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FREE SUBGROUPS OF THE GROUP OF FORMAL POWER SERIES AND THE CENTER PROBLEM FOR ODES

To my father, Professor Yuri Brudnyi, with love and gratitude.

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Presented by Pierre Milman, FRSC

Abstract. The paper belongs to the area related to the famous Poincaré center-focus problem and contains a new necessary and sufficient condition for existence of a center for ordinary differential equations with coefficients derived algebraically from a certain "basic" class. This class consists of families of equations $\frac{dv}{dx} = \sum_{j=1}^{\infty} a_j(x)\, v^{j+1}$ whose first return maps generate free subgroups of the group of formal power series. It is shown that such families form a sufficiently "massive" subset in the set of all possible equations as above. The paper contains various characterizations of this "basic" class. It follows the lines of the author's approach to the center-focus problem (involving modern algebraic techniques) that already deepened the understanding of the problem.

Résumé. Cet article porte sur le fameux problème du centre-foyer de Poincaré et contient une nouvelle condition nécessaire et suffisante pour l'existence d'un centre pour les équations diffèrentielles ordinaires avec des coéfficients derivés algébriquement d'une certaine classe de "base". Cette classe consiste en des familles d'équations de la forme $\frac{dv}{dx} = \sum_{j=1}^{\infty} a_j(x)\, v^{j+1}$ dont les premières fonctions de retour engendrent des sous groupes libres d'un groupe de séries entières formelles. On démontre que de telles familles forment un sous ensemble suffisamment "massif" dans l'ensemble de toutes les équations possible ci-dessus. L'article contient des diverses caractérisations de cette classe de "base". Il poursuit les directions de l'auteur sur le problème du centre-foyer (selon les techniques algébraiques modernes) qui ont déjà approfondies les connaissances du problème.

1. **Introduction.** Let $\mathbb{F}$ be either field $\mathbb{R}$ or $\mathbb{C}$ and $G_{\mathbb{F}}[\![r]\!]$ be the group of formal power series of the form $r + \sum_{j=1}^{\infty} c_j r^{j+1}$, $c_j \in \mathbb{F}$, $j \in \mathbb{N}$, with the multiplication $\circ$ defined by the composition of series. For an element $f \in G_{\mathbb{F}}[\![r]\!]$ let $c_j : G_{\mathbb{F}}[\![r]\!] \to \mathbb{F}$ be such that $c_j(f)$ is the $(j+1)$-st coefficient in the series expansion of f. We equip $G_{\mathbb{F}}[\![r]\!]$ with the weakest topology τ in which all c_j are continuous functions. Then $(G_{\mathbb{F}}[\![r]\!], \circ, \tau)$ is a cont ractible, residually torsion free

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nilpotent (*i.e.*, finite-dimensional unipotent representations separate the points of $G_{\mathbb{F}}[\![r]\!]$), separable topological group. The Lie algebra of $G_{\mathbb{F}}[\![r]\!]$ is isomorphic to $W_1(1)$, the nilpotent part of the Witt algebra of formal $\mathbb{F}$-valued vector fields on $\mathbb{R}$.

The group $G_{\mathbb{F}}[\![r]\!]$ arises naturally as the target space of the first return maps of ordinary differential equations of the form

$$\frac{dv}{dx} = \sum_{j=1}^{\infty} a_j(x)\, v^{j+1} \tag{1.1}$$

with all $a_j \in L^\infty_{\mathbb{F}}(I_T)$, the Banach space of bounded measurable $\mathbb{F}$-valued functions on the interval $I_T := [0, T]$ equipped with the supremum norm. Solving (1.1) by Picard iteration one obtains a solution $v\colon I_T \to G_{\mathbb{F}}[\![r]\!]$ such that $v(0) = r$ and each coefficient in the series expansion of v is a Lipschitz function on I_T. We say that (1.1) determines a *center* if $v(T) = r$. The center problem is: *given a sequence* $a_1, a_2, \dots$ *in* $L^\infty_{\mathbb{F}}(I_T)$ *to find out whether the corresponding equation* (1.1) *determines a center*. It appears in the qualitative theory of the ordinary differential equations created by Poincaré. In particular, the classical Poincaré Center-Focus problem for planar vector fields

$$\frac{dx}{dt} = -y + F(x, y), \quad \frac{dy}{dt} = x + G(x, y),$$

where F and G are real polynomials of a given degree without constant and linear terms can be reduced by passing to polar coordinates to the center problem for equations (1.1) whose coefficients are trigonometric polynomials depending polynomially on coefficients of pairs (F, G).

