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A GEOMETRIZATION OF THE HAPPEL FUNCTOR

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Abstract. The Happel functor is a full and faithful exact functor from the derived category $\mathcal{D}^b(A)$ of bounded complexes over module category of a finite-dimensional algebra A to the stable category $\underline{\text{mod}}\,\hat{A}$ of the repetitive algebra $\hat{A}$ of A. If A has finite global dimension, this functor is even an equivalence of triangulated categories. Xiao, Xu, and Zhang defined topological spaces associated with $\mathcal{D}^b(A)$. In this paper, we attach some topological spaces for $\underline{\text{mod}}\,\hat{A}$ and construct maps between two kinds of topological spaces as a geometric characterization of the Happel functor.

Résumé. Le foncteur Happel est un foncteur plein, fidèle, et exact de la categorie derivée $\mathcal{D}^b(A)$ des complexes bornés sur la categorie des modules d'une algèbre A de dimension finie dans la categorie stable $\underline{\text{mod}}\,\hat{A}$ de l'algèbre répétitive $\hat{A}$ de A. Si A est de dimension finie (en dimension globale), ce foncteur sera même une équivalence des categories triangulées. Les espaces topologiques associés à $\mathcal{D}^b(A)$ étaient définés par Xiao, Xu et Zhang. Dans cette article, nous associons quelques espaces topologiques à la categorie $\underline{\text{mod}}\,\hat{A}$, et nous construisons des applications entre deux sortes des espaces topologiques comme une caractérisation géometrique du foncteur Happel.

1. **Introduction.** Let A be a basic finite-dimensional algebra over the complex field $\mathbb{C}$ and $\text{mod}\,A$ the category of all finite-dimensional left A-modules. For any dimension vector $\underline{d}$, we can associate a module variety $\mathbb{E}_{\underline{d}}(A)$ to the subset of $\text{mod}\,A$ consisting of modules with dimension vector $\underline{d}$ (see [4] or Section 2.2 for details). In the same spirit, the geometry over $\text{mod}\,A$ can be generalized to $\mathcal{C}^b(A)$, the category of bounded differential complexes of A-modules (see [3]), *i.e.*, we can associate an algebraic variety $\mathcal{C}^b(A, \underline{\mathbf{d}})$ to the subset of $\mathcal{C}^b(A)$ consisting of complexes with dimension vector sequence $\underline{\mathbf{d}}$ (see Section 2.2). The geometrization of $\mathcal{C}^b(A)$ naturally induces a geometrization of $\mathcal{P}^b(A)$, the subcategory of $\mathcal{C}^b(A)$ of bounded projective complexes. In view of these constructions, we can associate some topological spaces to the derived category $\mathcal{D}^b(A)$ of bounded complexes over $\text{mod}\,A$ and the homotopy category $\mathcal{K}^b\big(\mathcal{P}(A)\big)$ of bounded projective complexes (see [7] or Section 2.2). Then when A has finite global dimension, this deduces a geometrization of the equivalence $\mathcal{D}^b(A) \cong \mathcal{K}^b\big(\mathcal{P}(A)\big)$. Note that

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the constructible functions over these topological spaces can be applied to give realization of Kac–Moody algebras (see [7]).

On the other hand, one can construct the repetitive algebra $\hat{A}$ for A. In [2], D. Happel discovered the following important connection between the derived category of A and the stable module category of $\hat{A}$.

THEOREM 1.1 (The Happel Functor). *There exists an exact functor*

$$\mathfrak{F}\colon \mathcal{D}^b(A) \to \underline{\mathrm{mod}}\ \hat{A}$$

of triangulated categories which is full and faithful such that $\mathfrak{F}\mid_{\mathrm{mod}\,A} = \mathrm{id}$. *In particular, if* $\mathrm{gl.dim} A < \infty$*, then* $\mathfrak{F}$ *is also dense.*

The aim of this paper is to associate topological spaces to $\mathcal{D}^b(A)$ and $\underline{\mathrm{mod}}\,\hat{A}$, respectively, and to convert the above categorical equivalence into locally constructible maps (see Definition 3.2) which can be viewed as a geometrization of the Happel functor $\mathfrak{F}$. As a byproduct, it also gives an explicit realization of the Happel functor $\mathfrak{F}$.

2. Some preliminary results.

2.1. Repetitive algebra. Let A be a basic finite-dimensional $\mathbb{C}$-algebra. Its repetitive algebra, denoted by $\hat{A}$, is an infinite-dimensional Frobenius algebra. The underlying vector space of $\hat{A}$ is given by

$$\hat{A} = \Big(\bigoplus_{i\in\mathbb{Z}} A\Big) \oplus \Big(\bigoplus_{i\in\mathbb{Z}} DA\Big),$$

where $D = \mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$ is the standard duality. We denote the elements of $\hat{A}$ by $(a_i,\varphi_i)_i$, where $a_i \in A$, $\varphi_i \in DA$, and almost all a_i,φ_i are zero. The multiplication is defined by

$$(a_i,\varphi_i)_i \cdot (b_i,\psi_i)_i = (a_ib_i, a_{i+1}\psi_i + \varphi_ib_i)_i.$$

Denote by $\mathrm{mod}\,\hat{A}$ the category of all finitely generated $\hat{A}$–modules. The $\hat{A}$-module can be written in the following way:

$$(M_i,f_i)_i = \cdots \sim M_i \overset{f_i}{\sim} M_{i+1} \sim \cdots$$

where $M_i \in \mathrm{mod}\,A$ are all but finitely many zero, and

$$f_i\colon M_i \longrightarrow \mathrm{Hom}_A(DA, M_{i+1})$$

are A-linear maps such that $\mathrm{Hom}_A(DA, f_{i+1})\cdot f_i = 0$ for all $i \in \mathbb{Z}$. Let $\{P^l \mid 1 \le l \le n\}$ be a complete set of indecomposable projective A-modules up to isomorphism, the indecomposable projective and injective $\hat{A}$-module has the form

$$P_{l_j} = \cdots \sim 0 \sim P^l \sim DA \otimes P^l \sim 0 \sim \cdots$$

where P^l is an indecomposable projective A-module and is put as the j-th term of P_{l_j} for any $j \in \mathbb{Z}$. We also often use the adjoint form

$$I_{l_j} = \cdots \sim 0 \sim \mathrm{Hom}_A(DA, I^l) \sim K^l \sim 0 \sim \cdots$$

where $I^l = \mathrm{Hom}_A(A, P^l)$ is an indecomposable injective A-module and is put as the $j+1$-th term of I_{l_j} for any $j \in \mathbb{Z}$. Note that $P_{l_j} = I_{l_j}$ for any $1 \leq l \leq n$ and $j \in \mathbb{Z}$.

