C. R. Math. Rep. Acad. Sci. Canada Vol. **32** (3) 2010, pp. 81–96

GEOMETRIC THEORY OF PARSHIN RESIDUES

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Presented by Pierre Milman, FRSC

Abstract. This is a brief summary of results. More detailed papers are in preparation. Preliminary versions of the detailed papers are available on the `arxiv.org` [MM1], [MM2]. We study the theory of Parshin residues from the geometric point of view. In particular, the residue is expressed in terms of an integral over a smooth cycle. The Parshin–Lomadze Reciprocity Law for residues in complex case is proved via a homological relation on these cycles.

The paper consist of two parts. In the first part the theory of Leray coboundary operators for stratified spaces is developed. These operators are used to construct the cycle and prove the homological relation. In the second part resolution of singularities techniques are applied to study local geometry near a complete flag of subvarieties. We give a short introduction to the theory of Parshin residues in the Introduction.

All the constructions are valid both in complex algebraic and complex analytic cases. However, for simplicity of presentation we restrict ourselves to the algebraic case.

Résumé. Ceci est un bref résumé de résultats. Des articles plus détaillés sont en préparation. Des versions préliminaires de ces articles sont disponibles sur `arxiv.org` [MM1], [MM2]. Nous étudions la théorie des résidus de Parshin d'un point de vue géométrique. En particulier, le résidu est exprimé sous la forme d'une intégrale sur un cycle lisse, et la Loi de Réciprocité de Parshin–Lomadze pour les résidus dans le cas complexe est démontrée par l'intermédiaire d'une relation homologique sur ces cycles. Cet article comporte deux parties. Dans la première la théorie des opérateurs de cobords pour espaces stratifiés est développée et est utilisée pour construire le cycle et démontrer la relation d'homologie. Dans la seconde partie, les techniques de résolution de singularités sont utilisées pour étudier la géométrie locale près d'un drapeau complet de sous-variétés. Nous donnons une courte introduction à la théorie des résidus de Parshin dans l'introduction. Toutes les constructions sont valables dans le cas algébrique complexe et dans le cas analytique complexe. Cependant pour la simplicité de l'exposé on s'est restreint au cas algébrique.

Introduction.

0.1. Overview. Let X be a compact complex algebraic curve and ω be a rational 1-form on X. Let $x \in X$ be a point in X, such that X is locally irreducible at x.

Received by the editors on October 9, 2009.
AMS Subject Classification: Primary: 47A12; secondary: 46C20.
Keywords: numerical range, indefinite inner product, J-Hermitian matrix, noninterlacing.

In an open neighborhood of x one can write

$$\omega = f(t)dt, \quad f(t) = \sum_{i>N} \lambda_i t^i,$$

where t is a local normalizing parameter at x and N is an integer. The coefficient λ_{-1} in the above series does not depend on the choice of parameter t and is called the residue of ω at x. We denote it $\operatorname{res}_x \omega$.

Note that this definition is purely algebraic and works over any algebraically closed field.

Let now X be locally reducible at x. Let $X^1, X^2, \dots, X^k$ be the local irreducible components of X at x. Following the above procedure, one can define the residue of ω at x along each X^i. We denote such a residue $\operatorname{res}_{\{x,X^i\}} \omega$. One defines the residue of ω at x by taking the sum of residues along local irreducible components:

$$\operatorname{res}_x \omega = \sum_{i=1}^{k} \operatorname{res}_{\{x,X^i\}} \omega.$$

The well-known residue formula says that the total sum of residues of ω is zero:

$$\sum_{x \in X} \operatorname{res}_x \omega = 0.$$

More precisely, the theorem says that only finitely many summands in this sum are not equal to zero, and the sum of non-zero residues is zero.

One can prove this theorem for curves over any algebraically closed field, not necessarily the complex numbers (see, for example, [S], [T]).

However, in the complex case one can prove the residue formula in a very simple topological way:

Let $\Sigma \subset X$ be the subset consisting of singularities of X and poles of ω. Note that Σ is finite. The proof can be done in the following steps:

(1) Let X^i be a local irreducible component of X at $x \in X$. One can consider a small cycle $\gamma_{\{x,X^i\}} \subset X \backslash \Sigma$, going one time around x on X^i counterclockwise. Then

$$\operatorname{res}_{\{x,X^i\}} \omega = \frac{1}{2\pi i} \int_{\gamma_{\{x,X^i\}}} \omega.$$

We denote by γ_x the sum of cycles γ_{x,X^i} over all local irreducible components X^i of X at x.

(2) If $x \notin \Sigma$, then γ_x is homologous to zero in $X \backslash \Sigma$. Indeed, γ_x is the boundary of a small neighborhood of x in $X \backslash \Sigma$.

(3) The sum of cycles γ_x over all points $x \in \Sigma$ is homologous to zero in $X \backslash \Sigma$. Indeed, for every $x \in \Sigma$, γ_x is the boundary of a small neighborhood U_x of x in X. Then $X \backslash \bigcup_{x \in \Sigma} U_x$ is a 2-chain in $X \backslash \Sigma$, which boundary is exactly $-(\sum_{x\in\Sigma} \gamma_x)$.

Since in $X \backslash \Sigma$ the form ω is holomorphic and $d\omega = 0$, the residue formula now follows from Stokes Theorem.