Let $\big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ be the set of coefficients $(a_1, a_2, \dots)$ with all $a_j \in L^\infty_{\mathbb{F}}(I_T)$ of equations (1.1). For $a \in \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ let $v(x; r; a)$, $x \in I_T$, be the solution of equation (1.1) corresponding to a with initial value $v(0; r; a) = r$. We set $P(a) := v(T; \cdot; a) \in G_{\mathbb{F}}[\![r]\!]$ (*i.e.*, $P(a)$ is the *first return map* of (1.1)).

The purpose of the paper is to investigate some properties of subsets of $\big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ subject to the following definition.

Definition 1.1. We say that a subset $S \subset \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ determines a trivial center if the family of first return maps $\{P(s)\}_{s\in S} \subset G_{\mathbb{F}}[\![r]\!]$ are generators of a free (nonabelian) subgroup of $G_{\mathbb{F}}[\![r]\!]$.

The class of such sets S will be denoted by $\mathcal{C}_t$. The basic example of a subset from this class is given in [Br2, Theorem 2.1].

Example 1.2. Assume that $S = \{s_1, s_2, \dots\}$ where $s_n = (s_{1n}, s_{2n}, \dots) \in \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ is such that $s_{kn} = 0$ for $k \neq n$ and $c_n := \frac{1}{T}\int_0^T s_{nn}(t)\, dt \neq 0$, $n \in \mathbb{N}$.

Then $S \in \mathcal{C}_t$. Indeed, one can easily show that

$$P(s_n)(r) = \frac{r}{\sqrt[n]{1 - nc_n r^n}}$$
$$= r + \sum_{j=1}^{\infty} \frac{(-1)^j \big(n(j-1)+1\big)\big(n(j-2)+1\big)\cdots 1}{j!}\, c_n^j\, r^{nj+1}.$$

The fact that $\{P(s_n)\}_{n\in\mathbb{N}}$ is the set of generators of a free subgroup of $G_{\mathbb{F}}[\![r]\!]$ follows from the deep result of Cohen [C].

2. **Formulation of main results.**

In this section we formulate some properties of subsets from $\mathcal{C}_t$.

2.1. Characteristic property of elements of $\mathcal{C}_t$. Let us equip $\big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ with operations $*$ and $^{-1}$ given for $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ by

$$a*b = (a_1*b_1, a_2*b_2, \dots) \in (L^\infty_{\mathbb{F}}(I_T))^\infty \text{ and } a^{-1} = (a_1^{-1}, a_2^{-1}, \dots) \in (L^\infty_{\mathbb{F}}(I_T))^\infty,$$

where for $i \in \mathbb{N}$

$$(a_i * b_i)(x) = \begin{cases} 2b_i(2x) & \text{if } 0 \le x \le T/2 \\ 2a_i(2x-T) & \text{if } T/2 < x \le T \end{cases}$$

and

$$a_i^{-1}(x) = -a_i(T-x), \quad 0 \le x \le T.$$

The operation $*$ is clearly nonassociative. For elements $q_1, \dots, q_n \in \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ we define $q_1 * \cdots * q_n$ using the recursive formula

$$q_1 * \cdots * q_n := (q_1 * \cdots * q_{n-1}) * q_n. \tag{2.1}$$

Let $S \subset \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ belong to the class $\mathcal{C}_t$. For $n \in \mathbb{N}$, $n \le \#S$ consider a word $w(X_1, \dots, X_n) := X_{j_1}^{k_1} \cdots X_{j_l}^{k_l}$ in free noncommutative variables $X_1, \dots, X_n$ where each $k_s \in \mathbb{Z} \setminus \{0\}$ and each $j_s \in \{1, \dots, n\}$. Choose pairwise distinct elements $s_1, \dots, s_n \in S$ and consider an element $w(s_1, \dots, s_n) \in \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ defined by the formula

$$w(s_1, \dots, s_n) := \underbrace{s_{j_1}^{\operatorname{sgn}(k_1)} * \cdots * s_{j_1}^{\operatorname{sgn}(k_1)}}_{|k_1| \text{ times}} * \cdots * \underbrace{s_{j_l}^{\operatorname{sgn}(k_l)} * \cdots * s_{j_l}^{\operatorname{sgn}(k_l)}}_{|k_l| \text{ times}}.$$