For two $\hat{A}$-modules X and Y, let $\mathcal{P}(X,Y)$ be the space of morphisms from X to Y which factor through a projective $\hat{A}$-module. Then the stable category $\underline{\mathrm{mod}}\,\hat{A}$, whose objects are the same as those of $\mathrm{mod}\,\hat{A}$ and whose morphisms are given by the quotient spaces $\underline{\mathrm{Hom}}(X,Y) = \mathrm{Hom}_{\hat{A}}(X,Y)/\mathcal{P}(X,Y)$ carries a natural structure of triangulated category (see [2]).

We have a canonical embedding of $\mathrm{mod}\,A$ into $\underline{\mathrm{mod}}\,\hat{A}$ as the composition of the embedding from $\mathrm{mod}\,A$ into $\mathrm{mod}\,\hat{A}$ and the canonical functor $\mathrm{mod}\,\hat{A} \to \underline{\mathrm{mod}}\,\hat{A}$ which sends $M \in \mathrm{mod}\,A$ onto $(M_i, f_i)_i \in \underline{\mathrm{mod}}\,\hat{A}$ where $M_0 = M$ and $M_i = 0$ for $i \neq 0$. For any $M \in \mathrm{mod}\,A$, its injective hull and projective cover in $\mathrm{mod}\,\hat{A}$ are described as follows:

$$\cdots \sim 0 \sim \mathrm{Hom}_A\big(DA, I(M)\big) \sim I(M) \sim 0 \sim \cdots$$

and

$$\cdots \sim 0 \sim P(M) \sim DA \otimes P(M) \sim 0 \sim \cdots$$

where $I(M)$ and $P(M)$ are the injective hull and the projective cover of M in $\mathrm{mod}\,A$, respectively.

2.2. Topological spaces for derived categories. Recall that $A \simeq \mathbb{C}Q/J$, where Q is a quiver, $\mathbb{C}Q$ is the path algebra of Q and J is an admissible ideal generated by a set R of relations in Q. Let I and Q_1 be the sets of vertices and arrows of the quiver Q, respectively, and let $s, t\colon Q_1 \to I$ be maps such that any arrow α starts at $s(\alpha)$ and terminates at $t(\alpha)$. For any dimension vector $\underline{d} = \sum_i a_i i \in \mathbb{N}I$, we consider the affine space over $\mathbb{C}$:

$$\mathbb{E}_{\underline{d}}(Q) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{a_{s(\alpha)}}, \mathbb{C}^{a_{t(\alpha)}}).$$

Any element $x = (x_\alpha)_{\alpha \in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q)$ defines a representation $M(x)$ of Q with $\underline{\dim}\, M(x) = \underline{d}$ in a natural way. A relation in Q is a linear combination $\sum_{i=1}^r \lambda_i p_i$, where $\lambda_i \in \mathbb{C}$ and p_i are paths of length at least two with $s(p_i) = s(p_j)$ and $t(p_i) = t(p_j)$ for all $1 \leq i, j \leq r$. For any $x = (x_\alpha)_{\alpha \in Q_1} \in \mathbb{E}_{\underline{d}}(Q)$ and any path $p = \alpha_1\alpha_2\cdots\alpha_m$ in Q, set $x_p = x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_m}$. Then x satisfies a relation $\sum_{i=1}^r \lambda_i p_i$ if $\sum_{i=1}^r \lambda_i x_{p_i} = 0$. If R is a set of relations in Q, then let $\mathbb{E}_{\underline{d}}(Q,R)$ be the closed subvariety of $\mathbb{E}_{\underline{d}}(Q)$ which consists of all elements satisfying all relations in R. Any element $x = (x_\alpha)_{\alpha \in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q,R)$ defines in a natural way a module $M(x)$ of $A \simeq \mathbb{C}Q/J$ with $\underline{\dim}\, M(x) = \underline{d}$ (see [4]). For simplicity, define $\mathbb{E}_{\underline{d}}(A) = \mathbb{E}_{\underline{d}}(Q,R)$.

For a dimension vector $\underline{d}$, set the I-graded $\mathbb{C}$-space $V^{\underline{d}} = \bigoplus_{i\in I} \mathbb{C}^{a_i}$. For a sequence of dimension vectors $\underline{\mathbf{d}}_A = (\dots, \underline{d}^i_A, \dots)$ with only finitely many non-zero entries, define $\mathcal{C}^b(A, \underline{\mathbf{d}}_A)$ to be the subset of

$$\prod_{i\in\mathbb{Z}} \mathbb{E}_{\underline{d}^i_A}(A) \times \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(V^{\underline{d}^i_A}, V^{\underline{d}^{i+1}_A})$$

which consists of elements $(x_i, \partial_i)_i$, where $x_i \in \mathbb{E}_{\underline{d}^i_A}(A)$, $M(x_i)$ is the corresponding A-module on the space $V^{\underline{d}^i_A}$ and $\partial_i \in \mathrm{Hom}_{\mathbb{C}}(V^{\underline{d}^i_A}, V^{\underline{d}^{i+1}_A})$ is an A-module homomorphism from $M(x_i)$ to $M(x_{i+1})$ with the property $\partial_{i+1}\partial_i = 0$. In fact, $\big(M(x_i), \partial_i\big)_i$, or simply denoted by $(x_i, \partial_i)_i$, is a complex of A-modules and $\underline{\mathbf{d}}_A$ is called its dimension vector sequence (see [3], [7]).