In the late 1970s, A. Parshin introduced his notion of multidimensional residue for a rational n-form ω on an n-dimensional algebraic variety V_n. (In [P1] Parshin mostly deals with the two-dimensional case, then A. Beilinson and V. Lomadze in [B] and [L] generalized Parshin's ideas to the multidimensional case). The main difference between the Parshin residue and the classical one-dimensional residue is that in higher dimensions one computes the residue not at a point but at a complete flag of subvarieties $F = \{V_n \supset \cdots \supset V_0\}$, $\dim V_k = k$. Similarly to the one-dimensional case, the flag F could be "locally reducible". In that case, one defines the residue at every "local irreducible component" of F and then sum up over these components. We call the "local irreducible components" of the flag F *Parshin points.*

Parshin, Beilinson, and Lomadze proved the Reciprocity Law for multidimensional residues, which generalizes the classical residue formula:

Fix a partial flag of irreducible subvarieties $\{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$, where V_k is omitted $(0 < k < n)$. Then

$$\sum_{V_{k+1} \supset X \supset V_{k-1}} \mathrm{res}_{V_n \supset \cdots \supset X \supset \cdots \supset X_0}(\omega) = 0,$$

where the sum is taken over all irreducible k-dimensional subvarieties X, such that $V_{k+1} \supset X \supset V_{k-1}$.

Similarly to the one-dimensional case, the precise statement of the theorem is that there are only finitely many summands which are not zeros, and the sum of non-zero summands is zero.

In addition, if V_1 is proper (compact in the complex case), then one has the same relation for $k = 0$:

$$\sum_{x \in V_1} \mathrm{res}_{V_n \supset \cdots \supset X_1 \supset \{x\}}(\omega) = 0.$$

Again, there are finitely many non-zero summands and their sum is zero.

We review the definition of the Parshin residues and the formulation of the Reciprocity Law in the next section.

The methods used by Parshin, Beilinson, and Lomadze are purely algebraic and applicable in very general situations, not only over complex numbers. However, in the complex case one would expect a more geometric variant of the theory. J.-L. Brylinski and D. A. McLaughlin give a more topological treatment of the complex case in [BrM]. Given a flag $F = \{V_n \supset \cdots \supset V_0\}$ they introduce *flag-localized* homology groups $H_*^{V_i}(V_n; F)$ and a homology class $k_F \in H_n^{V_n}(V_n; F)$, such that

$$\mathrm{res}_F\, \omega = \frac{1}{(2\pi i)^n} \int_{k_F} \omega$$

for any meromorphic n-form ω. The class k_F is obtained from of the fundamental class $c_{V_0} \in H_{2n}(V_n, V_n \backslash V_0)$ by applying the boundary homomorphisms in the appropriate flag-localized homology groups n times. J.-L. Brylinski and D. A. McLaughlin mention, that the class k_F could be constructed in a more geometric way, so that it is naturally represented by a union of certain real n-tori. However, they only give such a construction in the case when all elements of the flag F are smooth.

In the first part of the paper we use the geometry of stratified spaces (see [MJ]) to generalize the geometric construction of the class k_F to the case of singular flags. This allows us to generalize the topological proof of the residue formula to the multidimensional case.

Let $\Sigma \subset V_n$ be the union of the singular locus of V_n and the poles of ω. Note that $V_n \backslash \Sigma$ is smooth, ω is holomorphic in $V_n \backslash \Sigma$, and $d\omega = 0$ in $V_n \backslash \Sigma$ (by dimension). Our proof follows exactly the same steps as in the one-dimensional case:

(1) Let $F = \{V_n \supset \cdots \supset V_0\}$, $\dim V_k = k$ be a complete flag of irreducible subvarieties of V_n. We construct a smooth cycle $\gamma_F \subset V_n \backslash \Sigma$, such that

$$\operatorname{res}_F \omega = \frac{1}{(2\pi i)^n} \int_{\gamma_F} \omega.$$

More precisely, γ_F is the union of smooth real n-dimensional tori, one for each "local irreducible component" of the flag F (Theorem 3).

(2) Consider any Whitney Stratification $\mathbf{S}$ of V_n, such that Σ is a union of strata. If at least one element of the flag F is not the closure of a stratum from $\mathbf{S}$, then γ_F is homologous to zero in $V_n \backslash \Sigma$ (Theorem 4).

(3) Fix a partial flag of irreducible subvarieties $F^p = \{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$, $n > k > 0$. Suppose, that all the elements of F^p are closures of strata from $\mathbf{S}$. Let $S_1, \ldots, S_k$ be all k-dimensional strata from $\mathbf{S}$, such that $V_{k+1} \supset \overline{S_i} \supset V_{k-1}$. Let $F_1, \ldots, F_k$ be complete flags of irreducible subvarieties in V_n, such that the k-dimensional element of F_i is $\overline{S_i}$, and all other elements for all F_i coincide with the elements of the partial flag F^p. Then the sum of cycles $\sum_{i=1}^k \gamma_{F_i}$ is homologous to zero in $V_n \backslash \Sigma$. In addition, if V_1 is compact, then the same relation holds for $k = 0$ (Theorems 2 and 3).

Similarly to the one-dimensional case, the Reciprocity Law now follows from Stokes Theorem.

It follows from step 2, that for a given n-form ω there could be only finitely many non-trivial Parshin residues.