Let $f\colon I_T \to I_T$ be a Lipschitz map satisfying $f(0) = 0$ and $f(T) = T$. For $w(s_1, \dots, s_n) := (w_1, w_2, \dots) \in \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty$ we define

$$\big(w(s_1, \dots, s_n) \circ f\big) \cdot f' := \big((w_1 \circ f) \cdot f', (w_2 \circ f) \cdot f', \dots\big) \in \big(L^\infty_{\mathbb{F}}(I_T)\big)^\infty.$$

THEOREM 2.1. *Equation* (1.1) *with the vector of coefficients*

$$\big(w(s_1,\dots s_n)\circ f\big)\cdot f'$$

determines a center if and only if the word $w(X_1,\dots,X_n)$ *is reducible to the unit.*

In particular, the statement of the theorem implies that centers of equations with coefficients corresponding to $(w(s_1,\dots,s_n)\circ f)\cdot f'$, $s_1,\dots,s_n\in S$, are universal, *i.e.*, they are determined by vanishing of basic iterated integrals that in some cases is equivalent to a certain composition condition, see, *e.g.*, [Br1], [BFY], [FY], [Y]. It is worth remarking also that the order of brackets in the definition (2.1) is inessential for the statement of the theorem because the sets of elements of the form $(w(s_1,\dots,s_n)\circ f)\cdot f'$ for different choices of brackets in (2.1) coincide.

EXAMPLE 2.2. Consider Abel equations $\frac{dv}{dx}=c_1v^2+c_2v^3$ and $\frac{dv}{dx}=d_1v^2+d_2v^3$ on $I_1:=[0,1]$, where c_1,c_2,d_1,d_2 are algebraically independent over $\mathbb{Q}$ real numbers. It will be shown in Example 2.5 that the first return maps of these equations generate a free subgroup with two generators in $G_{\mathbb{R}}[\![r]\!]$. Consider a word $w(X_1,X_2):=X_{j_1}^{k_1}\cdots X_{j_l}^{k_l}$ in free noncommutative variables X_1,X_2 where each $k_s\in\mathbb{Z}\setminus\{0\}$ and each $j_s\in\{1,2\}$. We set $k:=|k_1|+\cdots+|k_l|$ and consider a vector-function (h_1,h_2) on $[0,k]$ defined as follows:

(∗) (h_1,h_2) is equal to $\operatorname{sgn}(k_m)\cdot(c_1,c_2)$ if $X_{j_m}=X_1$ or $\operatorname{sgn}(k_m)\cdot(d_1,d_2)$ if $X_{j_m}=X_2$ on the interval $\big[\sum_{0\le s\le m-1}|k_s|,\sum_{0\le s\le m}|k_s|\big)$, $1\le m\le l$; here $k_0:=0$.

Let $\phi\colon[0,1]\to[0,k]$ be a piecewise linear Lipschitz map such that $\phi(0)=0$ and $\phi(1)=k$. Consider Abel equation $\frac{dv}{dx}=(h_1\circ\phi)\cdot\phi'\cdot v^2+(h_2\circ\phi)\cdot\phi'\cdot v^3$ on I_1 (with piecewise constant coefficients). Then this equation determines a center if and only if the word $w(X_1,X_2)$ is reducible to the unit. Indeed, one can easily show that $((h_1\circ\phi)\cdot\phi',(h_2\circ\phi)\cdot\phi')$ can be written as $\big(w\big((c_1,c_2),(d_1,d_2)\big)\circ f\big)\cdot f'$ for a piecewise linear Lipschitz map $f\colon I_1\to I_1$ such that $f(0)=0$ and $f(1)=1$.

