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DIOPHANTINE INEQUALITY FOR EQUICHARACTERISTIC EXCELLENT HENSELIAN LOCAL DOMAINS

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Abstract. G. Rond has proved a Diophantine type inequality for the field of quotients of the convergent or formal power series ring in multivariables. We generalize his theorem to the field of the quotients of an excellent Henselian local domain in equicharacteristic case.

Résumé. G. Rond a démontré une inégalité de type diophantien pour le corps des quotients de séries convergentes (ou formelles) à plusieurs variables. On fait ici une généralisation de son théorème au corps des quotients d'un anneau local intégral henselien excellent dans le cas équicaractéristique.

1. **Introduction.** An important topic of Diophantine approximation is the problem of approximation of a real algebraic number by rational ones. The crucial result is Roth's theorem:

If $z \in \mathbb{R} \setminus \mathbb{Q}$ is an algebraic number,

$$\forall \epsilon > 0\, \exists c(z,\epsilon) > 0\, \forall x \in \mathbb{Z}\, \forall y \in \mathbb{Z}^* : \left| z - \frac{x}{y} \right| > c(z,\epsilon)|y|^{-2-\epsilon}.$$

There are quite similar results for the Laurent series field in a single variable (*cf.* [L]). It is also known that there are deep analogous results on the global function fields on certain special varieties in connection with Nevanlinna's theory (*cf.* [Ru]).

Rond [Ro2] obtained a Diophantine inequality for the field of quotients of the convergent or formal power series ring in multivariables in connection with the *linear Artin approximation property* (Spivakovsky, *cf.* [Ro1]). He used the *product inequality* [Iz1] for the order function ν on an analytic integral domain.

In this paper we assert that Diophantine inequality holds for the field of quotients of an equicharacteristic excellent Henselian local domain. For the proof, we need *Rees's inequality* [Re4] *for* $\mathfrak{m}$*-valuations* on complete local rings, a variant of the product inequality. To be precise, we use its generalization to excellent domains by Hübl–Swanson [HS].

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An inequality on the order function was once used for zero-estimate of elements transcendental over the polynomial ring generated by a parameter system in a local ring in [Iz2]. This time we are concerned with elements algebraic over a local ring.

Let us give a precise description of our theorem. Let K be a (commutative) field. We call a mapping $\boldsymbol{v}\colon K \longrightarrow \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$) a *valuation* when it satisfies the following:
(1) $\boldsymbol{v}(xy) = \boldsymbol{v}(x) + \boldsymbol{v}(y)$,
(2) $\boldsymbol{v}(x+y) \geq \min\{\boldsymbol{v}(x), \boldsymbol{v}(y)\}$,
(3) $\boldsymbol{v}(0) = \infty$.
We can define the absolute value $|\cdot|_{\boldsymbol{v}}\colon K \longrightarrow \mathbb{R}$ by $|w|_{\boldsymbol{v}} := \exp\bigl(-\boldsymbol{v}(w)\bigr)$ ($|0|_{\boldsymbol{v}} := 0$). Then K is a metric space defined by the absolute value of the difference. This defines a topology compatible with the field operations. We endow $\mathbb{R}$ with the discrete topology. Then $\boldsymbol{v}\colon K^* \longrightarrow \mathbb{R}$ ($K^* := K \setminus 0$) is continuous. We put:

$V_{\boldsymbol{v}} := \{z \in K : \boldsymbol{v}(z) \geq 0\}$: the *valuation ring* of $\boldsymbol{v}$,
$\mathfrak{m}_{\boldsymbol{v}} := \{z \in K : \boldsymbol{v}(z) > 0\}$: the *maximal ideal* of $V_{\boldsymbol{v}}$,
$k_{\boldsymbol{v}} := V_{\boldsymbol{v}}/\mathfrak{m}_{\boldsymbol{v}}$: the *residue field* of $\boldsymbol{v}$,
$\hat{K}$: the completion of K with respect to $\boldsymbol{v}$, which has a natural structure of a field,
$\hat{\boldsymbol{v}}$: the continuous extension of $\boldsymbol{v}$ to $\hat{K}$, which is a valuation on $\hat{K}$,
$\hat{V}_{\boldsymbol{v}}$: the valuation ring of the extension $\hat{\boldsymbol{v}}$.

