

## ISOMORPHISMS OF JET SCHEMES

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**ABSTRACT.** If two schemes are isomorphic, then their  $m$ -jet schemes are isomorphic for all  $m$ . In this paper we consider the converse problem. We prove that if an isomorphism of the  $m$ -jet schemes is induced from a morphism of the base schemes, then the morphism of the base schemes is an isomorphism. But we also prove that just the existence of isomorphisms between  $m$ -jet schemes does not yield the existence of an isomorphism between the base schemes.

**RÉSUMÉ.** Si deux schémas sont isomorphes, alors pour tout  $m$  les schémas de leurs  $m$ -jets sont isomorphes. Dans cet article nous considérons la question inverse. Nous démontrons que si un isomorphisme des schémas de  $m$ -jets est induit par un morphisme des schémas de base, alors ce morphisme des schémas de base est un isomorphisme. Mais nous démontrons aussi que l'existence d'un isomorphisme entre schémas des  $m$ -jets n'implique pas l'existence d'un isomorphisme entre les schémas de base.

**1. Introduction.** From a given scheme  $X$  we obtain the  $m$ -jet scheme  $X_m$  for every  $m \in \mathbb{N} \cup \{\infty\}$  [9]. These jet schemes are considered to represent the nature of the scheme  $X$  (see for example [1], [4], [8]). It is clear that if we have an isomorphism  $f: X \xrightarrow{\sim} Y$  of schemes, then the induced morphism  $f_m: X_m \rightarrow Y_m$  is also an isomorphism for every  $m \in \mathbb{N} \cup \{\infty\}$ . Conversely, we consider the problem: if a morphism  $f: X \rightarrow Y$  induces an isomorphism  $f_m: X_m \xrightarrow{\sim} Y_m$  for some  $m \in \mathbb{N} \cup \{\infty\}$ , then is  $f$  an isomorphism? The answer is “yes” and the proof follows immediately from the structure of jet schemes (see Corollary 3.2). Consider then the problem without the assumption of the existence of a morphism  $f$ , *i.e.*, if there is an isomorphism  $X_m \simeq Y_m$  for some  $m$  or all  $m \in \mathbb{N} \cup \{\infty\}$ , is there then an isomorphism  $X \simeq Y$ ? The answer is “no”. We give an example of two non-isomorphic varieties  $X$  and  $Y$  with isomorphic  $m$ -jet schemes  $X_m \simeq Y_m$  for every  $m \in \mathbb{N} \cup \{\infty\}$ .

For jet schemes, we use the terminology and notation in [3]. In this paper a  $k$ -scheme is always a scheme of finite type over an algebraically closed field  $k$ .

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When we say variety, then it means an irreducible reduced separated scheme of finite type over  $k$ .

## 2. Glossary of jet schemes.

**Definition 2.1** Let  $X$  be a  $k$ -scheme and  $K \supset k$  a field extension. For  $m \in \mathbb{N}$ , a  $k$ -morphism  $\text{Spec } K[t]/(t^{m+1}) \rightarrow X$  is called an  $m$ -jet of  $X$  and a  $k$ -morphism  $\text{Spec } K[[t]] \rightarrow X$  is called an *arc* or  $\infty$ -jet of  $X$ . For  $m \in \mathbb{N} \cup \{\infty\}$ , the space of  $m$ -jets of  $X$  is denoted by  $X_m$  (for details, see [3]).

Let  $G = \mathbb{A}_k^1 \setminus \{0\} = \text{Spec } k[s, s^{-1}]$  be the algebraic group with the multiplication. Then  $X_m$  has a canonical action of  $G$  for every  $m \in \mathbb{N} \cup \{\infty\}$ . This action is induced from the ring homomorphism  $k[t]/(t^{m+1}) \rightarrow k[s, s^{-1}, t]/(t^{m+1})$ ,  $t \mapsto st$ .

The following is well known by experts (see, for example [5]).

**Lemma 2.2** *For every  $m \in \mathbb{N} \cup \{\infty\}$ , the canonical projection  $\pi_m: X_m \rightarrow X$  is the categorical quotient of the  $m$ -jet scheme  $X_m$  by the action of  $G$ .*

For the definition of the categorical quotient, see [7, Definition 0.5].