Now we associate topological spaces to $\mathcal{D}^b(A)$ as discussed in [7]. Let $K_0\big(\mathcal{D}^b(A)\big)$ be the Grothendieck group of the derived category $\mathcal{D}^b(A)$, and $\underline{\dim}\colon \mathcal{D}^b(A) \to K_0\big(\mathcal{D}^b(A)\big)$ the canonical surjection. Indeed, if $X^\bullet = (X_i, \partial_i)_i \in \mathcal{D}^b(A)$, we obtain $\underline{\dim}\, X^\bullet = \sum_{i\in\mathbb{Z}} (-1)^i \,\underline{\dim}\, X_i$. It induces a canonical surjection from the abelian group of dimension vector sequences to $K_0\big(\mathcal{D}^b(A)\big)$, we still denote it by $\underline{\dim}$. Note that the set $\mathcal{C}^b(A, \underline{\mathbf{d}}_A)$ of all complexes of fixed dimension vector sequence $\underline{\mathbf{d}}_A$ in $\mathcal{C}^b(A)$ is an affine variety. Define

$$\mathcal{C}^b(A, \mathbf{d}_A) = \bigcup_{\underline{\mathbf{d}}_A \in \underline{\dim}^{-1}(\mathbf{d}_A)} \mathcal{C}^b(A, \underline{\mathbf{d}}_A)$$

for any $\mathbf{d}_A \in K_0\big(\mathcal{D}^b(A)\big)$. In general, we have two common ways to define topology over $\mathcal{C}^b(A, \mathbf{d}_A)$ when we want that the topology over $\mathcal{C}^b(A, \mathbf{d}_A)$ is naturally induced by the topology over $\mathcal{C}^b(A, \underline{\mathbf{d}}_A)$. One is the weak topology, the other is the strong topology. Here we prefer to use the strong topology, *i.e.*, we choose the following set as topology base of closed subsets:

$$\mathbf{B} = \Big\{ \bigcup_{\underline{\mathbf{d}}_A \in \underline{\dim}^{-1}(\mathbf{d}_A)} C_{\underline{\mathbf{d}}_A} \;\Big|\; C_{\underline{\mathbf{d}}_A} \text{ is the closed set of } \mathcal{C}^b(A, \underline{\mathbf{d}}_A) \Big\}.$$

Moreover, we can define the quotient space $\mathcal{Q}^b(A, \mathbf{d}_A) = \mathcal{C}^b(A, \mathbf{d}_A)/\sim$, where $X^\bullet \sim Y^\bullet$ if and only if the complexes $X^\bullet$ and $Y^\bullet$ are quasi-isomorphic, *i.e.*, they are isomorphic in $\mathcal{D}^b(A)$. The topology of $\mathcal{Q}^b(A, \mathbf{d}_A)$ is an quotient topology, *i.e.*, let $\pi\colon \mathcal{C}^b(A, \mathbf{d}_A) \to \mathcal{Q}^b(A, \mathbf{d}_A)$ be the canonical surjection, then U is an open (or closed) set of $\mathcal{Q}^b(A, \mathbf{d}_A)$ if and only if $\pi^{-1}(U)$ is an open (or closed) set of $\mathcal{C}^b(A, \mathbf{d}_A)$.

2.3. Topological spaces involving a repetitive algebra. The same constructions discussed in Section 2.2 for algebra A can be applied to repetitive algebra $\hat{A}$. First, in a similar manner, we can define the module variety over $\hat{A}$. For any $M \in \mathrm{mod}\,\hat{A}$, let $M = (M_i, f_i)_i = \cdots \sim M_i \overset{f_i}{\sim} M_{i+1} \sim \cdots$, we call the sequence of dimension vectors of A-modules $(\dots, \underline{\dim}\, M_i, \dots)$ the dimension vector sequence of M, simply denoted by $\underline{d}_{\hat{A}} = (\dots, \underline{d}_{\hat{A}}{}^{i}, \dots) = \underline{d}_{\hat{A}}(M) = (\dots, \underline{\dim}\, M_i, \dots)$. Then $\mathbb{E}_{\underline{d}_{\hat{A}}}(\hat{A})$ is defined as the closed subset of

$$\prod_{i\in\mathbb{Z}} \mathbb{E}_{\underline{d}_A^i}(A) \times \mathrm{Hom}_{\mathbb{C}}\big(V^{\underline{d}_A^i}, \mathrm{Hom}_{\mathbb{C}}(DA, V^{\underline{d}_A^{i+1}})\big)$$

which consists of elements $(x_i, f_i)_i$, where $x_i \in \mathbb{E}_{\underline{d}_A^i}(A)$, $M(x_i)$ is the corresponding A-module on the space $V^{\underline{d}_A^i}$ and $f_i \in \mathrm{Hom}_{\mathbb{C}}\big(V^{\underline{d}_A^i}, \mathrm{Hom}_{\mathbb{C}}(DA, V^{\underline{d}_A^{i+1}})\big)$ is an A-module homomorphism from $M(x_i)$ to $\mathrm{Hom}_A\big(DA, M(x_{i+1})\big)$ with the property $\mathrm{Hom}_A(DA, f_{i+1}) \cdot f_i = 0$ for all $i \in \mathbb{Z}$. In fact, $(M(x_i), f_i)_i$, or simply denoted by $(x_i, f_i)_i$, is an $\hat{A}$-module. Note that the above product only contains finitely many non-zero terms so that the definition is well defined.

Next, we can construct topological space for $\mathcal{D}^b(\hat{A})$, the bounded derived category of mod $\hat{A}$. Let $\underline{\mathbf{d}}_{\hat{A}} = (\dots, \underline{d}_{\hat{A}}^{j}, \dots)$ be the sequence of dimension vector sequences of $\hat{A}$-modules with only finitely many non-zero entries, which will be called the dimension vector array of $\hat{A}$-modules. Define $\mathcal{C}^b(\hat{A}, \underline{\mathbf{d}}_{\hat{A}})$ to be the subset of

$$\prod_{j\in\mathbb{Z}} \mathbb{E}_{\underline{d}_{\hat{A}}^j}(\hat{A}) \times \prod_{i,j\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(V^{(\underline{d}_{\hat{A}}^j)^i}, V^{(\underline{d}_{\hat{A}}^{j+1})^i})$$

which consists of elements $(x_j, \partial_{ij})_{(i,j)}$ where $x_j \in \mathbb{E}_{\underline{d}_{\hat{A}}^j}(\hat{A})$, $M(x_j)$ is the corresponding $\hat{A}$-module,

$$\partial_{ij} \in \mathrm{Hom}_{\mathbb{C}}(V^{(\underline{d}_{\hat{A}}^j)^i}, V^{(\underline{d}_{\hat{A}}^{j+1})^i})$$

is an A-module homomorphism from $M(x_j)_i$ to $M(x_{j+1})_i$ and $\partial_j = (\partial_{ij})_{i\in\mathbb{Z}}$ is an $\hat{A}$-module homomorphism from $M(x_j)$ to $M(x_{j+1})$ with the property $\partial_{j+1}\partial_j = 0$. Note that $\big(M(x_j), \partial_{ij}\big)_{(i,j)}$, or simply denoted by $(x_j, \partial_{ij})_{(i,j)}$, is a complex of $\hat{A}$-modules and $\underline{\mathbf{d}}_{\hat{A}}$ is its dimension vector array.