For instance, if $w(X_1,X_2)=X_1X_2X_1^{-1}X_2^{-1}$ and

$$\phi(x):=\begin{cases}4x & \text{if } 0\le x\le\frac14\\ 1 & \text{if } \frac14\le x\le\frac12\\ 12x-5 & \text{if } \frac12\le x\le\frac34\\ 4 & \text{if } \frac34\le x\le 1,\end{cases}$$

then we obtain equation $\frac{dv}{dx}=(h_1\circ\phi)\cdot\phi'\cdot v^2+(h_2\circ\phi)\cdot\phi'\cdot v^3$ on I_1 which does not determine a center; here for $i = 1, 2$

$$(h_i \circ \phi)(x) \cdot \phi'(x) := \begin{cases} 4c_i & \text{if } 0 \le x < \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} \le x < \frac{1}{2} \\ 12d_i & \text{if } \frac{1}{2} \le x < \frac{7}{12} \\ -12c_i & \text{if } \frac{7}{12} \le x < \frac{2}{3} \\ -12d_i & \text{if } \frac{2}{3} < x \le \frac{3}{4} \\ 0 & \text{if } \frac{3}{4} \le x \le 1. \end{cases}$$

2.2. Algebraic properties of elements of $\mathcal{C}_t$. Let $I \subset \mathbb{F}$ be a subset of algebraically independent over $\mathbb{Q}$ numbers. Consider a subset $C \subset (\mathbb{F} \setminus \{0\})^\infty$ such that

(a) For each $c = (c_1, c_2, \dots) \in C$ the set $I_c := \{c_j T\}_{j \in \mathbb{N}}$ consists of mutually distinct elements of I.
(b) The family of sets I_c, $c \in C$, forms a partition of I.

If I is a maximal subset of algebraically independent over $\mathbb{Q}$ numbers, then the set C is of the cardinality of the continuum. Also, each $c = (c_1, c_2, \dots) \in C$ is the vector of coefficients of the equation with constant coefficients

$$\frac{dv}{dx} = \sum_{j=1}^{\infty} c_j v^{j+1}.$$

THEOREM 2.3. *The set $C \subset \mathbb{F}^\infty \subset \left(L^\infty_{\mathbb{F}}(I_T)\right)^\infty$ belongs to $\mathcal{C}_t$.*

Next, we show how to construct new elements of the class $\mathcal{C}_t$ from the known ones.

Let $S \subset (L^\infty_{\mathbb{F}}(I_T))^\infty$ determine a trivial center and let $L_S \subset L^\infty_{\mathbb{F}}(I_T)$ consist of all possible coordinates of vectors from S. Consider a subset $K \subset L^\infty_{\mathbb{F}}(I_T)$ such that the set of all iterated integrals of the form

$$\int \cdots \int_{0 \le t_1 \le \cdots \le t_m \le T} a_m(t_m) \cdots a_1(t_1)\, dt_m \cdots dt_1$$

for all possible $a_1, \dots, a_m$ from the set $L_S \cup K$ and all $m \in \mathbb{N}$ belongs to a proper numerical subfield $\mathbb{E} \subset \mathbb{F}$. Let us fix a family $C \subset \mathbb{F}$ of algebraically independent over $\mathbb{E}$ numbers. We associate with each $s = (s_1, s_2, \dots) \in S$ a number $m(s) \in \mathbb{N}$, sequences of elements $k_i(s) = (k_{i1}(s), k_{i2}(s), \dots) \in (L^\infty_{\mathbb{F}}(I_T))^\infty$ with all $k_{ij}(s) \in K$, $1 \le i \le m(s)$, a sequence of numbers $c_1(s), c_2(s), \dots$ from C and sequences of polynomials $p_{i1}(s), p_{i2}(s), \dots$, $1 \le i \le m(s)$, with coefficients from $\mathbb{E}$ in variables $x_1, x_2, \dots$ without constant terms.

Now, we define $\widetilde{S} \subset (L^\infty_{\mathbb{F}}(I_T))^\infty$ as the set of elements $\widetilde{s} = (\widetilde{s}_1, \widetilde{s}_2, \dots)$ from $(L^\infty_{\mathbb{F}}(I_T))^\infty$ such that

$$\widetilde{s}_j := s_j + \sum_{i=1}^{m(s)} p_{ij}(s)(c_1(s), c_2(s), \dots) \cdot k_{ij}(s), \quad s = (s_1, s_2, \dots) \in S. \tag{2.2}$$

