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CLASSIFICATION OF COMPACT HOMOGENEOUS MANIFOLDS WITH PSEUDO-KÄHLERIAN STRUCTURES

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Abstract. In this note we apply a modification theorem for compact homogeneous solvmanifolds to compact complex homogeneous manifolds with pseudo-Kählerian structures. We are then finally able to classify these compact pseudo-Kählerian manifolds as certain products of projective rational homogeneous spaces, tori, and simple and double reduced primitive pseudo-Kähler spaces.

Résumé. Dans cette note, nous appliquons un théorème de modification pour des "solv-variétés" compactes et homogènes aux variétés compactes complexes equipées d'une structure pseudo-kählérienne. Nous obtenons une classification de ces variétés compactes pseudo-kählériennes sous la forme de certains produits d'espaces projectifs rationnels et homogènes, de tores, et d'espaces pseudo-kählériens réduits et primitifs simples ou doubles.

Compact complex homogeneous spaces with invariant Kähler structures were classified by Matsushima [Mt]. Compact complex homogeneous spaces with Kähler structures (not necessary invariant) were classified by Borel and Remmert [BR]. Compact complex homogeneous spaces with invariant pseudo-Kählerian structures were classified by Dorfmeister and Guan [DG]. In this note, I shall give a complete classification of compact complex homogeneous manifolds with (not necessarily invariant) pseudo-kählerian structures.

I previously reduced the classification to the case of a solvable parallelizable action in [Gu4] (see the Main Theorem 3 therein), following the plan proposed in [Gu1] and [Gu2].

An application of the modification argument in [Gu4, Corollary 1] is the so-called *complex parallelizable right invariant pseudo Kählerian algebra*. This is what has made the present classification possible.

Moreover, at the group level, I proved [Gu5] that the given group G can be decomposed as $G = AN$ with two abelian subgroups A and N (for partial results, see also [DG, Corollary 1, p. 509], [Gu4, Corollary 2], [Ym2], and the action of A on N is an algebraic group action of a product of $\mathbf{C}^*$s.

If the Lie algebra of A acts on N with only one pair of eigenvalue functions k_1 and $k_2 = -k_1$, then we will call the given compact complex parallelizable

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homogeneous manifold with a pseudo-Kählerian structure a *primary pseudo-Kähler manifold*.

THEOREM 1. *Every compact complex parallelizable homogeneous manifold with a pseudo-Kählerian structure is a pseudo-Kählerian torus bundle over a pseudo-Kählerian torus and, up to a finite covering of the fiber, is a torus bundle which is the bundle product of several primary pseudo-Kählerian manifolds.*

The bundle product in Theorem 1 is the fiberwise product. More details concerning the statement of this result can be found in [Gu5].

Let us call a primary pseudo-Kählerian manifold a *reduced primary pseudo-Kählerian manifold* if the action of A on N is effective. Notice that we may always change a given invariant form ω_1 on A to any other such form. As a consequence we have the following.

THEOREM 2. *After modifying* ω_1*, if necessary, any primary pseudo-Kählerian manifold, up to a finite covering, is the product of a torus and a reduced primary pseudo-Kählerian manifold. Moreover,* $\dim_{\mathbf{C}} A = 1$ *and* $\dim_{\mathbf{C}} N = 2m$ *with* m *the complex dimension of the eigenspaces for a reduced primary pseudo-Kählerian manifold. In particular, the index of the pseudo-Kählerian structure for a reduced primary pseudo-Kähler space is either* 1 *or* -1.

For any reduced primary pseudo-Kähler space, we have $k_1(z) = z$ and $k_2(z) = -z$. The fiber torus up to a finite covering can be split into complex irreducible ones with respect to the A action. For a primary pseudo-Kähler space, if the fiber is also an irreducible complex torus with respect to the A action, then we will call it a *primitive pseudo-Kähler space*. If the A action is also effective, then we will call it a *reduced primitive pseudo-Kähler space*. By changing ω_1 on A we can always obtain any primary pseudo-Kähler space, up to a finite covering, from a torus bundle product of primitive ones.

For any reduced primitive pseudo-Kählerian space, the rational module generated by the discrete subgroup $N_{\mathbf{Z}}$ of N, as a rational representation of $\mathbf{F} = \Gamma N/N$, can be split into two-dimensional representations, but in this case m can be any positive integer.

In general, set $A_{\mathbf{F}} = \Gamma N/N$ and $T = A/A_{\mathbf{F}}$.

THEOREM 3. *Any compact complex parallelizable homogeneous space with a pseudo-Kähler structure is a pseudo-Kählerian torus bundle over a pseudo-Kählerian torus* T *and, up to a finite covering of the fiber, is a torus bundle over* T *which is the bundle product of several primitive spaces. Moreover, any primitive space is, up to a finite covering, a product of a torus and a reduced primitive space. For a primitive pseudo-Kählerian manifold,* m *can be any given positive integer.*

Let us call a primitive pseudo-Kähler space a *simple primitive pseudo-Kähler space* if $N/N_{\mathbf{F}}$ is a simple complex torus. Let us call a primitive pseudo-Kähler space a *double primitive pseudo-Kähler space* if $N/N_{\mathbf{F}}$ is isogenous to the product of two identical complex tori. The example in [Ym1] is a double reduced primitive pseudo-Kählerian space.

THEOREM 4. *Let M be a reduced primitive pseudo-Kähler space. If M is not a simple primitive pseudo-Kähler space as defined above, then it is a double reduced primitive pseudo-Kähler space.*

THEOREM 5. *Within the class of reduced primitive pseudo-Kähler spaces, the simple ones are generic.*

Combining these results with our splitting theorem [Gu4, Main Theorem 3] and our results in [Gu2], we have the following.

THEOREM 6. *Every compact complex homogeneous manifold with a pseudo-Kählerian structure is the product of a projective rational homogeneous space and a solvable compact complex parallelizable pseudo-Kähler space. In particular, any compact complex parallelizable pseudo-Kähler space is solvable. Moreover, any compact complex parallelizable pseudo-Kähler space is a pseudo-Kählerian torus bundle over a pseudo-Kählerian torus T and, up to a finite covering of the fiber, is a torus bundle over T which is the bundle product over T of several simple and double primitive pseudo-Kähler spaces.*

Background remarks. A possible classification of compact complex homogeneous spaces with pseudo-Kählerian structures was proposed in [Gu1] and [Gu2].[1] However, the classification turns out to be much more complicated than suggested therein [Gu4], [Ha], [Ym1], [Ym2]. I received a reprint of [Ym2] only after submitting [Gu4], and a preprint of [Ha] only after completing [Gu5].[2] In this paper, I provided the last piece of this puzzle, thereby completely solving this problem. What was missing in [Gu1] and [Gu2] was the calculation of the cohomology group of a compact solvmanifold. That problem was resolved in [Gu4], using techniques in [Gu3]. The complete proof, which is quite technical, is given in [Gu5].

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