A valuation is called a *discrete valuation* if the value group $\boldsymbol{v}(K^*)$ is isomorphic to $\mathbb{Z}$ as an ordered group. In this case the valuation ring $V_{\boldsymbol{v}}$ is a discrete valuation ring (DVR) and $\boldsymbol{v}$ coincides with the $\mathfrak{m}_{\boldsymbol{v}}$-adic order on $V_{\boldsymbol{v}}$ and we have $K = Q(V_{\boldsymbol{v}})$. The completion $\hat{B}$ of some subset $B \subset K$ can be identified with its closure in $\hat{K}$. Our main result is the following.

Let $(A,\ \mathfrak{m})$ *be an equicharacteristic analytically irreducible excellent Henselian local domain and* $\boldsymbol{v}$ *an* $\mathfrak{m}$*-valuation (defined in the next section) on the field* $K := Q(A)$ *of quotients of* A*. If* $z \in \hat{K} \setminus K$ *is algebraic over* K*, then we have the following:*

$$\exists a > 0\, \exists c > 0\, \forall x \in A\, \forall y \in A^* : \left|z - \frac{x}{y}\right|_{\hat{\boldsymbol{v}}} > c|y|_{\boldsymbol{v}}^{a}.$$

Note that $\hat{K}$ is not generally the field quotients of $\hat{A}$ (*cf.* [Ro1, 2.4]). The essential point of the proof is reducing inequality on the valuation to inequality on the maximal-ideal-adic order in the same way as in [Ro2, Section 4, (v)]. In our general case, we need Rees's valuation theorem [Re2], [Re3] to connect valuations to the order.

Contrary to the case of algebraic numbers, the exponent on the right of this inequality is not uniformly bounded. Rond [Ro1, 2.4] has given a sequence of elements $z_i \in \hat{K}$ of degree 2 over K with unbounded exponents.

2. $\mathfrak{m}$-valuations on local domains. Let $(A, \mathfrak{m})$ be a local domain whose field of quotients $Q(A)$ is K. Let $k := A/\mathfrak{m}$ denote the residue field. A valuation $\boldsymbol{v}$ on K is called an $\mathfrak{m}$-*valuation*, if it satisfies the following:
(a) $x \in A \Longrightarrow \boldsymbol{v}(x) \geq 0$,
(b) $x \in \mathfrak{m} \Longrightarrow \boldsymbol{v}(x) > 0$,
(c) $\mathrm{trdeg}_k\, k_{\boldsymbol{v}} = \dim A - 1$,
(d) the value group $\boldsymbol{v}(K^*)$ is isomorphic to $\mathbb{Z}$ (as an ordered group).

Let us recall some key facts on valuations which are used in the proof. The first one is Rees's strong valuation theorem [Re3]. We state only the special case which we need later. We define the $\mathfrak{m}$-*adic order* $\nu_{\mathfrak{m}} \colon A \longrightarrow \overline{\mathbb{R}}$ on A by $\nu_{\mathfrak{m}}(f) := \max\{p : f \in \mathfrak{m}^p\}$. This is not necessarily a valuation. It satisfies formulae
(1′) $\nu_{\mathfrak{m}}(fg) \geq \nu_{\mathfrak{m}}(f) + \nu_{\mathfrak{m}}(g)$,
(2) $\nu_{\mathfrak{m}}(f+g) \geq \min\{\nu_{\mathfrak{m}}(f), \nu_{\mathfrak{m}}(g)\}$,
(3′) $\nu_{\mathfrak{m}}(0) = \infty$, $\nu_{\mathfrak{m}}(1) = 0$.

Let us stabilize $\nu_{\mathfrak{m}}$ by Samuel's idea: $\overline{\nu}_{\mathfrak{m}}(f) := \lim_{k\to\infty} \nu_{\mathfrak{m}}(f^k)/k$. This limit always exists and satisfies formulae (1′), (2), (3′) and the homogeneity formula
(4) $\overline{\nu}_{\mathfrak{m}}(f^n) = n\overline{\nu}_{\mathfrak{m}}(f)$ $(n \in \mathbb{N})$
also (see [Re1]). The following is Rees's *Strong valuation theorem*.

