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HYPERGROUPS WITH UNIQUE α-MEANS

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ABSTRACT. Let K be a commutative hypergroup and $\alpha \in \widehat{K}$. We show that K is α-amenable with the unique α-mean m_α if and only if $m_\alpha \in L^1(K) \cap L^2(K)$ and α is isolated in $\widehat{K}$. In contrast to the case of amenable noncompact locally compact groups, examples of polynomial hypergroups with unique α-means ($\alpha \neq 1$) are given. Further examples emphasize that the α-amenability of hypergroups depends heavily on the asymptotic behavior of Haar measures and characters.

RÉSUMÉ. Soit K un hypergroupe commutatif et $\alpha \in \widehat{K}$. Nous montrons que K est α-moyennable avec unicité de l'α-moyenne m_α si et seulement si $m_\alpha \in L^1(K) \cap L^2(K)$ et α est isolé dans $\widehat{K}$. Contrairement au cas des groupes moyennables localement compacts mais non compacts, des exemples d'hyper-groupes polynomiaux avec unicité des α-moyennes ($\alpha \neq 1$) sont donnés. Nous montrons à l'aide d'autres examples que l'α-moyennabilité des hypergroupes dépend fortement de leurs mesures de Haar ainsi que du comportement des caractères.

1. **Introduction.** Recently the notion of α-amenable hypergroups was introduced and studied in [8]. Let K be a commutative locally compact hypergroup and let $L^1(K)$ denote the hypergroup algebra. Assume that $\alpha \in \widehat{K}$ and denote by $I(\alpha)$ the maximal ideal in $L^1(K)$ generated by α. As shown in [8], K is α-amenable if and only if either $I(\alpha)$ has a b.a.i. (bounded approximate identity) or K satisfies the modified Reiter's condition of P_1-type in α. Commutative hypergroups are always 1-amenable [14], whereas a large class of non α-amenable hypergroups, $\alpha \neq 1$, are given in [1], [8]. It is worthwhile to mention that $1 \in \operatorname{supp} \pi_K$ does not hold in general, where $\operatorname{supp} \pi_K$ denotes the support of the Plancherel measure on $\widehat{K}$ [10], [14].

As in the case of locally compact groups [13], if K is a noncompact locally compact amenable hypergroup, then the cardinality of (1-)means is 2^{2^d}, where d is the smallest cardinality of a cover of K by compact sets [14]. However, it is well known that K has a unique (1-)mean if and only if K is compact [13], [14]. Hence, $\operatorname{supp} \pi_K = \widehat{K}$ and K is α-amenable for every $\alpha \in \widehat{K}$ [2], [8].

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For a α-amenable hypergroup K with a unique α-mean, one can pose the natural question of whether K is compact or K is β-amenable when $\alpha \neq \beta \in \widehat{K}$. Theorem 3.1 answers this question completely. In addition, examples of polynomial hypergroups show that the α-amenability of hypergroups depends on the asymptotic behavior of the Haar measures and characters. Furthermore, the α-amenability of K with a unique α-mean ($\alpha \neq 1$), even in every $\alpha \in \widehat{K} \setminus \{1\}$, does not imply the compactness of K; see Section 4.

Different axioms for hypergroups are given in [7], [10], [15]. However, in this paper we refer to Jewett's axioms in [10].

2. Preliminaries.

Let $(K, \omega, \sim)$ be a locally compact hypergroup, where $\omega\colon K \times K \to M^1(K)$ defined by $(x, y) \mapsto \omega(x, y)$, and $\sim\colon K \to K$ defined by $x \mapsto \tilde{x}$, denote the convolution and involution on K, where $M^1(K)$ stands for the set of all probability measures on K. The hypergroup K is called commutative if $\omega(x, y) = \omega(y, x)$ for every $x, y \in K$.