Theorem 2.4. *The set $\widetilde{S}$ determines a trivial center.*

Example 2.5. Consider Abel equations $\frac{dv}{dx} = c_1 v^2 + c_2 v^3$ and $\frac{dv}{dx} = d_1 v^2 + d_2 v^3$ on $I_1 := [0,1]$, where c_1, c_2, d_1, d_2 are algebraically independent over $\mathbb{Q}$ real numbers. Then the first return maps of these equations generate a free subgroup with two generators in $G_{\mathbb{R}}[\![r]\!]$. To obtain the required statement we choose in Theorem 2.4

$$S := \{(1,0,\dots),(0,1,0,\dots)\} \subset \mathbb{R}^\infty \subset \left(L^\infty_{\mathbb{R}}(I_T)\right)^\infty;$$

$$K := \{1\} \in L^\infty_{\mathbb{R}}(I_T), \quad k_j(s) = 1 \quad \text{for all } j \in \mathbb{N},\ s \in S;$$

$$C := \{c_1 - 1, c_2, d_1, d_2 - 1\}; \quad c_1((1,0,\dots)) := c_1 - 1,$$

$$c_2((1,0,\dots)) := c_2, c_1((0,1,0,\dots)) := d_1, c_2((0,1,0,\dots)) := d_2 - 1;$$

$$p_i((1,0,\dots))(x_1,x_2,\dots) := x_i, p_i((0,1,0,\dots))(x_1,x_2,\dots) := x_i,\ i = 1,2,$$

$$p_j(s) = 0 \quad \text{for all } j \geq 3,\ s \in S.$$

Similarly one obtains that the first return maps of Abel equations on $I_1 := [0,1]$,

$$\frac{dv}{dx} = p_1(x)v^2 + q_1(x)v^3 \quad \text{and} \quad \frac{dv}{dx} = p_2(x)v^2 + q_2(x)v^3$$

$$p_i(x) = \sum_{j=0}^{l} c_{ij}x^j \quad \text{and} \quad q_i(x) = \sum_{j=0}^{n} d_{ij}x^j, \quad i = 1,2,$$

such that all c_{ij}, d_{ij} are algebraically independent over $\mathbb{Q}$ real numbers, generate a free subgroup with two generators in $G_{\mathbb{R}}[\![r]\!]$.

2.3. Massivity of the class $\mathcal{C}_t$. In what follows β denotes one of the cardinal numbers $1, 2, 3, \dots, \aleph_0$. Let $N = \{n_1 < n_2 < n_3 < \cdots\} \subset \mathbb{N}$ be a subset of cardinality β. Consider the set of equations

$$\frac{dv}{dx} = \sum_{n \in N} c_n(x) v^{n+1}.$$

The coefficients $c = (c_1, c_2, \dots)$ are elements of $(L^\infty_{\mathbb{F}}(I_T))^\beta$, the direct product of β copies of $L^\infty_{\mathbb{F}}(I_T)$. We equip $(L^\infty_{\mathbb{F}}(I_T))^\beta$ with the product topology. Consider now the space $L^\beta_N := ((L^\infty_{\mathbb{F}}(I_T))^\beta)^\beta$ equipped with the product topology. Then L^β_N is a complete metrizable topological vector space.

A continuous function $p\colon L^\beta_N \to \mathbb{F}$ is called a *polynomial of degree d* if its restriction to any finite-dimensional subspace $L \subset L^\beta_N$ is a polynomial of degree at most d on L. For a subset $X \subset L^\beta_N$ we say that $V \subset X$ is an *algebraic subvariety* if it is defined as the set of common zeros in X of a system $\{p_\alpha\}_{\alpha \in \Lambda}$

of $\mathbb{F}$-valued polynomials on L_N^β. Note that if $X \subset L_N^\beta$ is a closed vector subspace, then each proper algebraic subvariety of X is closed and nowhere dense.

Let $M \subset (L_{\mathbb{F}}^\infty(I_T))^\beta$ be a subset containing vectors of coefficients of equations $\frac{dv}{dx} = c_n v^{n+1}$ with $\int_0^T c_n(x)\,dx \neq 0$ for all $n \in N$. By $M^\beta \subset L_N^\beta$ we denote the direct product of β copies of M.