It is well known that for a  $k$ -scheme  $X$ , the  $m$ -jet scheme  $X_m$  is affine over  $X$  and has a graded structure, *i.e.*, if we write  $X_m = \text{Spec } \mathcal{R}^{(m)}$ , then  $\mathcal{R}^{(m)}$  has a structure of graded  $\mathcal{O}_X$ -algebra  $\mathcal{R}^{(m)} = \bigoplus_{i \geq 0} \mathcal{R}_i^{(m)}$  with  $\mathcal{R}_0 = \mathcal{O}_X$ . The truncation morphism  $\psi_{m,m'}: X_m \rightarrow X_{m'}$  ( $m' < m$ ) corresponds to a homomorphism of graded algebras  $\psi_{m,m'}^*: \mathcal{R}^{(m')} \rightarrow \mathcal{R}^{(m)}$ . In particular, by  $\psi_{m,m-1}^*$ ,  $\mathcal{R}^{(m)}$  is an  $\mathcal{R}^{(m-1)}$ -algebra and generated by  $m$ -th homogeneous part  $\mathcal{R}_m^{(m)}$  over  $\mathcal{R}^{(m-1)}$ . Therefore, the restriction  $\mathcal{R}_i^{(m-1)} \rightarrow \mathcal{R}_i^{(m)}$  of  $\psi_{m,m-1}^*$  is surjective for  $i \leq m-1$ .

**Lemma 2.3** *Let  $X$  be a  $k$ -scheme. Under the notation above, let  $\mathcal{J}^{(m)} \subset \mathcal{R}^{(m)}$  be the  $\mathcal{R}^{(m)}$ -ideal defining  $\psi_{m,m-1}^{-1}(\sigma_{m-1}(X))$ , where  $\psi_{m,m-1}: X_m \rightarrow X_{m-1}$  is the truncation morphism and  $\sigma_{m-1}: X \rightarrow X_{m-1}$  is the canonical section of the projection  $\pi_{m-1}: X_{m-1} \rightarrow X$ . Let  $\mathcal{J}^{(m)} = \bigoplus_{i \geq 0} \mathcal{J}_i^{(m)}$  be the homogeneous decomposition. Then,  $\mathcal{J}^{(m)}$  is generated by  $\bigoplus_{i=1}^{m-1} \mathcal{R}_i^{(m-1)}$  and there exists an exact sequence of  $\mathcal{O}_X$ -modules:*

$$0 \longrightarrow \mathcal{J}_m^{(m)} \longrightarrow \mathcal{R}_m^{(m)} \longrightarrow \Omega_{X/k} \longrightarrow 0.$$

**PROOF.** The closed subscheme  $\sigma_{m-1}(X) \subset X_{m-1}$  is defined by the ideal  $\mathcal{I} = \bigoplus_{i \geq 1} \mathcal{R}_i^{(m-1)}$ . Here we note that  $\mathcal{I}$  is generated by  $\bigoplus_{i=1}^{m-1} \mathcal{R}_i^{(m-1)}$ , since  $\mathcal{R}^{(m-1)}$  is generated by  $\bigoplus_{i=1}^{m-1} \mathcal{R}_i^{(m-1)}$  as an  $\mathcal{O}_X$ -algebra. Therefore, we obtain that  $\mathcal{J}^{(m)}$  is generated by  $\bigoplus_{i=1}^{m-1} \mathcal{R}_i^{(m-1)}$ .

For the next statement, note that there is a canonical isomorphism

$$\psi_{m,m-1}^{-1}(\sigma(X)) \simeq \text{Spec } S(\Omega_{X/k}),$$

where  $S(\Omega_{X/k})$  is the symmetric  $\mathcal{O}_X$ -algebra defined by  $\Omega_{X/k}$  (see for example [5], where the proof is similar to that in [2, Example 2.5]). Then it follows an isomorphism  $\mathcal{R}^{(m)}/\mathcal{J}^{(m)} \simeq S(\Omega_{X/k})$  of graded  $\mathcal{O}_X$ -algebras. Here we note that the isomorphism sends the part of degree  $mi$  of  $\mathcal{R}^{(m)}/\mathcal{J}^{(m)}$  to the part of degree  $i$  of  $S(\Omega_{X/k})$ . By taking the generating parts of the both graded algebras, we have  $\mathcal{R}_m^{(m)}/\mathcal{J}_m^{(m)} \simeq \Omega_{X/k}$ .  $\square$

**Proposition 2.4** *Let  $X$  be an affine variety with  $\Omega_{X/k} \simeq \mathcal{O}_X^{\oplus n}$ . Then,  $X_m \simeq X \times \mathbb{A}_k^{mn}$  for every  $m \in \mathbb{N}$ .*