Theorem 2, which is the main ingredient in our proof of the Reciprocity Law, is applicable to any abstract stratified space, not only complex variety. In a sense, it generalizes the Reciprocity Law to abstract stratified spaces.

In the second part of the paper we use the resolution of singularity techniques to study the analytic properties of multidimensional Laurent expansions of meromorphic functions near an "irreducible component" of a flag of subvarieties $F = \{V_n \supset \cdots \supset V_0\}$. The Parshin's local parameters $(u_1, \ldots, u_n)$ (involved in the original Parshin's constructions, see Definition 3) play the role of coordinates in such a neighborhood.

We define a neighborhood of an "irreducible component" of F to be a holomorphic map $\phi\colon U \to V_n$, where U is a polydisk in $\mathbb{C}^n$ with center at the origin, satisfying the following properties:

(1) the flag of coordinate subspaces in U is mapped to F (*i.e.*, $\phi(\{x_n = \cdots = x_{k+1} = 0\}) \subset V_k$, where $(x_1, \dots, x_n)$ is a fixed system of coordinates in U);
(2) the restriction of ϕ to the complement to the union of coordinate hyperplanes $U^n = \{(x_1, \dots, x_n) \in U : x_1 x_2 \cdots x_n \neq 0\}$ is an isomorphism to the image;
(3) the inverse map ϕ^{-1} is given (where defined) by an n-tuple of monomials in local parameters $(u_1, \dots, u_n)$;
(4) a technical condition, which distinguishes different "irreducible components" of the flag F (see Section 2.4 for more details).

We show that for any meromorphic function f, any "irreducible component" of F, and any Parshin's local parameters $(u_1, \dots, u_n)$ there exists a neighborhood $\phi\colon U \to V_n$ of this irreducible component, such that f is holomorphic in $\phi(U^n)$. Therefore, f expands into a Laurent series in $(u_1, \dots, u_n)$, normally converging in $\phi(U^n)$ (Theorem 8 and Corollary).

It follows, that for a meromorphic form $\omega = f du_1 \wedge \cdots \wedge du_n$ the residue of ω at the "irreducible component" of F is equal to the coefficient of the expansion of f at $u_1^{-1} u_2^{-1} \cdots u_n^{-1}$. Alternatively, one can choose a small enough real positive ϵ, so that the equations $|u_1| = \cdots = |u_n| = \epsilon$ define a real smooth n-torus τ in $\phi(U^n)$. Then the residue is equal to $\frac{1}{(2\pi i)^n} \int_\tau \omega$.

0.2. Parshin residues and the Reciprocity Law. Let V_n be an algebraic variety of dimension n. Let $V_n \supset \cdots \supset V_0$ be a flag of subvarieties of dimensions $\dim V_k = k$.

Consider the diagram in Figure 1, where

(1) $p_n\colon \widetilde{V}_n \to V_n$ is the normalization;
(2) $W_{n-1} \subset \widetilde{V}_n$ is the union of $(n-1)$-dimensional irreducible components of the preimage of V_{n-1};
(3) for every $k = 1, 2 \dots, n-1$
    (a) $p_k\colon \widetilde{W}_k \to W_k$ is the normalization;
    (b) $W_{k-1} \subset \widetilde{W}_k$ is the union of $(k-1)$-dimensional irreducible components of the preimage of V_{k-1}.

Definition 1. We call Figure 1 the *normalization diagram* of the flag $V_n \supset \cdots \supset V_0$.

Definition 2. The flag $V_n \supset \cdots \supset V_0$ of irreducible subvarieties together with a choice of a point $a_\alpha \in W_0$ is called a *Parshin point*.

Choosing a point $a_\alpha \in W_0$ is equivalent to choosing irreducible components in every W_i, $i = n-1, \dots, 0$. Indeed, $\widetilde{W}_i$ is normal and, therefore, locally irreducible at every point. In particular, it is locally irreducible at the image

$$\begin{array}{ccccccccc}
V_n & \supset & V_{n-1} & \supset & \cdots & \supset & V_1 & \supset & V_0 \\
\uparrow{\scriptstyle p_n} & & \uparrow{\scriptstyle p_n} & & & & & & \\
\widetilde{V}_n & \supset & W_{n-1} & & & & & & \\
& & \uparrow{\scriptstyle p_{n-1}} & & & & & & \\
& & \widetilde{W}_{n-1} & \supset & \cdots & & & & \\
& & & & & & & & \\
& & & & \cdots & \supset & W_1 & & \\
& & & & & & \uparrow{\scriptstyle p_1} & & \\
& & & & & & \widetilde{W}_1 & \supset & W_0
\end{array}$$

Figure 1

of a_α. Let $\widetilde{W}_i^\alpha$ be the irreducible component of $\widetilde{W}_i$, containing the image of a_α. Let $W_i^\alpha = p_i(\widetilde{W}_i^\alpha)$. Note that W_i^α is an irreducible component of W_i.

In order to define the Parshin residue, one needs to define the *local parameters* at a Parshin point, which play the role of the normalizing parameter in one-dimensional case. After that, one uses these parameters to define a sequence of residual meromorphic forms $\omega_{n-1}, \dots, \omega_0$ on $W_{n-1}, \dots, W_0$.