THEOREM 2.6. *There exists a subset $V_M^\beta \subset M^\beta$ defined as countable union of proper algebraic subvarieties of M^β such that $M^\beta \setminus V_M^\beta \neq \varnothing$ and coordinates of vectors from $V_M^\beta \subset M^\beta$ form subsets of $(L_{\mathbb{F}}^\infty(I_T))^\beta$ determining trivial centers.*

If M is a closed vector subspace of $(L_{\mathbb{F}}^\infty(I_T))^\beta$, then M^β is a complete metrizable space and from Theorem 2.6 and the Baire theorem we obtain that for any open subset $U \subset M^\beta$ the set of points from U whose coordinates form subsets from the class $\mathcal{C}_t$ cannot be presented as countable union of nowhere dense subsets of U. In particular, if $\beta < \infty$ and M is a finite-dimensional vector subspace of $(L_{\mathbb{F}}^\infty(I_T))^\beta$ satisfying conditions of the theorem, then for almost each point $s = (s_1, \ldots, s_\beta)$ of M^β (with respect to the Lebesgue measure on M^β defined by an identification of M^β with some $\mathbb{R}^d$) the set $\{s_1, \ldots, s_\beta\} \subset (L_{\mathbb{F}}^\infty(I_T))^\infty$ determines a trivial center.

EXAMPLE 2.7. Consider the set of Abel equations $\frac{dv}{dx} = p(x)v^2 + q(x)v^3$ on $I_{2\pi}$ where p and q are elements of the spaces of real trigonometric polynomials of degrees $d(p)$ and $d(q)$, respectively. The space of pairs of such equations is parametrized by $\mathbb{R}^d \times \mathbb{R}^d$, $d := 2d(p) + 2d(q) + 2$, the vectors of coefficients of trigonometric polynomials in these equations. Then for almost each point $(v_1, v_2) \in \mathbb{R}^d \times \mathbb{R}^d$ the corresponding pair of Abel equations determines a trivial center.

3. Proofs.

Proof of Theorem 2.1. For the word $w(X_1, \ldots, X_n)$ we consider equation (1.1) with the set of coefficients $w(s_1, \ldots, s_n) = (w_1, w_2, \ldots) \in (L_{\mathbb{F}}^\infty(I_T))^\infty$. Let $v(x; r; w(s_1, \ldots, s_n))$ be the solution of this equation satisfying

$$v(0; r; w(s_1, \ldots, s_n)) = r.$$

By the definition of $w(s_1, \ldots, s_n)$ we have

$$P\big(w(s_1, \ldots, s_n)\big) = \underbrace{P(s_{j_1})^{\mathrm{sgn}(k_1)} \circ \cdots \circ P(s_{j_1})^{\mathrm{sgn}(k_1)}}_{|k_1|\text{times}} \circ \cdots$$
$$\cdots \circ \underbrace{P(s_{j_l})^{\mathrm{sgn}(k_l)} \circ \cdots \circ P(s_{j_l})^{\mathrm{sgn}(k_l)}}_{|k_l|\text{times}}.$$

Since $S \in \mathcal{C}_t$, the elements $P(s_1), \ldots, P(s_n)$ are generators of a free subgroup of $G_{\mathbb{F}}[\![r]\!]$. In particular, $P(w(s_1, \ldots, s_n)) = 1$, the unit of $G_{\mathbb{F}}[\![r]\!]$, if and only if the word $w(X_1, \ldots, X_n)$ is reducible to the unit.

Further, for equation (1.1) with the vector of coefficients $(w(s_1,\dots,s_n)\circ f)\cdot f'$ we have

$$v\big(x;r;\big(w(s_1,\dots,s_n)\circ f\big)\cdot f'\big)=v\big(f(x);r;w(s_1,\dots,s_n)\big).$$

Since $f(T)=T$, from the previous identity we obtain

$$P\big(\big(w(s_1,\dots,s_n)\circ f\big)\cdot f'\big)=P\big(w(s_1,\dots,s_n)\big).$$

This identity and the described above property of $P\big(w(s_1,\dots,s_n)\big)$ give the required statement. □