Throughout this paper K is a commutative hypergroup. Let $C_c(K)$ be the spaces of all continuous functions on K with compact support. The translation of $f \in C_c(K)$ at the point $x \in K$, $T_x f$, is defined by

$$T_x f(y) := \int_K f(t)\, d\omega(x, y)(t), \quad \text{for every } y \in K.$$

Commutativity of K ensures the existence of a Haar measure m on K which is unique up to a multiplicative constant [15]. Let $\big(L^p(K), \|\cdot\|_p\big)$ $(p = 1, 2)$ denote the usual Banach space of Borel measurable functions on K [10, 6.2]. For $f, g \in L^1(K)$ we may define the convolution and involution by $f * g(x) := \int_K f(y) T_{\tilde{y}} g(x)\, dm(y)$ (m-a.e. on K) and $f^*(x) = \overline{f(\tilde{x})}$, respectively, that $\big(L^1(K), \|\cdot\|_1\big)$ becomes a Banach $*$-algebra. If K is discrete, then $L^1(K)$ has an identity element. Otherwise $L^1(K)$ has a b.a.i., *i.e.*, there exists a net $\{e_i\}_i$ of functions in $L^1(K)$ with $\|e_i\|_1 \leq M$, for some $M > 0$, such that $\|f * e_i - f\|_1 \to 0$ as $i \to \infty$ [2]. The set of all multiplicative linear functionals on $L^1(K)$, *i.e.*, the maximal ideal space of $L^1(K)$ [3], can be identified with

$$\mathfrak{X}^b(K) := \{\alpha \in C^b(K) : \alpha \neq 0, \omega(x, y)(\alpha) = \alpha(x)\alpha(y), \forall x, y \in K\}$$

via $\varphi_\alpha(f) := \int_K f(x)\overline{\alpha(x)}\, dm(x)$, for every $f \in L^1(K)$. $\mathfrak{X}^b(K)$ is a locally compact Hausdorff space with the compact-open topology [2]. $\mathfrak{X}^b(K)$ and its subset

$$\widehat{K} := \{\alpha \in \mathfrak{X}^b(K) : \alpha(\tilde{x}) = \overline{\alpha(x)}, \forall x \in K\}$$

are considered as the character spaces of K. The maximal ideal in $L^1(K)$ generated by the character α is $I(\alpha) := \{f \in L^1(K) : \varphi_\alpha(f) = 0\}$. The Fourier transform of $f \in L^1(K)$, $\widehat{f} \in C_0(\widehat{K})$, is $\widehat{f}(\alpha) := \varphi_\alpha(f)$ for every $\alpha \in \widehat{K}$. There exists a unique (up to a multiplicative constant) regular positive Borel measure π_K on $\widehat{K}$ with $\operatorname{supp} \pi_K = \mathcal{S}$ such that $\int_K |f(x)|^2\, dm(x) = \int_{\mathcal{S}} |\widehat{f}(\alpha)|^2\, d\pi_K(\alpha)$ for all $f \in L^1(K) \cap L^2(K)$ [2]. The extension of the Fourier transform defined on $L^1(K) \cap L^2(K)$ to all of $L^2(K)$ onto $L^2(\widehat{K})$ is the Plancherel transform which is an isometric isomorphism. Observe that $\mathcal{S}$ is a nonvoid closed subset of $\widehat{K}$, and the constant function 1 is in general not contained in $\mathcal{S}$ [10, 9.5].

The inverse Fourier transform for $\varphi \in L^1(\widehat{K})$ is given by

$$\check{\varphi}(x) = \int_{\mathcal{S}} \varphi(\alpha)\alpha(x)\, d\pi_K(\alpha)$$

for every $x \in K$. Then $\check{\varphi} \in C_0(K)$ and if $\check{\varphi} \in L^1(K)$ then $\widehat{\check{\varphi}} = \varphi$ [2].

Let $L^1(K)^*$ and $L^1(K)^{**}$ denote the dual and the bidual spaces of $L^1(K)$ respectively. As usual, $L^1(K)^*$ can be identified with the space $L^\infty(K)$ of essentially bounded Borel measurable complex-valued functions on K. We may define the Arens product on $L^1(K)^{**}$ as follows:

$$\langle m \cdot m', f\rangle = \langle m, m' \cdot f\rangle$$

in which $\langle m' \cdot f, g\rangle = \langle m', f \cdot g\rangle$ and $\langle f \cdot g, h\rangle = \langle f, g * h\rangle$ for all $m, m' \in L^1(K)^{**}$, $f \in L^\infty(K)$ and $g, h \in L^1(K)$. Then $L^1(K)^{**}$ with the Arens product is a noncommutative Banach algebra in general [3], [5]. From the definitions of the Arens product and the convolution, we may have $g \cdot f = g^* * f$ and $m \cdot (f \cdot g) = (m \cdot f) \cdot g$.

