcal{W} = \left\{ \text{Tr}(CgKg^{-1}) : g \in \mathcal{U}(r, n-r), \text{Tr}(CgHg^{-1}) = \sum_{j=1}^n c_j \lambda_j \right\}$$

is connected.

PROOF. Let $H = (\lambda'_1 I_{n_1}) \oplus \dots \oplus (\lambda'_s I_{n_s})$, where the λ'_j are all distinct. Consider $C = C_1 \oplus \dots \oplus C_s$, $J = J_1 \oplus \dots \oplus J_s$, and let K be the J -Hermitian block matrix

$$K = \begin{pmatrix} K_{11} & \dots & K_{1s} \\ \vdots & \ddots & \vdots \\ K_{s1} & \dots & K_{ss} \end{pmatrix},$$

where $C_j, J_j, K_{jj} \in M_{n_j}$, $j = 1, \dots, s$. If $\text{Tr}(CgHg^{-1}) = \sum_{j=1}^n c_j \lambda_j$, then $g = WV$, where $W, V \in \mathcal{U}(r, n-r)$, $WCW^{-1} = C$ and $VHV^{-1} = H$. Obviously, $V = V_1 \oplus \dots \oplus V_s$, $V_j \in M_{n_j}$, and $V_j J_j V_j^* = J_j$. Moreover,

$$VKV^{-1} = \begin{pmatrix} V_1 K_{11} V_1^{-1} & \cdots & V_1 K_{1s} V_s^{-1} \\ \vdots & \ddots & \vdots \\ V_s K_{s1} V_1^{-1} & \cdots & V_s K_{ss} V_s^{-1} \end{pmatrix}$$

and

$$\text{Tr}(CgKg^{-1}) = \sum_{j=1}^n \text{Tr}(C_j V_j K_{jj} V_j^{-1}).$$

Thus

$$(3.2) \quad \mathcal{W} = W_C^{J_1}(K_{11}) + W_{C_2}^{J_2}(K_{22}) + \cdots + W_{C_s}^{J_s}(K_{ss}).$$

Since each summand in the Minkowski sum in the right hand side of (3.2) is connected, the result follows. \square

4. Proof of Theorem 1.1.

LEMMA 4.1 (cf. [3], [6]). *Let M be an m -dimensional connected C^∞ -manifold ($m \geq 2$) and let f be a real-valued C^∞ -function on M . Assume that f satisfies the following conditions:*

- (i) *the function f attains a minimum value;*
- (ii) *the set $\{x \in M : a \leq f(x) \leq b\}$ is compact for every $-\infty < a < b < +\infty$;*
- (iii) *the number of critical points x_1, x_2, \dots, x_n of f is finite, and the critical values $f(x_1), f(x_2), \dots, f(x_n)$ are all distinct;*
- (iv) *the Hessian matrix of f at each critical point x_j is non-singular;*
- (v) *the number of the negative eigenvalues of the Hessian matrix of f at each critical point x_j is neither 1 nor $m - 1$.*

Then the level set $\{x \in M : f(x) = a\}$ is connected for every $a \in \mathbf{R}$.

Some critical values of the function f_0 given by (1.2) at distinct critical points may coincide. Then we apply a perturbative method. We choose a vector (y_1, \dots, y_n) satisfying $\sum_{k=1}^n y_k (c_{\sigma(k)} - c_{\tau(k)}) \neq 0$ for any distinct pair $\sigma, \tau \in \mathcal{S}_{r, n-r}$. We can find $t_0 \in \mathbf{R}$ with sufficiently small modulus such that $\lambda_k + y_k t_0$ satisfy the condition (iii) of Lemma 4.1.

Now, we prove Theorem 1.1. First we show the simple connectedness of $W_C^J(T)$ under the assumptions of the theorem. Consider the case $c_n > c_1$, $\lambda_r > \lambda_{r+1}$. The remaining cases reduce to this one by replacing C by $-C$ and/or T_θ by $T_{\theta+\pi}$. By a rotation in the Gaussian plane \mathbf{C} , we assume that $\theta = 0$ and so $H = T_0 = (T + T^\#)/2$ is a J -unitarily diagonalizable J -Hermitian matrix

with noninterlacing eigenvalues $\lambda_r > \lambda_{r+1}$. Define the following sequences of J -Hermitian matrices:

$$C_m = \text{diag}(c_1^{(m)}, \dots, c_n^{(m)}), \quad H_m = \text{diag}(\lambda_1^{(m)}, \dots, \lambda_n^{(m)}),$$