Proof of Theorem 2.3. We first recall a formula for the first return map $P(a)$ of equation (1.1) with coefficients $a=(a_1,a_2,\dots)\in(L^\infty_{\mathbb{F}}(I_T))^\infty$, see [Br1]:

$$P(a)=r+\sum_{i=1}^{\infty}\Big(\sum_{i_1+\cdots+i_k=i}c_{i_1,\dots,i_k}(i)\cdot I_{i_1,\dots,i_k}(a)\Big)r^{i+1}, \tag{3.1}$$

where

$$c_{s_{i_1},\dots,s_{i_k}}(i):=(i-i_1+1)(i-i_1-i_2+1)(i-i_1-i_2-i_3+1)\cdots 1 \tag{3.2}$$

and

$$I_{i_1,\dots,i_k}(a):=\int\cdots\int_{0\le t_1\le\cdots\le t_k\le T}a_{i_k}(t_k)\cdots a_{i_1}(t_1)\,dt_k\cdots dt_1. \tag{3.3}$$

Let $w(X_1,\dots,X_n)$ be a word in free noncommutative variables $X_1,\dots,X_n$ and let $v_1,\dots,v_n\in\mathbb{F}^\infty\subset(L^\infty_{\mathbb{F}}(I_T))^\infty$. According to formulas (3.1), (3.2), (3.3), and the formulas for operations $\circ$ and $^{-1}$ in $G_{\mathbb{F}}[\![r]\!]$,

$$w\big(P(v_1),\dots,P(v_n)\big)(r)=r+\sum_{j=1}^{\infty}p_j(v_1T,\dots,v_nT)r^{j+1},$$

where p_j are certain polynomials with rational coefficients in coordinates of the vectors $v_1T,\dots,v_nT$. Now, if $c_1,\dots,c_n$ are mutually distinct elements of the set C, then due to properties (a) and (b) all coordinates of vectors $c_1T,\dots,c_nT$ are algebraically independent over $\mathbb{Q}$. This implies that if

$$w\big(P(c_1),\dots,P(c_n)\big)(r)=r,$$

then all polynomials p_j are identically zeros. In particular,

$$w\big(P(\delta_1),\dots,P(\delta_n)\big)(r)=r$$

where $\delta_k := (0, \dots, 0, 1, 0, \dots)$ with 1 at the k-th place, $k \in \mathbb{N}$. But

$$P(\delta_1), \dots, P(\delta_n)$$

are generators of a free subgroup of $G_{\mathbb{F}}[\![r]\!]$, see Example 1.2. Hence,

$$w\big(P(\delta_1), \dots, P(\delta_n)\big)(r) = r$$

if and only if the word $w(X_1, \dots, X_n)$ is reducible to the unit. From here we obtain that $P(c_1), \dots, P(c_n)$ are generators of a free subgroup of $G_{\mathbb{F}}[\![r]\!]$ for all subsets of mutually distinct elements $c_1, \dots, c_n \in C$, $n \in \mathbb{N}$. Thus $C \in \mathcal{C}_t$. □

Proof of Theorem 2.4. Let $w(X_1, \dots, X_n)$ be a word in free noncommutative variables $X_1, \dots, X_n$, $n \in \mathbb{N}$, $n \leq \#S$. Consider mutually distinct elements $\widetilde{s}_1, \dots, \widetilde{s}_n \in \widetilde{S}$. Then from formulas (3.1)–(3.3) and (2.2) we obtain that

$$w\big(P(\widetilde{s}_1), \dots, P(\widetilde{s}_n)\big)(r) = r + \sum_{j=1}^{\infty} q_j(\widetilde{s}_1, \dots, \widetilde{s}_n) r^{j+1},$$

where q_j are certain polynomials with coefficients from the field $\mathbb{E}$ in mutually distinct coordinates of vectors $(c_1(s_j), c_2(s_j), \dots)$, $1 \leq j \leq n$. By the definition these coordinates are algebraically independent over $\mathbb{E}$. Hence, if $w(P(\widetilde{s}_1), \dots, P(\widetilde{s}_n))(r) = r$, then all q_j are identically zeros. In particular, the constant terms q_{0j} of polynomials q_j are zeros. But according to our definitions

$$w\big(P(s_1), \dots, P(s_n)\big)(r) = r + \sum_{j=1}^{\infty} q_{0j} r^{j+1}.$$

Since $S \in \mathcal{C}_t$, the latter implies that $w\big(P(\widetilde{s}_1), \dots, P(\widetilde{s}_n)\big)(r) = r$ if and only if the word $w(X_1, \dots, X_n)$ is reducible to the unit. This shows that $P(\widetilde{s}_1), \dots, P(\widetilde{s}_n)$ are generators of a free subgroup of $G_{\mathbb{F}}[\![r]\!]$.