By using a similar way, we can also define $\mathcal{C}^b(\hat{A}, \mathbf{d}_{\hat{A}})$ and $\mathcal{Q}^b(\hat{A}, \mathbf{d}_{\hat{A}})$ for any $\mathbf{d}_{\hat{A}} \in K_0\big(\mathcal{D}^b(\hat{A})\big)$.

We now fix a complete set of indecomposable projective-injective $\hat{A}$-modules (up to isomorphism) $\{P_{l_j} \mid 1 \le l \le n, j \in \mathbb{Z}\}$ $(= \{I_{l_j}\})$. Let $\mathcal{H}^b(\hat{A})$ be the full subcategory of $\mathcal{C}^b(\hat{A})$ which consists of almost projective-injective complexes (special case of 'almost projective complexes' defined by Saorin and Huisgen-Zimmermann in [3]), *i.e.*, the complexes $H^\bullet = (H^k, \partial_k)$ such that: for $k \neq 0$, $H^k \cong \bigoplus e_{l_j}^k P_{l_j}$ where $e_{l_j}^k$ are nonnegative integers, all but finitely many being zero; for $k = 0$, H^0 is an $\hat{A}$-module. For $k \neq 0$, denote by $\underline{L}(H^k)$ the vector $(e_{l_j}^k)$. The sequence $\underline{\mathbf{L}}(H^\bullet) = \big(\dots, \underline{L}(H^{-1}), \underline{d}_{\hat{A}}(H^0), \underline{L}(H^1), \dots\big)$ is called the almost projective-injective dimension sequence of $H^\bullet$. For a fixed almost projective-injective dimension sequence $\underline{\mathbf{L}} = \underline{\mathbf{L}}(H^\bullet)$, the corresponding dimension vector array is $\underline{\mathbf{d}}_{\hat{A}}(\underline{\mathbf{L}}) = \big(\underline{d}_{\hat{A}}(H^k)\big)$, we define $\mathcal{H}^b(\hat{A}, \underline{\mathbf{L}})$ to be a (locally closed) subset of $\mathcal{C}^b\big(\hat{A}, \underline{\mathbf{d}}_{\hat{A}}(\underline{\mathbf{L}})\big)$ which consists of elements $(x_j, \partial_{ij})_{(i,j)}$ where $x_j \in \mathbb{E}_{\underline{d}_{\hat{A}}^j}(\hat{A})$ for any $j \in \mathbb{Z}$, and $M(x_j)$ is the corresponding $\hat{A}$-module isomorphic to H^j for any $j \neq 0$. Recall that the canonical surjection $\underline{\dim}\colon \mathcal{D}^b(\hat{A}) \to K_0\big(\mathcal{D}^b(\hat{A})\big)$ can induce a canonical surjection from the abelian group of dimension vector arrays of $\hat{A}$-modules to $K_0\big(\mathcal{D}^b(\hat{A})\big)$, we still denote it by $\underline{\dim}$. We may define

$$\mathcal{H}^b(\hat{A}, \mathbf{L}) = \bigcup_{\underline{\mathbf{d}}_{\hat{A}}(\underline{\mathbf{L}}) \in \underline{\dim}^{-1}(\mathbf{L})} \mathcal{H}^b(\hat{A}, \underline{\mathbf{L}})$$

for any $\mathbf{L} \in K_0\big(\mathcal{D}^b(\hat{A})\big)$.

Then $\mathcal{H}^b(\hat{A}, \mathbf{L})$ has the natural topological structure with the topological base of closed subsets given by

$$\mathbf{B} = \Big\{ \bigcup_{\underline{\mathbf{d}}_{\hat{A}}(\underline{\mathbf{L}}) \in \underline{\dim}^{-1}(\mathbf{L})} B_{\underline{\mathbf{L}}} \;\Big|\; B_{\underline{\mathbf{L}}} \text{ is a closed set of } \mathcal{H}^b(\hat{A}, \underline{\mathbf{L}}) \Big\}.$$

We also have the quotient space $\mathcal{QH}^b(\hat{A}, \mathbf{L}) = \mathcal{H}^b(\hat{A}, \mathbf{L})/\sim$, where $X^\bullet \sim Y^\bullet$ in $\mathcal{H}^b(\hat{A}, \mathbf{L})$ if and only if $X^\bullet$ and $Y^\bullet$ are quasi-isomorphic, *i.e.*, they are isomorphic in $\mathcal{D}^b(\hat{A})$. The topology of $\mathcal{QH}^b(\hat{A}, \mathbf{L})$ is the quotient topology derived from the topology of $\mathcal{H}^b(\hat{A}, \mathbf{L})$.

For $X, Y \in \underline{\mathrm{mod}}\,\hat{A}$, we say $X \sim Y$ if there exists $u\colon X \to Y$ such that the mapping cone $\mathrm{Cone}(u)$ of the map u is an injective $\hat{A}$-module or $\mathrm{Cone}(u) = 0$. This condition is equivalent to that there exists an isomorphism of $\hat{A}$-modules $X \oplus I' \simeq Y \oplus I''$ for some injective $\hat{A}$-modules I' and I''.