Definition 2.1. Let K be a commutative hypergroup and $\alpha \in \widehat{K}$. K is called α-amenable if there exists a bounded linear functional m_α on $L^\infty(K)$ with the following properties:
(i) $m_\alpha(\alpha) = 1$,
(ii) $m_\alpha(\delta_{\tilde{x}} * f) = \alpha(x)m_\alpha(f)$, for every $f \in L^\infty(K)$ and $x \in K$.

For example, if K is compact or $L^1(K)$ is amenable, then K is α-amenable, for every $\alpha \in \widehat{K}$ [8], [14].

3. Main theorem.

Theorem 3.1. *Let K be a hypergroup and $\alpha \in \widehat{K}$. Suppose K is α-amenable with the unique α-mean m_α. Then the following hold.*
(i) *m_α and α belong to $L^1(K) \cap L^2(K)$ and $\alpha \in \mathcal{S}$ is isolated. Further, $m_\alpha^2 = m_\alpha$.*
(ii) *$m_\alpha = \pi(\alpha)/\|\alpha\|_2^2$, where $\pi\colon L^1(K) \to L^1(K)^{**}$ is the canonical embedding.*
(iii) *If α is positive, then $\alpha = 1$, hence K is compact.*

Proof. (i) Since K is α-amenable with the unique α-mean m_α, $m_\alpha(\alpha) = 1$, and $f \cdot g = g \cdot f$, we have

$$\begin{aligned}\langle m_\alpha, f \cdot g\rangle &= \langle m_\alpha, g^* * f\rangle \\ &= \left\langle m_\alpha, \int_K (\delta_x * f) g^*(x) dm(x)\right\rangle \\ &= \int_K \langle m_\alpha, \delta_x * f\rangle g^*(x)\, dm(x) \\ &= \widehat{g^*}(\alpha)\langle m_\alpha, f\rangle,\end{aligned}$$

for every $f \in L^\infty(K)$ and $g \in L^1(K)$. Moreover, if $n \in L^1(K)^{**}$ and $h \in L^1(K)$, then

$$\begin{aligned}\langle m_\alpha \cdot n, f \cdot g\rangle &= \langle m_\alpha, n \cdot (f \cdot g)\rangle = \langle m_\alpha, (n \cdot f) \cdot g\rangle \\ &= \widehat{g^*}(\alpha)\langle m_\alpha, n \cdot f\rangle = \widehat{g^*}(\alpha)\langle m_\alpha \cdot n, f\rangle.\end{aligned}$$

Since the α-mean m_α is unique and the associated functional to α on $L^1(K)^{**}$ is multiplicative [5], $m_\alpha \cdot n = \lambda_n \cdot m_\alpha$, where $\lambda_n = \langle n, \alpha\rangle$. Let (n_i) be a net in $L^1(K)^{**}$ converging to n in the w^*-topology. Then the convergence $\lambda_{n_i} \to \lambda_n$, as $i \to \infty$, implies that the mapping $n \to m_\alpha \cdot n$ is w^*-w^* continuous on $L^1(K)^{**}$, hence m_α is in $L^1(K)$, the topological centre of $L^1(K)^{**}$ [11]. In that $\widehat{m_\alpha}(\alpha) = 1$, $g \cdot m_\alpha = \widehat{g^*}(\alpha) m_\alpha$ for every $g \in L^1(K)$, and the Arens product is continuous in the first variable, then $m_\alpha^2 = m_\alpha$.