The proof of the theorem is complete. □

Proof of Theorem 2.6. We will prove the theorem for $\beta = \aleph_0$ only. In this case we write ∞ instead of $\aleph_0$. Proofs for other values of β are similar.

Consider a map $P_N \colon L_N^\infty \to (G_{\mathbb{F}}[\![r]\!])^\infty$, $L_N^\infty := \big((L_{\mathbb{F}}^\infty(I_T))^\infty\big)^\infty$, defined by the formula

$$P_N(\widehat{a}) := \big(P(\hat{a}_1), P(\hat{a}_2), \dots\big) \quad \text{for } \widehat{a} = (\hat{a}_1, \hat{a}_2, \dots) \in L_N^\infty.$$

We equip $(G_{\mathbb{F}}[\![r]\!])^\infty$ with the product topology. Then by the definitions of the map P and of topologies on L_N^∞ and $(G_{\mathbb{F}}[\![r]\!])^\infty$, the map P_N is continuous. Moreover, according to the definition of M^∞ and Example 1.2 the image of the restriction $P_{M^\infty} := P_N|_{M^\infty} \colon M^\infty \to (G_{\mathbb{F}}[\![r]\!])^\infty$ contains a point

$g = (g_1, g_2, \dots) \in (G_{\mathbb{F}}[\![r]\!])^\infty$ such that $g_1, g_2, \dots$ are generators of a free subgroup of $G_{\mathbb{F}}[\![r]\!]$ with countable number of generators.

Let $w(X_1, \dots, X_n)$ be an irreducible word in free noncommutative variables $X_1, \dots, X_n$. We associate with w a continuous map $\widehat{w}\colon (G_{\mathbb{F}}[\![r]\!])^\infty \to G_{\mathbb{F}}[\![r]\!]$ given for $f = (f_1, f_2, \dots) \in (G_{\mathbb{F}}[\![r]\!])^\infty$ by the formula

$$\widehat{w}(f) := w(f_1, \dots, f_n) = r + \sum_{j=1}^{\infty} w_j(f_1, \dots, f_n) r^{j+1}.$$

Due to formulas (3.1)–(3.3) the function $\widehat{w} \circ P_{M^\infty} : M^\infty \to G_{\mathbb{F}}[\![r]\!]$ is continuous and such that $\widehat{w} \circ P_{M^\infty} = r + \sum_{j=1}^{\infty} \hat{w}_j r^{j+1}$ where each $\hat{w}_j$ is the restriction to M^∞ of a polynomial on L_N^∞. Also, $\hat{w}_j = w_j \circ P_{M^\infty}$.

Let $\mu(w) \in \mathbb{N}$ be the minimal number such that $w_{\mu(w)}(g_1, \dots, g_n) \neq 0$. Consider proper algebraic subvarieties $\widehat{N}_w := P_{M^\infty}^{-1}(N_w) \subset M^\infty$ where N_w is defined as the set of common zeros on $(G_{\mathbb{F}}[\![r]\!])^\infty$ of polynomials $w_1, \dots, w_{\mu(w)}$ (*i.e.*, $\widehat{N}_w$ is defined as the set of common zeros of polynomials $\hat{w}_1, \dots, \hat{w}_{\mu(w)}$). Properness of these varieties follows from definitions of numbers $\mu(w)$. Let $\mathcal{W}_\infty$ be the set of all irreducible words in variables $X_1, X_2, \dots$. We set

$$V_M^\infty := \bigcup_{w \in \mathcal{W}_\infty} \widehat{N}_w.$$

Then the sets of coordinates of vectors of elements from $M^\infty \setminus V_M^\infty$ determine trivial centers. In fact, for $\widehat{a} = (\hat{a}_1, \hat{a}_2, \dots) \in M^\infty \setminus V_M^\infty$ we have $P_{M^\infty}(\widehat{a}) \notin \bigcup_{w \in \mathcal{W}_\infty} N_w$. This implies that the subgroup of $G_{\mathbb{F}}[\![r]\!]$ generated by $P(\hat{a}_1), P(\hat{a}_2), \dots$ is free, *i.e.*, $\widehat{a} \in \mathcal{C}_t$. Clearly, $M^\infty \setminus V_M^\infty \neq \varnothing$ (because the image of P_{M^∞} contains the point g). □

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