Let $K_0(\underline{\mathrm{mod}}\,\hat{A})$ be the Grothendieck group of triangulated category $\underline{\mathrm{mod}}\,\hat{A}$ and $\underline{\mathrm{Dim}}\colon \underline{\mathrm{mod}}\,\hat{A} \to K_0(\underline{\mathrm{mod}}\,\hat{A})$ be the canonical surjective. It induces a canonical surjective map from the abelian group of dimension vector sequences $\underline{d}_{\hat{A}}$ to $K_0(\underline{\mathrm{mod}}\,\hat{A})$ which is still denoted by $\underline{\mathrm{Dim}}$. In the same way as used in Section 2.2, we can define:

$$\mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}}) = \bigcup_{\underline{d}_{\hat{A}} \in \underline{\mathrm{Dim}}^{-1}(\mathbf{d}'_{\hat{A}})} \mathbb{E}_{\underline{d}_{\hat{A}}}(\hat{A}) \quad \text{and} \quad \mathcal{Q}\mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}}) = \mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}})/\sim$$

for any $\mathbf{d}'_{\hat{A}} \in K_0(\underline{\mathrm{mod}}\,\hat{A})$, where "$\sim$" is the equivalence relation defined on $\underline{\mathrm{mod}}\,\hat{A}$.

3. Main theorem.

In this section, we will prove the main result of Theorem 3.3. Throughout, we assume that A has finite global dimension.

DEFINITION 3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic varieties over $\mathbb{C}$. A map $f_c\colon \mathcal{X} \to \mathcal{Y}$ is called constructible if there exists a finite stratification of locally closed subsets $\mathcal{X} = \bigsqcup_{l=1}^{t} \mathcal{X}_l$ such that the restriction of f_c to each $\mathcal{X}_i$ is a morphism of varieties.

DEFINITION 3.2. Let $\mathcal{X} = \bigsqcup_{i \in S} \mathcal{X}_i$ and $\mathcal{Y} = \bigsqcup_{j \in T} \mathcal{Y}_j$ be disjoint unions of algebraic varieties $\mathcal{X}_i$ and $\mathcal{Y}_j$ for countable sets S and T, respectively. Let

$\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ be the quotient spaces with respect to some equivalence relations, respectively. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called locally constructible with respect to the above disjoint unions if $f\mid_{\mathcal{X}_i}$ is a constructible map and the induced quotient map $\mathfrak{q}f$ is a bijection. If $g\colon \mathcal{Y} \to \mathcal{X}$ is a locally constructible map and $\mathfrak{q}g$ is the inverse of $\mathfrak{q}f$, then we say f and g constitute a pair of locally constructible maps with respect to disjoint unions.

Note that the composition of two locally constructible maps is still locally constructible.

THEOREM 3.3. *For any $\mathbf{d}_A \in K_0(\mathcal{D}^b(A))$, there exist $\mathbf{d}'_{\hat{A}} \in K_0(\underline{\mathrm{mod}}\,\hat{A})$ and a pair of locally constructible maps*

$$H_1\colon \mathcal{C}^b(A, \mathbf{d}_A) \to \mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}}) \quad \textit{and} \quad H_2\colon \mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}}) \to \mathcal{C}^b(A, \mathbf{d}_A)$$

with respect to disjoint unions:

$$\mathcal{C}^b(A, \mathbf{d}_A) = \bigcup_{\underline{\mathbf{d}}_A \in \underline{\mathbf{dim}}^{-1}(\mathbf{d}_A)} \mathcal{C}^b(A, \underline{\mathbf{d}}_A) \quad \textit{and} \quad \mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}}) = \bigcup_{\underline{d}_{\hat{A}} \in \underline{\mathrm{Dim}}^{-1}(\mathbf{d}'_{\hat{A}})} \mathbb{E}_{\underline{d}_{\hat{A}}}(\hat{A}).$$

The proof will be made along the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C}^b(A, \mathbf{d}_A) \arrow[r, "\theta"] \arrow[d, "\pi_A"'] & \mathcal{C}^b(\hat{A}, \mathbf{d}_{\hat{A}}) \arrow[r, "\tilde{f}"] \arrow[d, "\pi_{\hat{A}}"'] & \mathcal{H}^b(\hat{A}, \mathbf{L}) \arrow[r, "p"] \arrow[d, "\pi_{\hat{A}}"'] & \mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}}) \arrow[d, "\underline{\pi}_{\hat{A}}"'] \\
\mathcal{Q}^b(A, \mathbf{d}_A) \arrow[r, "\mathfrak{q}\theta"] & \mathcal{Q}^b(\hat{A}, \mathbf{d}_{\hat{A}}) \arrow[r, "\mathfrak{q}\tilde{f}"] & \mathcal{QH}^b(\hat{A}, \mathbf{L}) \arrow[r, "\mathfrak{q}p"] & \mathcal{Q}\mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}})
\end{tikzcd} \tag{3.1}$$

where θ is derived from the canonical embedding from $\mathrm{mod}\,A$ to $\mathrm{mod}\,\hat{A}$, $\tilde{f}$ and p will be discussed later.

In the following, all statements involving disjoint unions and equivalence relations have been described in the previous section. For simplicity, we will not emphasis the expression “with respect to disjoint unions”. In general, we cannot guarantee the maps in diagram (3.1) are morphisms of varieties. Instead, we consider the constructible maps [6].

The following result is an immediate consequence of the knowledge of linear algebra.

LEMMA 3.4. *Let A be a basic finite-dimensional $\mathbb{C}$-algebra. Define $\phi\colon \mathbb{N}I \to \mathbb{N}$ which maps $\underline{d} = \sum_i a_i i$ to $\sum_i a_i$. Let $\mathbb{E}_d(A) = \bigcup_{\underline{d} \in \phi^{-1}(d)} \mathbb{E}_{\underline{d}}(A)$ be the module variety of all A-modules M with $\dim_{\mathbb{C}} M = d$. For any $d_1, d_2 \in \mathbb{N}$, let $V(d_1, d_2)$ be the subset of $\mathbb{E}_{d_1}(A) \times \mathbb{E}_{d_2}(A) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^d_1, \mathbb{C}^{d_2})$ consisting of triple (M_1, M_2, f)*

such that $M_1 \in \mathbb{E}_{d_1}(A)$, $M_2 \in \mathbb{E}_{d_2}(A)$ and $f \in \mathrm{Hom}_A(M_1, M_2)$. Then there exist constructible maps:

$$\mathfrak{Ker}\colon V(d_1, d_2) \to \bigsqcup_{d \leq d_1} \mathbb{E}_d(A) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^d, \mathbb{C}^{d_1})$$

and

$$\mathfrak{Coker}\colon V(d_1, d_2) \to \bigsqcup_{d \leq d_2} \mathbb{E}_d(A) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_2}, \mathbb{C}^d)$$

such that $\mathfrak{Ker}(M_1, M_2, f) = (\mathrm{Ker}\, f, e)$ and $\mathfrak{Coker}(M_1, M_2, f) = (\mathrm{Coker}\, f, \pi)$, where $e\colon \mathrm{Ker}\, f \to M_1$ and $\pi\colon M_2 \to \mathrm{Coker}\, f$ are the natural embedding and projection, respectively.