The local parameters are defined as follows: $W_{i-1}^\alpha \subset \widetilde{W}_i$ is a hypersurface in a normal variety. It follows that there exists a (meromorphic) function u_i on $\widetilde{W}_i$ which has zero of order 1 at a generic point of W_{i-1}^α. Since meromorphic functions are the same on W_i and $\widetilde{W}_i$, one can consider u_i as a function on W_i. Then one can extend (in an arbitrary way) u_i to $\widetilde{W}_{i+1}$ and so on. For simplicity, we denote all these functions by u_i. Now u_i is defined on V_n, and can be consecutively restricted to W_j for $j \geq i$.

Definition 3. Functions $(u_1, \dots, u_n)$ are called *local parameters* at the Parshin point $P = \{V_n \supset \cdots \supset V_0, a_\alpha\}$.

Let ω be a meromorphic n-form on V_n. One can show that the differentials $du_1, \dots, du_n$ are linearly independent at a generic point of V_n. Therefore, one can write

$$\omega = f du_1 \wedge \cdots \wedge du_n,$$

where f is a meromorphic function on V_n.

Now we define the forms ω_i:

Take a generic point $p \in W_{n-1}$. Both $\widetilde{V}_n$ and W_{n-1} are smooth at p. Moreover, parameters $u_1, \dots, u_n$ provide an isomorphism of a neighborhood of p to an open subset in $\mathbb{C}^n$ and W_{n-1} is given by the equation $u_n = 0$ in this neighborhood. Restrict the function f to the transversal section to W_{n-1} at p, given by fixing the parameters $u_1, \dots, u_{n-1}$. The restriction can be expanded into a Laurent series in u_n. It is easy to see that the coefficients of this expansion depend analytically on p. Moreover, one can see that the coefficients are meromorphic functions on W_{n-1}. Let f_{-1} be the coefficient at u_n^{-1} in this expansion. Then $\omega_{n-1} = f_{-1} du_1 \wedge \dots \wedge du_{n-1}$ is a meromorphic $(n-1)$-form on W_{n-1}.

Repeating this procedure one more time one gets a meromorphic $(n-2)$-form on W_{n-2}. Finally, after n steps, one gets a function ω_0 on the finite set W_0.

Definition 4. The *residue of* ω *at the Parshin point* $P = \{V_n \supset \dots \supset V_0, a_\alpha \in W_0\}$ is $\mathrm{res}_P(\omega) = \omega_0(a_\alpha)$.

Parshin proves that the residue is independent on the choice of local parameters.

Definition 5. The sum of residues over all $a \in W_0$ is called the *residue at the flag* $F = \{V_n \supset \dots \supset V_0\}$ and denoted $\mathrm{res}_F(\omega) = \sum_{a \in W_0} \mathrm{res}_{\{F,a\}}(\omega)$.

Theorem 1 ([P1], [B], [L]). *Let* ω *be a meromorphic* n*-form on* V_n*. Fix a partial flag of irreducible subvarieties* $\{V_n \supset \dots \supset \widehat{V}_k \supset \dots \supset V_0\}$*, where* V_k *is omitted* $(0 < k < n)$*. Then*

$$\sum_{V_{k+1} \supset X \supset V_{k-1}} \mathrm{res}_{V_n \supset \dots \supset X \supset \dots \supset X_0}(\omega) = 0,$$

where the sum is taken over all irreducible k*-dimensional subvarieties* X*, such that* $V_{k-1} \supset X \supset V_{k+1}$*. (In this formula only finitely many summands are non-zero.)*

In addition, if V_1 *is compact then one has the same relation for* $k = 0$*.*

1. Reciprocity Law via coboundary operators for stratified spaces.

1.1. Coboundary operators for stratified spaces and a relation between them.

Definition 6. Let M be a smooth manifold. Let V be a locally closed subset of M. By a *Whitney stratification* $\mathbf{S}$ of V, we mean a locally finite subdivision of V into smooth strata, such that:

(1) for each stratum $X \in \mathbf{S}$ its boundary $(\overline{X} \setminus X) \cap V$ is a union of strata;
(2) each pair (X, Y) of strata satisfies the Whitney condition **b**.

Let $\mathbf{S}$ be a Whitney stratification of V and let all strata of $\mathbf{S}$ be oriented.

Definition 7. Let $X, Y \in \mathbf{S}$. We say that $X < Y$ if $X \subset \overline{Y}$. We say that $X < Y$ are *consecutive strata* if there is no such Z that $X < Z < Y$.

Let $X < Y$ be consecutive strata, let $\dim X = n$ and $\dim Y = k$. Then one can define the Leray homomorphism $\phi_{X,Y} \colon H_*(X) \to H_{*+k-n-1}(Y)$ as follows.

Take a tubular neighborhood U_X of X in V. Let $U_{X,Y} := U_X \cap Y$. One can assume that the boundary $S_{X,Y}$ of $\overline{U_{X,Y}}$ is a smooth hypersurface in Y. Moreover, due to the local triviality of the stratification (see [MJ]), one can construct a smooth retraction $\pi_{X,Y} \colon \overline{U_{X,Y}} \to X$ which restricts to a smooth fibration π on the boundary $S_{X,Y}$. Since X and Y are consecutive strata, it follows that the fibers of π are compact.

Let $a \in H_*(X)$ be a homology class. Suppose that there is a smooth representative $A \subset X$ of a. Then $\phi_{X,Y}(a)$ is represented by $\pi^{-1}(A)$. One can extend this construction to a homomorphism in homology.