FACT 2.1 ([Re2, Re3]). Let $(A, \mathfrak{m})$ be a Noetherian local ring whose $\mathfrak{m}$-adic completion is reduced (has no non-zero nilpotent element). Then there exist a non-negative number C and a set of valuations $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ on K with the value group $\mathbb{Z}$ such that

$$\forall x \in A : \nu_{\mathfrak{m}}(x) \leq \overline{\nu}_{\mathfrak{m}}(x) \leq \nu_{\mathfrak{m}}(x) + C,$$

$$\forall x \in A : \overline{\nu}_{\mathfrak{m}}(x) = \min\{r_1\boldsymbol{v}_1(x), \ldots, r_p\boldsymbol{v}_p(x)\} \quad (r_i := 1/\min\{\boldsymbol{v}_i(y) : y \in \mathfrak{m}\}).$$

The set $\{\boldsymbol{v}_1(x), \ldots, \boldsymbol{v}_p(x)\}$ is unique, if it is taken to be irredundant.

We call the irredundant valuations $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ the *valuations associated with* $\mathfrak{m}$. We call a local ring *analytically irreducible* when its $\mathfrak{m}$-adic completion is an integral domain. Rees proves the following:

FACT 2.2 ([Re1, 5.9]). Let $(A, \mathfrak{m})$ be an equicharacteristic analytically irreducible local domain. Then the valuations associated with $\mathfrak{m}$ are all $\mathfrak{m}$-valuations.

In the proof of the regular analytic case, Rond [Ro2] uses the product inequality [Iz1] for analytic domain. Rees generalises this inequality and, in the complete domain case, gives a valuation theoretic form [Re4, E]. Hübl and Swanson generalise the latter to excellent domains as follows:

FACT 2.3 ([HS, 1.3]). Let $(A, \mathfrak{m})$ be an analytically irreducible excellent local domain. Then for any pair of $\mathfrak{m}$-valuations $\boldsymbol{v}$ and $\boldsymbol{v}'$, we have the following.

$$\exists d > 0\, \forall x \in A : \boldsymbol{v}(x) \leq d\boldsymbol{v}'(x).$$

The constant d can be chosen independently of $\boldsymbol{v}'$.

Combining these facts, we see the following:

FACT 2.4. Let $(A, \mathfrak{m})$ be an equicharacteristic analytically irreducible excellent local domain and let $\boldsymbol{v}$ be an $\mathfrak{m}$-valuation on A. Then we have:

$$\exists C > 0\, \exists s > 0\, \exists t > 0\, \forall x \in A : s\boldsymbol{v}(x) \leq \overline{\nu}_{\mathfrak{m}}(x) \leq \nu_{\mathfrak{m}}(x) + C \leq t\boldsymbol{v}(x) + C.$$

3. Main theorem. With the notation in the introduction, our main theorem is the following.

THEOREM 3.1. *Let $(A, \mathfrak{m})$ be an equicharacteristic analytically irreducible excellent Henselian local domain and let $K := Q(A)$ denote its field of quotients and let $\boldsymbol{v}\colon K \longrightarrow \overline{\mathbb{R}}$ be an $\mathfrak{m}$-valuation. If $z \in \hat{K} \setminus K$ is algebraic over K, then we have*

$$\exists a > 0\, \exists c > 0\, \forall x \in A\, \forall y \in A^* : \left| z - \frac{x}{y} \right|_{\hat{\boldsymbol{v}}} > c|y|_{\boldsymbol{v}}^{a}.$$

Just in the same way as Rond [Ro2, 3.1] (see also [Ro1, 2.1]), our Theorem 3.1 implies the following.

COROLLARY 3.2. *Let $(A, \mathfrak{m})$ be an equicharacteristic analytically irreducible excellent Henselian domain and let $P(X, Y) \in A[X, Y]$ be a homogeneous polynomial. Then the Artin function of $P(X, Y)$ is majorised by an affine function,* i.e.*,*

$$\begin{aligned} &\exists \alpha\, \exists \beta\, \forall x \in A\, \forall y \in A : \nu_{\mathfrak{m}}\big(P(x, y)\big) \geq \alpha i + \beta \\ &\qquad \Longrightarrow \exists \overline{x} \in A\ \exists \overline{y} \in A : \nu_{\mathfrak{m}}(\overline{x} - x) \geq i, \nu_{\mathfrak{m}}(\overline{y} - y) \geq i, P(\overline{x}, \overline{y}) = 0. \end{aligned}$$

This corollary reminds us of the theorem that an excellent Henselian local ring has the strong Artin approximation property (*cf.* [P]). The case $P(X, Y) = xy$ is nothing but the product inequality.