Let $X^\bullet \in \mathcal{C}^b(\hat{A})$ be the complex as follows:

$$0 \longrightarrow X^{-n} \xrightarrow{\partial_{-n}} \cdots \longrightarrow X^{-1} \xrightarrow{\partial_{-1}} X^0 \xrightarrow{\partial_0} X^1 \longrightarrow \cdots \xrightarrow{\partial_{m-1}} X^m \longrightarrow 0.$$

As discussed in [1], we can construct an almost projective-injective complex quasi-isomorphic to $X^\bullet$. We start with the following commutative diagram:

$$0 \longrightarrow I_{-n} \longrightarrow I_{-n+1} \longrightarrow \cdots \longrightarrow Z^0 \xrightarrow{\partial_0} \cdots \longrightarrow P_{m-1} \longrightarrow P_m \longrightarrow 0$$

where I_{-n} is the injective hull of X^{-n}, Y^{-n+1} is the corresponding pushout, P_m is the projective cover of X^m and Y^{m-1} is the corresponding pullback. The below complex is quasi-isomorphic to the above complex $X^\bullet$ (see Lemma 2.6 in [7]). By iterating this process, we can obtain the complex with the following form:

$$0 \longrightarrow I_{-n} \longrightarrow I_{-n+1} \longrightarrow \cdots \longrightarrow Z^0 \xrightarrow{\partial_0} \cdots \longrightarrow P_{m-1} \longrightarrow P_m \longrightarrow 0$$

where the terms on the left-hand side of the 0-th term are injective modules and the terms on the right-hand side of the 0-th term are projective modules. This complex is quasi-isomorphic to $X^\bullet$. We set $F(X^\bullet)$ to be this complex. Using a routine method, we can easily verify that F is a functor from $\mathcal{C}^b(\hat{A})$ to $\mathcal{C}^b(\hat{A})$.

Proposition 3.5. *For a fixed dimension vector array $\underline{\mathbf{d}}_{\hat{A}}$ of $\hat{A}$-modules, the functor $F\colon \mathcal{C}^b(\hat{A}) \to \mathcal{C}^b(\hat{A})$ induces a constructible map*

$$f\colon \mathcal{C}^b(\hat{A}, \underline{\mathbf{d}}_{\hat{A}}) \to \mathcal{H}^b(\hat{A}, \underline{\mathbf{L}})$$

for some almost projective-injective dimension sequence $\underline{\mathbf{L}}$.

Proof. For any $M^\bullet \in \mathcal{C}^b(\hat{A}, \underline{\mathbf{d}}_{\hat{A}})$ with $\underline{\mathbf{d}}_{\hat{A}} = (\underline{d}^j_{\hat{A}})_{j\in\mathbb{Z}}$ and $\underline{d}^j_{\hat{A}} = ((\underline{d}^j_{\hat{A}})^i)_{i\in\mathbb{Z}}$, as a complex, assume that $M^\bullet$ has the following form:

$$\cdots \longrightarrow M^j \longrightarrow M^{j+1} \longrightarrow \cdots.$$

Each M^j as an $\hat{A}$-module has the form as follows:

$$\cdots \sim M_i^j \sim M_{i+1}^j \sim \cdots$$

with $\underline{\dim}\, M_i^j = (\underline{d}_{\hat{A}}^j)^i$. There are standard surjective A-morphisms: $A^{d_i^j} \twoheadrightarrow M_i^j$ where $d_i^j = \dim_{\mathbb{C}} M_i^j$. For a given A-module M_i^j, denote by $M_i^j(i)$ the $\hat{A}$-module whose i-th term is M_i^j and all other terms are 0. Therefore, the above standard surjective A-morphism induces a standard surjective $\hat{A}$-morphism: $P_i^j \to M_i^j(i)$ where $P_i^j = \cdots \sim 0 \sim A^{d_i^j} \sim DA \otimes_A A^{d_i^j} \sim 0 \sim \cdots$ is a projective-injective $\hat{A}$-module. Let $P_{\hat{A}}^j = \bigoplus P_i^j$, then there is a surjective $\hat{A}$-morphism from $P_{\hat{A}}^j$ to M^j which is called standard projective cover. Dually, we can write down the standard injective hull $I_{\hat{A}}^j$ for M^j. Then we obtain a complex $N^\bullet \in \mathcal{C}^b(\hat{A}, \underline{\mathbf{d}}'_{\hat{A}})$ for some dimension vector array $\underline{\mathbf{d}}'_{\hat{A}}$ with the following commutative diagram

$$\begin{tikzcd}
N^\bullet \arrow[d] & \cdots \arrow[r] \arrow[d, equal] & N^{j-1} \arrow[r] \arrow[d] & P_{\hat{A}}^j \arrow[r] \arrow[d] & \cdots \arrow[d, equal] \\
M^\bullet : & \cdots \arrow[r] & M^{j-1} \arrow[r] & M^j \arrow[r] & \cdots
\end{tikzcd}$$

where N^{j-1} is the pull-back. Note that the process of taking pull-back only involves the operation of taking kernel of some morphism, therefore by Lemma 3.4, the above construction induces a constructible map from $\mathcal{C}^b(\hat{A}, \underline{\mathbf{d}}_{\hat{A}})$ to $\mathcal{C}^b(\hat{A}, \underline{\mathbf{d}}'_{\hat{A}})$. By iterating this construction as the way of constructing $F(X^\bullet)$, we can obtain a constructible map $f\colon \mathcal{C}^b(\hat{A}, \underline{\mathbf{d}}_{\hat{A}}) \to \mathcal{H}^b(\hat{A}, \underline{\mathbf{L}})$ for some almost projective-injective dimension sequence $\underline{\mathbf{L}}$. $\square$

PROPOSITION 3.6. *For any* $\mathbf{d}_{\hat{A}} \in K_0\big(\mathcal{D}^b(\hat{A})\big)$*, under the quotient topology, the constructible map* f *discussed in Proposition* 3.5 *induces a locally constructible map*

$$\tilde{f}\colon \mathcal{C}^b(\hat{A}, \mathbf{d}_{\hat{A}}) \to \mathcal{H}^b(\hat{A}, \mathbf{L})$$

where $\mathbf{L} = \mathbf{d}_{\hat{A}}$.