Remark. The homomorphism $\phi_{X,Y}$ can *a priori* depend on the choice of the tubular neighborhood and the projection. However, we were able to prove that for homology with rational coefficients $\phi_{X,Y}$ doesn't depend on the choice of $U_{X,Y}$ and $\pi_{X,Y}$.

Definition 8. We call a sequence of consecutive strata $X_1 < \cdots < X_n$ a *chain* of length n.

Theorem 2. *Let $X < Y$ be two strata such that all the chains from X to Y are of length* 3. *Let $Z_1, \dots, Z_n$ be all strata such that $X < Z_i < Y$. Then*

$$\phi_{Z_1,Y} \circ \phi_{X,Z_1} + \phi_{Z_2,Y} \circ \phi_{X,Z_2} + \cdots + \phi_{Z_k,Y} \circ \phi_{X,Z_k} = 0.$$

Example. Let $M_1, \dots, M_k$ be submanifolds in $\mathbb{R}^N$, passing through the origin, such that $M_i \cap M_j = \{(0, \dots, 0) \in \mathbb{R}^N\}$ for $i \neq j$. Consider Whitney stratification of $\mathbb{R}^N$ consisting of $k+2$ elements:

$$\begin{aligned} X &= \{(0, \dots, 0)\}, \\ Z_1 &= M_1 \backslash (0, \dots, 0), \\ &\vdots \\ Z_k &= M_k \backslash (0, \dots, 0), \\ Y &= \mathbb{R}^n \backslash \bigcup M_i. \end{aligned}$$

Note that $X, Z_1, \dots, Z_k$, and Y satisfy the conditions of Theorem 2.

Consider a small sphere S^{N-1} in $\mathbb{R}^N$ with center at the origin. It intersects the stratum Z_i along the hypersurface S_i, homeomorphic to sphere. Note that this sphere represents the homology class $\phi_{X,Z_i}([X]) \in H_{\dim Z_i - 1}(Z_i)$.

Let D_i be the boundary of a small tubular neighborhood U_i of S_i in S^{N-1}. It is easy to see that D_i represents the class $\phi_{Z_i,Y} \circ \phi_{X,Z_i}([X]) \in H_{n-2}(Y)$. Then $\big(S^{N-1} \backslash (\bigcup U_i)\big) \subset Y$ is a compact oriented manifold with boundary $\bigcup D_i$. Therefore,

$$\phi_{Z_1,Y} \circ \phi_{X,Z_1}([X]) + \phi_{Z_2,Y} \circ \phi_{X,Z_2}([X]) + \cdots + \phi_{Z_k,Y} \circ \phi_{X,Z_k}([X]) = 0 \in H_{n-2}(Y).$$

1.2. Application to the Parshin residues: Reciprocity Law. In this section all stratifications considered have complex algebraic strata.

Let $F = \{V_n \supset \cdots \supset V_0\}$ be a flag of irreducible varieties. Let ω be a meromorphic n-form on V_n. Let $V_n = \bigsqcup_{X \in \mathbf{S}} X$ be a Whitney stratification of V_n, such that for any $k = 0, \dots, n$, V_k is a union of strata, and ω is regular on the stratum of dimension n. For all $k = 0, \dots, n$, let $\breve{V}_k \subset V_k$ be the stratum of dimension k in V_k.

Definition 9. Let $\Delta_F^{\mathbf{S}} = \phi_{\breve{V}_{n-1},\breve{V}_n} \circ \cdots \circ \phi_{\breve{V}_0,\breve{V}_1}([V_0]) \in H_n(\breve{V}_n)$.

Theorem 3.
$$\operatorname{res}_F(\omega) = \frac{1}{(2\pi i)^n} \int_{\Delta_F} \omega.$$

Remark. Note that, according to our construction, $\Delta_F^{\mathbf{S}}$ is represented by a smooth real compact n-dimensional submanifold $\gamma_F \subset \breve{V}_n$. Moreover, γ_F is a union of smooth n-dimensional tori (we start from a point and then on each step take the total space of a smooth oriented fibration with compact one-dimensional fiber). One can show, that these tori are in one-to-one correspondence with the points of W_0. Moreover, the residue at the Parshin point $\{V_n \supset \cdots \supset V_0, a_\alpha \in W_0\}$ is equal to the integral over the corresponding torus.

Fix a stratification $V_n = \bigsqcup_{X \in \mathbf{S}} X$. Let $F = \{V_n \supset \cdots \supset V_0\}$ be a flag of irreducible varieties. Suppose that at least one element of the flag F is not a union of strata of $\mathbf{S}$. Consider another stratification $V_n = \bigsqcup_{X' \in \mathbf{S}'} X'$, such that all strata of $\mathbf{S}$ and all the elements of F are unions of strata of $\mathbf{S}'$. Note that $\breve{V}'_n \subset \breve{V}_n$, where $\breve{V}'_n$ and $\breve{V}_n$ are the n-dimensional strata of $\mathbf{S}'$ and $\mathbf{S}$ respectively.