4. Proof of Theorem 3.1.

(I) REDUCTION TO NORMAL CASE.

We may assume that $\boldsymbol{v}(K^*) = \boldsymbol{v}(\hat{K}^*) = \mathbb{Z}$. This results in a change of the exponent a. Let $\tilde{A}$ denote the normalization (the integral closure of A in K) of A. Since A is a Henselian integral domain, $\tilde{A}$ is a local ring by [N, 43.11 and 43.20]. Since A is excellent,

(1) A is a G-ring and a Nagata (= pseudo-geometric) ring

by [M, 33.H]. Then $\tilde{A}$ is a finite A-module. Hence $\dim A = \dim \tilde{A}$ by a theorem of Cohen–Seidenberg (*cf.* [N, 10.10]) and $r\tilde{A} \subset A$ for some $r \in A^*$ (existence of a universal denominator). Then a Diophantine inequality for $\tilde{A}$ implies one

for A with the same exponent a. Finiteness also implies that $\tilde{A}$ is excellent and Henselian by [N, 43.16].

Let $\tilde{\mathfrak{m}}$ denote the maximal ideal of $\tilde{A}$. We claim that $\boldsymbol{v}$ is an $\tilde{m}$-valuation. If $x \in \tilde{A}$,

$$\exists p \in \mathbb{N}, \exists b_0, \ldots, b_{p-1} \in A : x^p = b_o + b_1 x + \cdots + b_{p-1} x^{p-1}.$$

Then we have $p\boldsymbol{v}(x) \geq \min\{i\boldsymbol{v}(x) : 0 \leq i \leq p-1\}$. This proves $\tilde{A} \subset V_{\boldsymbol{v}}$ and condition (a) for $(\tilde{A}, \tilde{\mathfrak{m}})$. Let us put $\overline{\mathfrak{m}} := \{x \in \tilde{A} : \boldsymbol{v}(x) > 0\}$. Then $\overline{\mathfrak{m}}$ is a prime ideal of $\tilde{A}$ and $\overline{\mathfrak{m}} \cap A = \mathfrak{m}$. This implies that $\overline{\mathfrak{m}} = \tilde{\mathfrak{m}}$ by [B, Chap. 5, 2.1, Proposition 1], and (b) holds. Since $\tilde{A}$ is a finite A-module, $\tilde{k} = \tilde{A}/\tilde{\mathfrak{m}}$ is a finite k-module ($k := A/\mathfrak{m}$), *i.e.*, $\tilde{k}$ is algebraic over k. This proves (c). The condition (d) is obvious. We have proved the claim and we may assume that

(2) A is an equicharacteristic, excellent, Henselian and normal local domain.

(II) Reduction of the minimal equation.

Let

$$\varphi(Z) := a_0 + a_1 Z + \cdots + a_d Z^d \quad (a_d \neq 0, d \geq 2)$$

be a minimal equation for z over A, that is, φ is a polynomial of the minimal degree in $A[Z]$ with $\varphi(z) = 0$. Now take $u \in A^*$ and put

$$\varphi_u(Z) := u^d a_d^{d-1} \varphi(Z/ua_d).$$

Then we have

$$\varphi_u(Z) = a_0 u^d a_d^{d-1} + a_1 u^{d-1} a_d^{d-2} Z + \cdots + Z^d \in A[Z]$$

and $w' \in \hat{K}$ is a root of $\varphi_u(Z)$ if and only if $w := w'/ua_d$ is a root of $\varphi(Z)$. If

$$\exists a \geq 0 \exists c > 0 \forall x \in A \forall y \in A : \left| z' - \frac{x}{y} \right|_{\hat{\boldsymbol{v}}} > c|y|_{\boldsymbol{v}}^a$$

holds for $z' := ua_d z$, we have

$$\exists a \geq 0\, \exists c > 0\, \forall x \in A\, \forall y \in A : \left| z - \frac{x}{y} \right|_{\hat{\boldsymbol{v}}} > \frac{c}{|ua_d|} |y|_{\boldsymbol{v}}^a.$$

The polynomial $\varphi_u(Z) \in A[Z]$ is minimal for z'. Thus, choosing u, we may assume that $z \in \hat{V}_{\boldsymbol{v}}$ and

$$(3) \qquad \varphi(Z) := a_0 + a_1 Z + \cdots + a_{d-1} Z^{d-1} + Z^d \quad (d \geq 2, a_i \in \mathfrak{m}^{d-i})$$

from the first.