Let $\beta \in \widehat{K}$. The equality $\beta(x) m_\alpha(\beta) = m_\alpha(T_x\beta) = \alpha(x) m_\alpha(\beta)$, for all $x \in K$, implies that $m_\alpha(\beta) = \delta_\alpha(\beta)$. Since $\widehat{m_\alpha} \in C_0(\widehat{K})$, α is isolated in $\widehat{K}$ and $\widehat{m_\alpha} \in L^1(\widehat{K})$. The inverse of Fourier theorem yields $m_\alpha = \widehat{m_\alpha}^\vee$, hence $\alpha \in \mathcal{S}$. Moreover, since the Plancherel transform is an isometric isomorphism of $L^2(K)$ onto $L^2(\widehat{K})$ and $\widehat{m_\alpha}(\beta) = \delta_\beta(\alpha)$, $m_\alpha \in L^2(K)$.

(ii) Plainly $\delta_x \cdot m_\alpha = \alpha(x) m_\alpha$, for every $x \in K$, so it follows from part (i) that $\alpha \in L^1(K) \cap L^2(K)$. Let $n_\alpha = \pi(\alpha)/\|\alpha\|_2^2$. We shall prove $m_\alpha = n_\alpha$. Apparently $\langle n_\alpha, \alpha\rangle = 1$, and for every $x \in K$ and $f \in L^\infty(K)$ we have $\langle n_\alpha, T_x f\rangle = \alpha(x)\langle n_\alpha, f\rangle$, hence n_α is a α-mean on $L^\infty(K)$. Since $m_\alpha \in L^1(K)^{**}$, there exists $(m_i)_i$ a net of functions in $L^1(K)$ such that $\pi(m_j) \xrightarrow{w^*} m_\alpha$, Goldstein's theorem [6]. Moreover, $m_j \cdot \alpha = \alpha \cdot m_j = \widehat{m_j}(\alpha)\alpha$ and $m_\alpha(\alpha) = 1$, so taking the w^*-limit yields $m_\alpha \cdot \pi(\alpha) = \pi(\alpha)$. Therefore, for every $f \in L^\infty(K)$ and $x \in K$ we have

$$\begin{aligned}\|\alpha\|_2^2\langle n_\alpha, f\rangle &= \langle \pi(\alpha), f\rangle = \langle m_\alpha \cdot \pi(\alpha), f\rangle \\ &= \langle m_\alpha, \pi(\alpha) \cdot f\rangle = \langle m_\alpha, \alpha \cdot f\rangle = \|\alpha\|_2^2\langle m_\alpha, f\rangle,\end{aligned}$$

hence $m_\alpha = n_\alpha$.

(iii) By (i), since $\alpha \in L^1(K) \cap L^2(K)$, we have

$$\alpha(x) \int_K \alpha(y)\, dm(y) = \int_K T_x\alpha(y)\, dm(y) = \int_K \alpha(y)\, dm(y),$$

which implies that $\alpha(x) = 1$ for every $x \in K$, hence K is compact [10]. $\square$

COROLLARY 3.2. *Let K be a α-amenable hypergroup with a unique α-mean in all $\alpha \in \widehat{K} \setminus \{1\}$. Then $1 \in \mathcal{S}$.*

REMARK 3.3. We observe that part (iii) of Theorem 3.1 can also be derived from part (i) and [16, Theorem 2.1].

4. Examples.

(I) **Symmetric hypergroup** [17]. For each $n \in \mathbb{N}$, let $b_n \in]0, 1]$, $c_0 = 1$, and define numbers c_n inductively by $c_n = \frac{1}{b_n}(c_0 + c_1 + \cdots + c_{n-1})$. A symmetric hypergroup structure on $\mathbb{N}_0$ is defined by $\varepsilon_n * \varepsilon_m = \varepsilon_m * \varepsilon_n = \varepsilon_n$ if $0 \leq m < n$ and

$$\varepsilon_n * \varepsilon_n = \frac{c_0}{c_n}\varepsilon_0 + \frac{c_1}{c_n}\varepsilon_1 + \cdots + \frac{c_{n-1}}{c_n}\varepsilon_n + (1 - b_n)\varepsilon_n.$$

$\mathbb{N}_0$ with the above convolution and an involution defined by the identity map is a commutative hypergroup with $\mathfrak{X}^b(\mathbb{N}_0) = \mathcal{S}$. Every nontrivial character α in $\widehat{\mathbb{N}_0}$ has a finite support, so $\alpha \in \ell^1(\mathbb{N}_0) \cap \ell^2(\mathbb{N}_0)$. Consequently by Theorem 3.1 we see that $\mathbb{N}_0$ is α-amenable with a unique α-mean if and only if $\alpha \neq 1$.