Theorem 4. *$i_*(\Delta_F^{\mathbf{S}'}) = 0 \in H_n(\breve{V}_n)$, where $i \colon \breve{V}'_n \hookrightarrow \breve{V}_n$ is the embedding.*

Corollary 1. *Let V_n be an n-dimensional variety and ω be a meromorphic n-form on V_n. Let $V_n = \bigsqcup_{X \in \mathbf{S}} X$ be any Whitney stratification of V_n such that the divisor of poles of ω is a union of strata. Let $F = \{V_n \supset \cdots \supset V_0\}$ be a flag of irreducible subvarieties, such that $\operatorname{res}_F(\omega) \neq 0$. Then all elements of the flag F are unions of strata of the stratification $\mathbf{S}$.*

More details about the material of Section 1 can be found in the preprint [MM1].

2. Resolution of singularities for flags and the neighborhoods of Parshin points.

2.1. Branched coverings and the proper preimage of a hypersurface.

Definition 10. Let X and Y be non-empty pure-dimensional algebraic varieties of the same dimension. An algebraic map $f\colon X \to Y$ is called a *branched covering* if it is proper, surjective, and for every irreducible component X^0 of X, $f(X^0)$ is an irreducible component of Y.

Equivalently, f is a branched covering if there exist an open smooth dense subset $Y_0 \subset Y$, such that $f|_{f^{-1}(Y_0)}\colon f^{-1}(Y_0) \to Y_0$ is an (unbranched) covering and $f^{-1}(Y_0)$ is dense in X.

Note that a composition of two branched coverings is again a branched covering.

Definition 11. Let $f\colon X \to Y$ be a branched covering. Let $H \subset Y$ be a hypersurface in Y. Then the *proper preimage* $H_f \subset X$ of H is the union of those irreducible components H_f^i of the full preimage $f^{-1}(H)$ for which $f(H_f^i)$ has codimension 1 in Y (and, therefore, is an irreducible component of H).

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be branched coverings. Let $H \subset Z$ be a hypersurface. Then, clearly, $(H_g)_f = H_{g\circ f}$.

Lemma 1. *$f|_{H_f}\colon H_f \to H$ is a branched covering.*

Note that the normalization map is always a branched covering. Note also that a degree one branched covering is a birational isomorphism. (And vice versa, a proper map which is a birational isomorphism is a degree one branched covering.)

We will need the following statement:

Theorem 5. *Let Y be a pure-dimensional algebraic variety. Let $\pi\colon X \to Y$ be a degree one branched covering and let X be normal. Let $p\colon \widetilde{Y} \to Y$ be the normalization. Let $H \subset Y$ be a hypersurface. Then there exists a natural map $\widetilde{\pi}\colon H_\pi \to H_p$ which is a degree one branched covering.*

In particular, the degree of $\pi|_{H_\pi}\colon H_\pi \to H$ is the same as the degree of $p|_{H_p}\colon H_p \to H$.

2.2. Resolution of singularities for flags.

Theorem 6. *Let X be a variety. Let $Y_1, \dots, Y_K$ be closed subvarieties in X. Then there exists a branched covering of degree one $\pi\colon \overline{X} \to X$ such that:*

(1) *$\overline{X}$ is smooth;*

(2) $\pi|_{\overline{X}\setminus D}$ *is an isomorphism to* $\operatorname{reg}(X)\setminus(Y_1 \cup \cdots \cup Y_K)$, *where* $D = H_1 \cup \cdots \cup H_N$ *is a union of smooth exceptional hypersurfaces* H_i, *which simultaneously have only normal crossings. We denote by* $\mathcal{D} = \{H_1, \ldots, H_N\}$ *the set of exceptional hypersurfaces (we always assume* $H_1, \ldots, H_N$ *irreducible);*
(3) *for any* $m = 1, \ldots, K$ *the preimage* $\pi^{-1}(Y_m)$ *is a union of exceptional hypersurfaces;*
(4) *let* $H_i, H_j \in \mathcal{D}$ *and* $H_i \cap H_j \neq \varnothing$. *Then either* $\pi(H_i) \subset \pi(H_j)$ *or* $\pi(H_i) \supset \pi(H_j)$;
(5) *let* $H_{i_1}, \ldots, H_{i_k} \in \mathcal{D}$, $C := H_{i_1} \cap \cdots \cap H_{i_k} \neq \varnothing$, *and* $\pi(H_{i_1}) \subset \cdots \subset \pi(H_{i_k})$. *Then* C *is irreducible and* $\pi(C) = \pi(H_{i_1})$.

The existence of a resolution satisfying conditions (1)–(3) follows from the classical Hironaka's Theorem about resolutions of singularities [Hi], [BiM]. In order to obtain conditions (4) and (5) one needs to do some additional blow-ups with centers in intersections of exceptional hypersurfaces.

Let $V_n \supset \cdots \supset V_0$ be a flag of irreducible subvarieties. Let us apply Theorem 6 to V_n with subvarieties $V_{n-1}, \ldots, V_0$. Let $\pi\colon \overline{V}_n \to V_n$ be the resolution and $\mathcal{D} = \{H_1, \ldots, H_N\}$ be the set of exceptional hypersurfaces.

Let $\mathcal{D}_k = \{H_i \in \mathcal{D} : \pi(H_i) = V_k\}$ and $D_k = \bigcup_{H_i \in \mathcal{D}_k} H_i$. Let also $\overline{V}_{n-1} \supset \cdots \supset \overline{V}_0$ be the flag of consecutive proper preimages of the flag $V_n \supset \cdots \supset V_0$ (*i.e.*, $\overline{V}_k$ is the proper preimage of V_k under $\pi|_{\overline{V}_{k+1}}$). Then one has the following:

Lemma 2. $\overline{V}_k = D_{n-1} \cap \cdots \cap D_k$ *and* $\overline{V}_k$ *is smooth for all* $k = 0, \ldots, n$.