(III) ORDER FUNCTION ON $A[z]$ (THE RING GENERATED BY z OVER A).

Let us consider the residue ring $B := A[Z]/\varphi(Z)A[Z]$. There is an isomorphism $\iota\colon B \longrightarrow A[z]$. The ring B is a finite A-module with basis $1, z, z^2, \ldots, z^{d-1}$. Since $A[z]$ is a subring of the field $\hat{K}$, B is an integral domain. Thus we have the following:

(4) $\qquad A[z] \cong B := A[Z]/\varphi(Z)A[Z]$ is an integral extension of A.

Since A is Henselian, B is a local ring by (4) and by [N, 43.12]. As a consequence of (II), $z^d \in \mathfrak{m}A[z]$. Hence the maximal ideal of B is $\mathfrak{n} := \mathfrak{m}B + ZB$ and its residue ring is the same as that of A: $k = A/\mathfrak{m} = B/\mathfrak{n}$. Let us define $\mu\colon A[Z] \longrightarrow \overline{\mathbb{R}}$ by

$$\mu\Big(\sum_{i=0}^{e} b_i Z^i\Big) := \min_i\{\nu_{\mathfrak{m}}(b_i) + i\} \quad (b_i \in A),$$

and $\nu_{\mathfrak{n}}\colon B \longrightarrow \overline{\mathbb{R}}$ as the $\mathfrak{n}$-adic order. The function μ is nothing but the restriction of the standard order on the formal power series ring $A[[Z]]$. We claim that $\nu_{\mathfrak{n}}(x)$ coincides with the μ-order of the unique representative of x in $A[Z]$ of degree less than d, *i.e.*,

$$\mu\Big(\sum_{i=0}^{d-1} b_i Z^i\Big) = \nu_{\mathfrak{n}}\Big(\sum_{i=0}^{d-1} b_i Z^i \mod \varphi(Z)A[Z]\Big).$$

We have only to show that inequality

$$\mu\Big(\sum_{i=0}^{d-1} b_i Z^i\Big) < \mu\Big(\sum_{i=0}^{d-1} b_i Z^i + \sum_{0}^{d} a_i Z^i \sum_{j=0}^{e} c_j Z^j\Big)$$

leads us to a contradiction. Let us develop the product $\sum_{i=0}^{d} a_i Z^i \sum_{i=0}^{e} c_j Z^j$ and reduce its degree in Z by repeated substitutions $Z^d = -\sum_{i=0}^{d-1} a_i Z^i$, beginning from the highest degree term. By the assumption $a_i \in \mathfrak{m}_A^{d-i}$, the substitutions do not lower the μ-order and we reach the left side. This contradicts the inequality we assumed.

The function $\nu_{\mathfrak{n}}$ induces $\nu := \nu_{\mathfrak{n}} \circ \iota^{-1}\colon A[z] \longrightarrow \overline{\mathbb{R}}$. Of course ν inherits the *non-cancellation property* from $\nu_{\mathfrak{n}}$:

$$\nu\Big(\sum_{i=0}^{d-1} b_i z^i\Big) = \min\{\nu(b_i) + i : 0 \le i \le d-1\} = \min\{\nu_{\mathfrak{m}}(b_i) + i : 0 \le i \le d-1\}.$$

In other words, there occurs no cancellation among summands of degree less than d.

(iv) $A[z]$ is analytically irreducible.

Since A is a normal G-ring, it is analytically normal, *i.e.,* the completion $\check{A}$ with respect to the $\mathfrak{m}$ topology is normal , by [M, 33.I]. Hence, by 2.4 we have

$$\hat{A} = \check{A} \text{ is normal.} \tag{5}$$

by [M, 33.I]. Let $\mathfrak{m}'$ denote the maximal ideal of $A[z]$. Taking the last equality of (III) into account, we see that the $\mathfrak{m}'$-adic completion of $A[z]$ is isomorphic to $\hat{A}[z] \subset \hat{K}$. Hence $A[z]$ is an analytically irreducible domain. (This can be also deduced from [N, 44.1] and (1), (4), (5).) Now we can apply 2.4 to $A[z]$.

(v) Diophantine inequality.