PROOF. Given any $X^\bullet \in \mathcal{C}^b(\hat{A})$. Since $F(X^\bullet)$ is quasi-isomorphic to $X^\bullet$, they correspond to the same element in $K_0\big(\mathcal{D}^b(\hat{A})\big)$. Therefore

$$\mathbf{L} = \underline{\dim}\big(\underline{\mathbf{d}}_{\hat{A}}\big(\underline{\mathbf{L}}\big(F(X^\bullet)\big)\big)\big) = \underline{\dim}\big(\underline{\mathbf{d}}_{\hat{A}}(X^\bullet)\big) = \mathbf{d}_{\hat{A}}.$$

The map $\tilde{f}$ can be derived from f naturally. There exists a canonical embedding $\tilde{e}\colon \mathcal{H}^b(\hat{A}, \mathbf{L}) \to \mathcal{C}^b(\hat{A}, \mathbf{d}_{\hat{A}})$. Denote by $\mathfrak{q}\tilde{f}$ and $\mathfrak{q}\tilde{e}$ the quotient maps of $\tilde{f}$ and $\tilde{e}$ (see Diagram 3.1), respectively. It is easy to verify that $\mathfrak{q}\tilde{f}$ and $\mathfrak{q}\tilde{e}$ are inverse to each other. The statement of the proposition will follow immediately. $\square$

For any $H^\bullet \in \mathcal{H}^b(\hat{A}, \mathbf{L})$, let H^0 be its 0-th term and $\mathbf{d}'_{\hat{A}} = \underline{\mathrm{Dim}}\big(\underline{d}_{\hat{A}}(H^0)\big)$. Then we can define a map $p\colon \mathcal{H}^b(\hat{A}, \mathbf{L}) \to \mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}})$ with $p(H^\bullet) = H^0$. For $X^\bullet, Y^\bullet \in \mathcal{H}^b(\hat{A}, \mathbf{L})$, if $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathcal{D}^b(\hat{A})$, then $p(X^\bullet) \sim p(Y^\bullet)$ in $\underline{\mathrm{mod}}\,\hat{A}$ (see Proposition 7 in [1]). Note that $X^\bullet$ is isomorphic to $p(X^\bullet)$ in $\mathcal{D}^b(\hat{A})/\mathcal{K}^b\big(\mathcal{P}(\hat{A})\big)$ which is equivalent as a triangulated category to $\underline{\mathrm{mod}}\,(\hat{A})$ (see [5]). Hence, the quotient map $\mathfrak{q}p'$ of the natural embedding $p'\colon \mathbb{E}(\hat{A}, \mathbf{d}'_{\hat{A}}) \to \mathcal{H}^b(\hat{A}, \mathbf{L})$ is the inverse of the quotient map $\mathfrak{q}p$ of p. It is clear that p and p' is a pair of locally constructible maps.

For any dimension vector sequence $\underline{d}_{\hat{A}} \in \underline{\mathrm{Dim}}^{-1}(\mathbf{d}'_{\hat{A}})$, we can construct the map

$$h\colon \mathbb{E}_{\underline{d}_{\hat{A}}}(\hat{A}) \longrightarrow \mathcal{C}^b(A, \underline{\mathbf{e}}_A)$$

by induction for some dimension vector sequence $\underline{\mathbf{e}}_A$ of A-modules.

Assume that the global dimension of A is m. Let M be an A-module. A projective resolution of M is called standard if it has the form

$$0 \longrightarrow P_{m+1} \longrightarrow A^{d_m} \xrightarrow{\pi_m} \cdots \longrightarrow A^{d_i} \xrightarrow{\pi_i} \cdots \longrightarrow A^{d_i} \xrightarrow{\pi_1} M \longrightarrow 0$$

where $d_1 = \dim_{\mathbb{C}} M$, $d_i = \dim_{\mathbb{C}} \mathrm{Ker}\,\pi_{i-1}$ and $\pi_i\colon A^{d_i} \to \mathrm{Ker}\,\pi_{i-1}$ is a standard projective cover for $i = 2, \ldots, m$. Dually, we can define the standard injective resolution of M.

The following easy result will be used to do the induction.

Lemma 3.7. *Assume that an $\hat{A}$-module $M_{[a,b+1]}$ has the form*

$$M_{[a,b+1]} = 0 \sim M_a \overset{f_a}{\sim} \cdots \overset{f_b}{\sim} M_{b+1} \sim 0$$

and let $M_{[a,b]} = 0 \sim M_a \overset{f_a}{\sim} \cdots \overset{f_{b-1}}{\sim} M_b \sim 0$. Assume that as an A-module, M_{b+1} has the standard injective resolution: $0 \to M_{b+1} \to I_1 \to I_2 \to \cdots \to I_t \to 0$. For $l = 1, \ldots, t$, let $\mathrm{Hom}_A(DA, I_l)(b)$ be the $\hat{A}$-module whose b-th term is $\mathrm{Hom}_A(DA, I_l)$ and all other terms are zero. Then, in $\mathcal{D}^b(\hat{A})$, $M_{[a,b+1]}$ as a stalk complex, is isomorphic to

$$0 \longrightarrow M_{[a,b]} \longrightarrow \mathrm{Hom}_A(DA, I_1)(b) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(DA, I_t)(b) \longrightarrow 0.$$