Definition 12. The flag $\overline{V}_n \supset \overline{V}_{n-1} \supset \cdots \supset \overline{V}_0$ together with the map $\pi\colon \overline{V}_n \to V_n$ is called a *resolution of singularities of the flag* $V_n \supset V_{n-1} \supset \cdots \supset V_0$.

2.3. Parshin point and residue via resolution of singularities. Consider the diagram in Figure 2, where the first line is the resolution of singularities of the flag $V_n \supset \cdots \supset V_0$ and the rest is the normalization diagram.

The following lemma follows immediately from Theorem 5:

Lemma 3. *For every* $k = 0, \ldots, n-1$ *there exists a natural degree one branched covering* $\varphi_k\colon \overline{V}_k \to \widetilde{W}_k$, *such that all the diagram commutes. In particular,* $\varphi_0\colon \overline{V}_0 \to \widetilde{W}_0 = W_0$ *is a bijection of finite sets.*

Let ω be a meromorphic n-form on V_n. By adding the poles of ω as another subvariety in Theorem 6, one can assume that $\pi^*(\omega)$ only has poles on exceptional hypersurfaces.

Let $P := \{V_n \supset \cdots \supset V_0, a_\alpha\}$ be a Parshin point. Let $b_\alpha = \varphi_0^{-1}(a_\alpha) \in \overline{V}_0$. Let $(x_1, \ldots, x_n)$ be coordinates in a neighborhood U of $b_\alpha \in \overline{V}_n$, such that in U the coordinate hyperplanes coincide with the exceptional hypersurfaces (note, that there are exactly n exceptional hypersurfaces meeting at b_α, one from each $\mathcal{D}_k$, $k = 0, \ldots, n-1$).

$$
\begin{array}{ccccccccc}
\overline{V}_n & \supset & \overline{V}_{n-1} & \supset & \cdots & \supset & \overline{V}_1 & \supset & \overline{V}_0 \\
\big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\
V_n & \supset & V_{n-1} & \supset & \cdots & \supset & V_1 & \supset & V_0 = \{a\} \\
\big\uparrow{\scriptstyle p_n} & & \big\uparrow{\scriptstyle p_n} & & & & & & \\
\widetilde{V}_n & \supset & W_{n-1} & & & & & & \\
& & \big\uparrow{\scriptstyle p_{n-1}} & & & & & & \\
& & \widetilde{W}_{n-1} & \supset & \cdots & & & & \\
& & & & & & & & \\
& & & & \cdots & \supset & W_1 & & \\
& & & & & & \big\uparrow{\scriptstyle p_1} & & \\
& & & & & & \widetilde{W}_1 & \supset & W_0
\end{array}
$$

Figure 2

Since $\pi^*(\omega)$ can only have poles on coordinate hyperplanes in U, one can write

$$\pi^*(\omega) = \Big(\sum_{d \in (\mathbb{Z}_{\geq -N})^n} w_d x^d \Big) dx_1 \wedge \cdots \wedge dx_n.$$

Theorem 7.

$$\operatorname{res}_P \omega = w_{-1\cdots-1} = \frac{1}{(2\pi i)^n} \int_{\{|x_1| = \cdots = |x_n| = \epsilon\}} \pi^*(\omega).$$

for a small enough ϵ.

This interpretation of the Parshin residue leads to another proof of the Reciprocity Law. We are planning to include it in a separate paper.

2.4. Neighborhoods of a Parshin point. Let $P := \{V_n \supset \cdots \supset V_0, a_\alpha\}$ be a Parshin point. Let $(u_1, \ldots, u_n)$ be local parameters (see Definition 3 in Section 0.2). Let $(v_1, \ldots, v_n)$ be meromorphic functions on V_n given by

$$v_1 = u_1;$$
$$v_2 = u_1^{c_{21}} u_2;$$
$$\vdots$$
$$v_n = u_1^{c_{n1}} \cdots u_{n-1}^{c_{nn-1}} u_n,$$

where $C = [c_{ij}]$ is a lower-triangular integer matrix with units on the diagonal.

Easy to check that $(v_1, \dots, v_n)$ are again local parameters.

Definition 13. We say that local parameters $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ are *similar.*

Now we give the definition of a neighborhood of the Parshin point P. Local parameters play the role of coordinates in this neighborhood.

Definition 14. Let $U \subset \mathbb{C}^n$ be a neighborhood of the origin. Let's fix coordinates $(x_1, \dots, x_n)$ in U. Let $U^k = \{x_n = 0, \dots, x_{k+1} = 0, x_k \neq 0, \dots, x_1 \neq 0\}$.

A holomorphic map $\phi\colon U \to V_n$, such that:

(1) $\phi(U^k) \subset V_k$ for $k = 0, \dots, n$;
(2) $\phi|_{U^n}$ is an isomorphism to the image;
(3) $\phi|_{\overline{U^k}}\colon \overline{U^k} \to V_k$ can be lifted to W_k, *i.e.*, there exists $\phi_k\colon \overline{U^k} \to W_k$, such that $\phi|_{\overline{U^k}} = p_n \circ \cdots \circ p_{k+1} \circ \phi_k$;
(4) $\phi_0(0) = a_\alpha$;
(5) the inverse map ϕ^{-1} is given (where defined) by local parameters, similar to $(u_1, \dots, u_n)$;

is called *a neighborhood of the Parshin point P with local parameters* $(u_1, \dots, u_n)$.