We claim that the restriction $\hat{\boldsymbol{v}}|Q(A[z])$ is an $\mathfrak{m}'$-valuation. By the reduction (II) we see that $z \in \hat{V}_{\boldsymbol{v}}$ using the argument in (I) and (a) follows. Since $\mathfrak{m}'$ is generated by $\mathfrak{m}$ and z, the condition (b) is satisfied. Take any element $x \in \hat{V}_{\boldsymbol{v}} \cap K[z]$. There exists a nontrivial polynomial relation

$$c_0 + c_1 x + \cdots + c_{p-1}x^{p-1} + c_p x^p = 0 \quad (c_i \in A).$$

If t is a generator of $\mathfrak{m}_{\boldsymbol{v}} \subset V_{\boldsymbol{v}}$, we have the expressions

$$c_i = c_i' t^{\alpha_i} \quad (\alpha_i \in \{0, 1, 2, \dots\},\ c_i' \in V_{\boldsymbol{v}} \setminus \mathfrak{m}_{\boldsymbol{v}}).$$

We may assume that some α_i is zero. Then the equation implies that $x \bmod \hat{\mathfrak{m}}_{\boldsymbol{v}} \cap K[z]$ is algebraic over $k_{\boldsymbol{v}} = V_{\boldsymbol{v}}/\mathfrak{m}_{\boldsymbol{v}}$. Therefore $(\hat{V}_{\boldsymbol{v}} \cap K[z])/(\hat{\mathfrak{m}}_{\boldsymbol{v}} \cap K[z])$ is algebraic over $k_{\boldsymbol{v}}$ and we have

$$\begin{aligned}\operatorname{trdeg}_{A[z]/\mathfrak{m}'}(\hat{V}_{\boldsymbol{v}} \cap K[z])/(\hat{\mathfrak{m}}_{\boldsymbol{v}} \cap K[z]) &= \operatorname{trdeg}_k(\hat{V}_{\boldsymbol{v}} \cap K[z])/(\hat{\mathfrak{m}}_{\boldsymbol{v}} \cap K[z]) \\ &= \operatorname{trdeg}_k k_{\boldsymbol{v}} = \dim A - 1 = \dim A[z] - 1.\end{aligned}$$

Here the third equality follows from (c) for $\boldsymbol{v}$. This equality implies (c) for $\hat{\boldsymbol{v}}|_{K[z]}$. The condition (d) is trivial. This completes the proof of the claim.

If $\hat{\boldsymbol{v}}(z - \frac{x}{y}) \leq \hat{\boldsymbol{v}}(z)$, we have $|z - \frac{x}{y}|_{\hat{\boldsymbol{v}}} \geq \exp\bigl(-\hat{\boldsymbol{v}}(z)\bigr)$ at once. Hence we may assume that $\hat{\boldsymbol{v}}(x - yz) - \boldsymbol{v}(y) > \hat{\boldsymbol{v}}(z)$. If $\boldsymbol{v}(x) \neq \hat{\boldsymbol{v}}(yz)$, we have a contradiction:

$$\hat{\boldsymbol{v}}(z) < \hat{\boldsymbol{v}}(x - yz) - \boldsymbol{v}(y) \leq \hat{\boldsymbol{v}}(yz) - \boldsymbol{v}(y) = \hat{\boldsymbol{v}}(z).$$

Hence we have only to consider the case

$$\boldsymbol{v}(x) = \hat{\boldsymbol{v}}(yz) = \boldsymbol{v}(y) + \hat{\boldsymbol{v}}(z).$$

Since $A[z]$ is analytically irreducible, applying the inequality 2.4 and the equality at the last part of (III), we have

$$s\hat{\boldsymbol{v}}(x - yz) \leq \nu_{\mathfrak{m}'}(x - yz) + C \leq \nu_{\mathfrak{m}'}(x) + C \leq t\boldsymbol{v}(x) + C.$$

It follows that

$$s\hat{\boldsymbol{v}}\Big(z-\frac{x}{y}\Big) \le t\boldsymbol{v}(x) - s\boldsymbol{v}(y) + C = (t-s)\boldsymbol{v}(y) + t\hat{\boldsymbol{v}}(z) + C.$$

This implies the inequality of our theorem.

If $a = 0$, z is isolated from K and cannot be in $\hat{K}$. Hence we see that $a > 0$.

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