where $c_n^{(m)} = c_n > c_1 = c_1^{(m)}$, $\lambda_r^{(m)} = \lambda_r > \lambda_{r+1} = \lambda_{r+1}^{(m)}$ for $m = 1, 2, 3, \dots$. Suppose that $|c_h^{(m)} - c_h| \rightarrow 0$, $|\lambda_h^{(m)} - \lambda_h| \rightarrow 0$ as $m \rightarrow \infty$; $c_h^{(m)} \neq c_j^{(m)}$, $\lambda_h^{(m)} \neq \lambda_j^{(m)}$ for each m , $1 \leq h < j \leq n$ and assume that the critical values $\sum_{h=1}^n c_h^{(m)} \lambda_{\sigma(h)}^{(m)}$ for any distinct permutations $\sigma \in \mathcal{S}_{r, n-r}$ are distinct. By Proposition 3.1 the set

$$\{g \in U(r, n-r) : \text{Tr}(C_m g H_m g^{-1}) \geq b\} \cup \{g \in U(r, n-r) : \text{Tr}(C g H g^{-1}) \geq b\}$$

is contained in a compact set B for any $b \in \mathbf{R}$. The pairs $\{C_m, H_m\}$ satisfy the conditions of Propositions 2.1 and 2.2. It follows that these pairs satisfy the conditions of Lemma 4.1. By Lemma 4.1, each set $\{g \in U(r, n-r) : \text{Tr}(C_m g H_m g^{-1}) = a - 1/2^m\}$ is a compact connected set. Hence for $T_m = H_m + i(T - T^\#)/(2i) \in M_n$ and $a \in \mathbf{R}$, $\mathcal{I}_m = \{z \in W_{C_m}^J(T_m) : \Re(z) = a - 1/2^m\}$ is a closed interval. For every $a \leq \sum_{h=1}^n c_h \lambda_h$, the set $\{z \in W_C^J(T) : \Re(z) = a\}$ is compact. Let

$$b_0 = \min\{y \in \mathbf{R} : a + iy \in W_C^J(T)\}, \quad c_0 = \max\{y \in \mathbf{R} : a + iy \in W_C^J(T)\},$$

$$b_m = \min\left\{y \in \mathbf{R} : a - \frac{1}{2^m} + iy \in W_{C_m}^J(T_m)\right\},$$

$$c_m = \max\left\{y \in \mathbf{R} : a - \frac{1}{2^m} + iy \in W_{C_m}^J(T_m)\right\}.$$

Let us take an arbitrary $0 < t_0 < 1$ and choose a point U_m in B satisfying

$$\text{Tr}(C_m U_m T_m U_m^{-1}) = a - \frac{1}{2^m} + i(t_0 b_m + (1 - t_0) c_m)$$

for each m . Since the set B is compact, we consider a subsequence U_{m_k} of U_m converging to a point U_0 of B . Then as $k \rightarrow \infty$

$$\text{Tr}(C_{m_k} U_{m_k} T_{m_k} U_{m_k}^{-1}) \rightarrow \text{Tr}(C U_0 T U_0^{-1}) = a + i(t_0 b_0 + (1 - t_0) c_0).$$

Hence the set

$$(4.1) \quad \{z \in W_C^J(T) : \Re(z) = a\},$$

is also a closed interval. We set $a_0 = \sum_{h=1}^n c_h \lambda_h$. Since the set (4.1) depends continuously on a , then

$$W_C^J(T) = \{x + iy : -\infty < x \leq a_0, \phi_1(x) \leq y \leq \phi_2(x)\}$$

for some continuous real-valued functions $\phi_1(x) \leq \phi_2(x)$ defined on the half-line $(-\infty, a_0]$, and so it is simply connected.

Now we prove that the boundary of $W_C^J(T)$, $\partial W_C^J(T)$, is convex. The set $W_C^J(T)$ has a point z_0 with $\Re(z_0) = a_0$ and the line $\Re(z) = a_0$ is a support line of $W_C^J(T)$. If $\partial W_C^J(T)$ is not convex, there exists a tangent line at some point $z_1 \in \partial W_C^J(T)$ expressed as $\Re(z_1 e^{-i\theta_1}) = b_0$ for some $\theta_1, b_0 \in \mathbf{R}$ satisfying $\Re(z_2 e^{-i\theta_1}) < b_0$, $\Re(z_3 e^{-i\theta_1}) > b_0$, where $z_2, z_3 \in W_C^J(T)$. Then we find a point z_4 on the arc of $\partial W_C^J(T)$ joining z_0, z_1 and a support line ℓ_θ passing through z_4 , $\Re(z e^{-i\theta_0}) = a_\theta$, such that $W_C^J(T) \cap \ell_\theta$ is not connected. However the last condition of ℓ_θ contradicts Proposition 3.3. So $\partial W_C^J(T)$ is convex. We take a point $x_0 < a_0$ at which ϕ_1, ϕ_2 are differentiable. Then we have $\phi_2'(x_0) \leq \phi_1'(x_0)$ and the cone $\{x+iy \in \mathbf{C} : \phi_1'(x_0)(x-x_0) + \phi_1(x_0) \leq y \leq \phi_2'(x_0)(x-x_0) + \phi_2(x_0)\}$ satisfies the last assertion of Theorem 1.1. This completes the proof of the theorem. \square

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