PROOF. As  $X_1 \simeq \text{Spec } S(\Omega_{X/k})$ , it follows that  $X_1 \simeq X \times \mathbb{A}_k^n$ . For  $m \geq 2$ , denote  $X_m$  by  $\text{Spec } \mathcal{R}^{(m)}$  and the  $\mathcal{R}^{(m)}$ -ideal defined in Lemma 2.3 by  $\mathcal{J}^{(m)}$ . Then, by Lemma 2.3, we have the exact sequence:

$$(1) \quad 0 \longrightarrow \mathcal{J}_2^{(2)} \xrightarrow{\iota} \mathcal{R}_2^{(2)} \xrightarrow{\psi} \Omega_{X/k} \longrightarrow 0.$$

Since  $\Omega_{X/k} \simeq \mathcal{O}_X^{\oplus n}$  and  $X$  is affine, there is a section  $s: \Omega_{X/k} \rightarrow \mathcal{R}_2^{(2)}$  of  $\psi$  and therefore the exact sequence (1) splits. Since  $X$  is a non-singular variety, the homomorphism  $\mathcal{R}_i^{(1)} \rightarrow \mathcal{R}_i^{(2)}$  ( $i = 0, 1$ ) is injective and therefore bijective by the note before Lemma 2.3, it follows that  $\mathcal{R}_i^{(2)} = \mathcal{R}_i^{(1)}$  for  $i = 0, 1$  with identifying by the bijection. Then, by Lemma 2.3

$$\mathcal{J}_2^{(2)} = \mathcal{R}_1^{(1)} \cdot \mathcal{R}_1^{(2)} + \mathcal{R}_2^{(1)} \cdot \mathcal{R}_0^{(2)} = \mathcal{R}_1^{(1)} \cdot \mathcal{R}_1^{(1)} + \mathcal{R}_2^{(1)} \cdot \mathcal{R}_0^{(1)} \subset \mathcal{R}^{(1)}.$$

Therefore, by the splitting exact sequence (1), an element of  $\mathcal{R}_2^{(2)}$  is the sum of an element of  $\mathcal{R}^{(1)}$  and an element of  $s(\Omega_{X/k})$  which is globally generated by  $n$  elements over  $\mathcal{O}_X$ . As  $\mathcal{R}^{(2)}$  is generated by  $\mathcal{R}_2^{(2)}$  over  $\mathcal{R}^{(1)}$ , we have a surjection

$$\mathcal{R}^{(1)}[\theta_1, \dots, \theta_n] \rightarrow \mathcal{R}^{(2)},$$

where  $\theta_1, \dots, \theta_n$  are indeterminates. Thus, we obtain a closed immersion  $X_2 \hookrightarrow X_1 \times \mathbb{A}^n$ . Considering the dimension of the both varieties, we have  $X_2 \simeq X_1 \times \mathbb{A}^n$ . For  $m > 2$ , similarly we obtain  $X_m \simeq X_{m-1} \times \mathbb{A}^n$ , which shows  $X_m \simeq X \times \mathbb{A}_k^{nm}$  for every  $m$ .  $\square$

**Remark 2.5** This proposition can be proved by using the fact that  $X_m$  has a  $X_1 \times_X X_{m-1}$ -torsor structure over  $X_{m-1}$ . But we prefer our present proof, because it is elementary.

**3. Isomorphism problems.** A  $G$ -equivariant isomorphism yields an isomorphism of the categorical quotients. Therefore, by Lemma 2.2 we obtain the following.

**Proposition 3.1** *Let  $X$  and  $Y$  be two schemes over  $k$ . If there exists a  $G$ -equivariant isomorphism  $X_m \xrightarrow{\sim} Y_m$  of  $m$ -jet schemes for some  $m \in \mathbb{N} \cup \{\infty\}$ , then there is an isomorphism  $X \xrightarrow{\sim} Y$ .*

The induced morphism  $f_m: X_m \rightarrow Y_m$  from a morphism  $f: X \rightarrow Y$  is  $G$ -equivariant. Therefore, by the previous proposition and the universality of the categorical quotient, we obtain the following.

**Corollary 3.2** *Let  $f: X \rightarrow Y$  be a morphism of schemes over  $k$ . If the induced morphism  $f_m: X_m \rightarrow Y_m$  is isomorphic for some  $m \in \mathbb{N} \cup \{\infty\}$ , then the morphism  $f$  is an isomorphism.*

**Remark 3.3** This corollary can be proved directly by using the fact that the morphism of the base spaces induces the morphism of the sections in the jet schemes.

Now let us be just given an isomorphism of  $m$ -jet schemes and we consider if it induces an isomorphism of base schemes. The following is a counterexample for this problem. We use the counterexample of the cancellation problem called Danielewski's example.