We use Theorem 6, Lemma 2, and Lemma 3 to prove the following result:

Theorem 8. *For any hypersurface $H \subset V^n$ there exists a neighborhood $\phi\colon U \to V_n$ of the Parshin point P, such that $H \cap \phi(U^n) = \varnothing$.*

Corollary 2. *Let f be a meromorphic function in a (usual) neighborhood of $a \in V^n$. One can choose a neighborhood $\phi\colon U \to V_n$ of the Parshin point P in such a way that $f \circ \phi$ is almost monomial in U (*i.e.*, $f \circ \phi$ is equal to a monomial multiplied by a non-vanishing regular function). Therefore, f can be expanded into a Laurent power series in $(u_1, \dots, u_n)$, normally converging in $\phi(U^n)$. Moreover, after switching to appropriate local parameters similar to $(u_1, \dots, u_n)$, the Newton polyhedron of the expansion will be a subset of the positive octant shifted by an integer vector.*

Corollary 3. *Let $\omega = f du_1 \wedge \cdots \wedge du_n$ be a meromorphic n-form on V_n. Let $f = \sum_{d \in \mathbb{Z}^n} f_d u^d$ be the Laurent power series expansion of f converging in $\phi(U_n)$. Then*

$$\operatorname{res}_P(\omega) = f_{-1 \cdots -1}.$$

2.5. Change of variables in a neighborhood of a Parshin point. Let $(v_1, \ldots, v_n)$ and $(u_1, \ldots, u_n)$ be similar local parameters and let $C = [c_{ij}]$ be the corresponding lower-triangular matrix, *i.e.*,

$$v_1 = u_1;$$
$$v_2 = u_1^{c_{21}} u_2;$$
$$\vdots$$
$$v_n = u_1^{c_{n1}} \ldots u_{n-1}^{c_{nn-1}} u_n.$$

Definition 15. There is a natural partial order on the set of all local parameters, similar to $(u_1, \ldots, u_n)$: one says that $(v_1, \ldots, v_n) \geq (u_1, \ldots, u_n)$ if $c_{ij} \geq 0$ for $i \neq j$.

Let $\phi\colon U \to V_n$ be a neighborhood of the Parshin point P, such that the inverse map is given exactly by $(v_1, \ldots, v_n)$. Let $(v_1, \ldots, v_n) \geq (u_1, \ldots, u_n)$ and $C = [c_{ij}]$ be the corresponding matrix. Then there exist another neighborhood $\phi'\colon U' \to V_n$ of P, such that the inverse map is given by $(u_1, \ldots, u_n)$ and $\phi'(U') \subset \phi(U)$.

Indeed, consider the map $d\colon \mathbb{C}^n \to \mathbb{C}^n$ given by

$$x_1 = y_1;$$
$$x_2 = y_1^{c_{21}} y_2;$$
$$\vdots$$
$$x_n = y_1^{c_{n1}} \cdots y_{n-1}^{c_{nn-1}} y_n,$$

where $(y_1, \ldots, y_n)$ and $(x_1, \ldots, x_n)$ are coordinates in, respectively, source and image.

Take U' to be any neighborhood of the origin in $\mathbb{C}^n$, such that $U' \subset d^{-1}(U)$. Take $\phi' = \phi \circ d|_{U'}$. Easy to see, that $\phi'\colon U' \to V_n$ is a neighborhood of P and that the inverse map is given by $(u_1, \ldots, u_n)$.

This procedure allows one to "shrink" a neighborhood of a Parshin point.

Suppose now that we have two sets of local parameters $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ at the Parshin point P. We describe how one can switch from one set of local parameters to another.

Theorem 9. *There exists a neighborhood $\phi\colon U \to V_n$ of the Parshin point P with local parameters $(u_1, \ldots, u_n)$, a neighborhood $\psi\colon W \to V_n$ of the Parshin point P with local parameters $(v_1, \ldots, v_n)$, and an isomorphism $\theta\colon U \to W$, such that $\phi = \psi \circ \theta$.*

Remark. Note that the inverse maps ϕ^{-1} and ψ^{-1} are given (where defined) by local parameters, similar to $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$. These local parameters may be "much" smaller then $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ respectively (*i.e.*, the corresponding matrices could have arbitrary large negative entries).

Remark. Theorem 9 gives another proof of the fact that the Parshin residue is independent on the choice of local parameters.

More details about the material of Section 2 can be found in the preprint [MM2]. There we also give a "toric" version of the neighborhoods of Parshin points. In this version we replace the neighborhood U of the origin in $\mathbb{C}^n$ by a neighborhood of the zero-dimensional orbit in an affine non-normal toric variety. The advantage of this approach is that in this case $\phi|_{U^k} : U^k \to V_k$ is an isomorphism to the image for all k, not only $k = n$. ($U^0, \ldots, U^n$ is a sequence of orbits, one in each dimension.) Thus, toric neighborhoods capture more geometric information about Parshin points.

Acknowledgments. We would like to thank Askold Khovanskii for raising the question and helpful discussions. We also thank Pierre Milman and Edward Bierstone for useful comments on the resolutions of singularities.

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