Proof. In $\mathcal{D}^b(\hat{A})$, $M_{[a,b+1]}$ is isomorphic to the complex

$$0 \longrightarrow M_{[a,b]} \xrightarrow{\bar{f}_b} \big(\cdots \sim 0 \sim \mathrm{Hom}_A(DA, I_1) \sim K_1 \sim 0 \sim \cdots\big) \longrightarrow 0$$

where $\bar{f}_b$ is naturally induced by the composition $M_b \to \mathrm{Hom}_A(DA, M_{b+1}) \to \mathrm{Hom}_A(DA, I_1)$ and K_1 is the 1-syzygy I_1/M_{b+1}. Similarly, in $\mathcal{D}^b(\hat{A})$, $(\cdots \sim 0 \sim \mathrm{Hom}_A(DA, I_1) \sim K_1 \sim 0 \sim \cdots)$ is isomorphic to

$$0 \longrightarrow \mathrm{Hom}_A(DA, I_1)(b) \longrightarrow \big(\cdots \sim 0 \sim \mathrm{Hom}_A(DA, I_2) \sim K_2 \sim 0 \sim \cdots\big) \longrightarrow 0$$

By induction on t, we can prove the lemma. $\square$

Proof of Theorem 3.3. Let $\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}_{\hat{A}})[a,b]$ be the subset of $\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}_{\hat{A}})$ consisting of complexes with the property such that each term has the form: $M_{[a,b]}=0\sim M_a \overset{f_a}{\sim}\cdots\overset{f_{b-1}}{\sim} M_b\sim 0$. Without loss of generality, we can assume $a\leq 0$ and $b\geq 0$.

For any dimension vector sequence $\underline{d}_{\hat{A}}$ of $\hat{A}$-module, let $\underline{\mathbf{d}}_{\hat{A}}=(\underline{d}^j_{\hat{A}})_{j\in\mathbb{Z}}$ be a special dimension vector array with $\underline{d}^j_{\hat{A}}=0$ for $j\neq 0$ and $\underline{d}^0_{\hat{A}}=\underline{d}_{\hat{A}}$. According to Lemmas 3.4 and 3.7, we know that there is a constructible map

$$\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}_{\hat{A}})[a,b+1]\to\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}'_{\hat{A}})[a,b]$$

for some $\underline{\mathbf{d}}'_{\hat{A}}$, since only kernels and cokernels of morphisms are involved in order to obtain the desired map. Note that $\mathbb{E}_{\underline{d}_{\hat{A}}}(\hat{A})=\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}_{\hat{A}})[a,b+1]$ for some a and b. Dually, if we consider the standard projective resolution of M_a, then we can obtain a constructible map

$$\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}_{\hat{A}})[a,b+1]\to\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}''_{\hat{A}})[a+1,b+1]$$

for some $\underline{\mathbf{d}}''_{\hat{A}}$. As for general $\underline{\mathbf{d}}_{\hat{A}}$, we can use the mapping cones of morphisms of complexes to obtain the above two constructible maps. For example, assume that $\underline{\mathbf{d}}_{\hat{A}}=(\underline{d}^j_{\hat{A}})_{j\in\mathbb{Z}}$ with $\underline{d}^j_{\hat{A}}=0$ for $j\neq 0,1$. Given $M^\bullet:=M^0\xrightarrow{g}M^1\in\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}_{\hat{A}})[a,b+1]$, note that M^i has the form

$$M^i=0\sim M^i_a\overset{f^i_a}{\sim}\cdots\overset{f^i_b}{\sim}M^i_{b+1}\sim 0$$

for $i=0,1$ and $g=(g_t)_{a\leq t\leq b+1}$. For $i=0,1$, assume that M^i_{b+1} as an A-module, has the standard injective resolution $0\to M^i_{b+1}\to I^i_1\to I^i_2\to\cdots\to I^i_l\to 0$. Then by Lemma 3.7, in $\mathcal{D}^b(\hat{A})$, $M^\bullet$ is isomorphic to the mapping cone $N^\bullet$ of the following map between two complexes:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M^0_{[a,b]} & \longrightarrow & \mathrm{Hom}_A(DA,I^0_1)(b) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_A(DA,I^0_l)(b) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle g_{[a,b]}} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^1_{[a,b]} & \longrightarrow & \mathrm{Hom}_A(DA,I^1_1)(b) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_A(DA,I^1_l)(b) & \longrightarrow & 0
\end{array}$$

where $M^i_{[a,b]}:=0\sim M^i_a\overset{f^i_a}{\sim}\cdots\overset{f^i_{b-1}}{\sim}M^i_b\sim 0$ for $i=0,1$ and $g_{[a,b]}=(g_t)_{a\leq t\leq b}$. Note that $N^\bullet\in\mathcal{C}^b(\hat{A},\underline{\mathbf{d}}'_{\hat{A}})[a,b]$. By iterating the above construction and by Lemma 3.4, we finally obtain a constructible map

$$h\colon\mathbb{E}_{\underline{d}_{\hat{A}}}(\hat{A})\to\mathcal{C}^b(\hat{A},\underline{\mathbf{e}}_{\hat{A}})[0,0]$$

for some dimension vector array $\underline{\mathbf{e}}_{\hat{A}}$ of $\hat{A}$-modules. Note that $\underline{\mathbf{e}}_{\hat{A}}$ can also be viewed as a dimension vector sequence for mod A which is denoted by $\underline{\mathbf{e}}_A$. Therefore, we have $\mathcal{C}^b(\hat{A}, \underline{\mathbf{e}}_{\hat{A}})[0,0] = \mathcal{C}^b(A, \underline{\mathbf{e}}_A)$.

Assume that $\underline{d}_{\hat{A}} \in \underline{\mathrm{Dim}}^{-1}(\mathbf{d}')$ for some $\mathbf{d}' \in K_0(\underline{\mathrm{mod}}\,\hat{A})$ and $\underline{\mathbf{e}}_A \in \underline{\dim}^{-1}(\mathbf{d}'')$ for some $\mathbf{d}'' \in K_0\big(\mathcal{D}^b(A)\big)$. Then we have a map from $\mathbb{E}(\hat{A}, \mathbf{d}')$ to $\mathcal{C}^b(A, \mathbf{d}'')$, which is still denoted by h. For any $X^\bullet \in \mathcal{C}^b(A, \mathbf{d}_A)$, by definition, $X^\bullet$ is isomorphic to $hp\tilde{f}\theta(X^\bullet)$ in $\mathcal{D}^b(\hat{A})/\mathcal{K}^b\big(\mathcal{P}(\hat{A})\big)$. Hence, we conclude that $\mathbf{d}'' = \mathbf{d}_A$ and h is a locally constructible map. Set $H_1 = p\tilde{f}\theta$ and $H_2 = h$. This completes the proof of Theorem 3.3. □

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