**Theorem 3.4** *Let  $X$  and  $Y$  be hypersurfaces in  $\mathbb{A}_{\mathbb{C}}^3$  defined by  $xz - y^2 + 1 = 0$  and  $x^2z - y^2 + 1 = 0$ , respectively. Then  $X \not\cong Y$  but  $X_m \simeq Y_m$  for every  $m \in \mathbb{N} \cup \{\infty\}$ .*

**PROOF.** By the work of Danielewski it is known that  $X, Y$  are non-singular,  $X \not\cong Y$  and  $X \times \mathbb{A}_{\mathbb{C}}^1 \simeq Y \times \mathbb{A}_{\mathbb{C}}^1$  (see for example, [6], [10]). Therefore, we have only to prove that  $X_m \simeq X \times \mathbb{A}_{\mathbb{C}}^{2m}$ ,  $Y_m \simeq Y \times \mathbb{A}_{\mathbb{C}}^{2m}$  for  $m < \infty$  and  $X_{\infty} \simeq Y_{\infty}$ . According to Danielewski's idea, define actions of the additive group  $\mathbb{A}_{\mathbb{C}}^1$  on  $X$  and  $Y$  as follows:

$$\begin{aligned} \mathbb{A}_{\mathbb{C}}^1 \times X &\longrightarrow X, & (t, x, y, z) &\mapsto (x, y + xt, z + 2yt + xt^2), \\ \mathbb{A}_{\mathbb{C}}^1 \times Y &\longrightarrow Y, & (t, x, y, z) &\mapsto (x, y + x^2t, z + 2yt + x^2t^2). \end{aligned}$$

Then the actions are free and the number of the orbits for fixed  $x \neq 0$  is one and for  $x = 0$  it is two for both actions. By this we have principal fiber bundles  $\varphi_1: X \rightarrow Z$ ,  $\varphi_2: Y \rightarrow Z$ , where  $Z = \mathbb{A}_{\mathbb{C}}^1 \cup \mathbb{A}_{\mathbb{C}}^1$  is the line with bug-eyes; *i.e.*, the union of the two lines  $\mathbb{A}_{\mathbb{C}}^1$  with patching by  $\text{id}: \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\} \simeq \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ . As the morphisms  $\varphi_i$  ( $i = 1, 2$ ) are smooth, we have surjections of tangent sheaves:

$$T_{X/\mathbb{C}} \longrightarrow \varphi_1^* T_{Z/\mathbb{C}}, \quad T_{Y/\mathbb{C}} \longrightarrow \varphi_2^* T_{Z/\mathbb{C}}.$$

Let  $L_i$  ( $i = 1, 2$ ) be their kernels, respectively. Since  $T_{Z/\mathbb{C}} \simeq \mathcal{O}_Z$  and  $X, Y$  are affine varieties, we have the splittings

$$T_{X/\mathbb{C}} \simeq \mathcal{O}_X \oplus L_1, \quad T_{Y/\mathbb{C}} \simeq \mathcal{O}_Y \oplus L_2.$$

By this, we obtain the canonical sheaves  $\omega_X \simeq L_1^{-1}$ ,  $\omega_Y \simeq L_2^{-1}$ . On the other hand, it is known that  $\omega_{\mathbb{A}_{\mathbb{C}}^3} \simeq \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^3}$ ,  $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^3}(X) \simeq \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^3}$  and  $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^3}(Y) \simeq \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^3}$ . Then, by the Adjunction Formula, we obtain

$$\omega_X \simeq \omega_{\mathbb{A}_{\mathbb{C}}^3}(X) \otimes \mathcal{O}_X \simeq \mathcal{O}_X, \quad \omega_Y \simeq \omega_{\mathbb{A}_{\mathbb{C}}^3}(Y) \otimes \mathcal{O}_Y \simeq \mathcal{O}_Y,$$

which shows that  $L_i$ 's are trivial. Hence, we have  $\Omega_{X/\mathbb{C}} \simeq \mathcal{O}_X^{\oplus 2}$  and  $\Omega_{Y/\mathbb{C}} \simeq \mathcal{O}_Y^{\oplus 2}$ . By Proposition 2.4, we have  $X_m \simeq X \times \mathbb{A}_{\mathbb{C}}^{2m}$ ,  $Y_m \simeq Y \times \mathbb{A}_{\mathbb{C}}^{2m}$  for every  $m \in \mathbb{N}$ . For the proof of  $X_{\infty} \simeq Y_{\infty}$ , we have only to take the projective limit of the isomorphisms  $X_m \simeq Y_m$  which are compatible with the truncation morphisms.  $\square$

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