(II) Let $\{p_n\}_{n \in \mathbb{N}_0}$ be a set of polynomials defined by a recursion relation

$$(*) \qquad p_1(x)p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$$

for $n \in \mathbb{N}$ and $p_0(x) = 1$, $p_1(x) = \frac{1}{a_0}(x - b_0)$, where $a_n > 0$, $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$ and $c_n > 0$ for $n \in \mathbb{N}$. There exists a probability measure $\pi \in M^1(\mathbb{R})$ such that $\int_{\mathbb{R}} p_n(x)p_m(x)\, d\pi(x) = \delta_{n,m}\mu_m$ $(\mu_m > 0)$ [4]. Assume that $p_n(1) \neq 0$, so after renorming, for $n \in \mathbb{N}_0$ we have $p_n(1) = 1$. The relation $(*)$ implies that $a_n + b_n + c_n = 1$ and $a_0 + b_0 = 1$. The polynomial set $\{p_n\}_{n \in \mathbb{N}_0}$ induces a hypergroup structure on $\mathbb{N}_0$, which is known as a polynomial hypergroup [2].

(i) **Hypergroups of compact type** [9]. If in the recursion formula $(*)$ $a_n, c_n \to 0$ and $b_n \to 1$ as $n \to \infty$, then the induced hypergroup $\mathbb{N}_0$ is called to be of compact type. In this case, $\mathcal{S} = \widehat{\mathbb{N}_0} = \mathfrak{X}^b(\mathbb{N}_0)$, 1 is the only accumulation point of $\widehat{\mathbb{N}_0}$ and nontrivial characters of $\mathbb{N}_0$ belong to $\ell^1(\mathbb{N}_0) \cap \ell^2(\mathbb{N}_0)$. By Theorem 3.1 we see that $\mathbb{N}_0$ is α-amenable with a unique α-mean if and only if $\alpha \neq 1$. For instance, the little q-Legendre polynomial hypergroup is of compact type.

(ii) **Hypergroups of Nevai classes**. Let $\{p_n\}_{n \in \mathbb{N}_0}$ define a hypergroup structure on $\mathbb{N}_0$ with the relations $(*)$. Consider the orthonormal polynomials $q_n(x) := \sqrt{h(n)}p_n(x)$, which by the recursion $(*)$ satisfy the following recursion formula

$$xq_n(x) = \lambda_{n+1}q_{n+1}(x) + \beta_n q_n(x) + \lambda_n q_{n-1}(x), \quad \forall n \in \mathbb{N}_0,$$

where $q_0(x) = 1$, $\lambda_n = a_0\sqrt{c_n a_{n-1}}$ for $n \geq 2$, $\lambda_1 = a_0\sqrt{c_1}$, $\lambda_0 = 0$, and $\beta_n = a_0 b_n + b_0$ for $n \geq 1$, with $\beta_0 = b_0$. The polynomial set $(q_n)_{n \in \mathbb{N}_0}$ is of the Nevai class $M(0,1)$ if $\lim_{n\to\infty} \lambda_n = \frac{1}{2}$ and $\lim_{n\to\infty} \beta_n = 0$. It has a bounded variation, $(q_n)_{n\in\mathbb{N}_0} \in BV$, if

$$\sum_{n=1}^{\infty} (|\lambda_{n+1} - \lambda_n| + |\beta_{n+1} - \beta_n|) < \infty.$$

THEOREM 4.1. *Let $(q_n)_{n\in\mathbb{N}_0} \in BV \cap M(0,1)$ and $\alpha_x \in \widehat{\mathbb{N}_0}$, where $\alpha_x(n) := q_n(x)$ for $n \in \mathbb{N}_0$. Then the following hold:*

(i) *$\mathcal{S} \cong [-1,1] \cup A$, where A is a nonempty countable set and $[-1,1] \cap A = \varnothing$.*
(ii) *If $x \in A$, then $\mathbb{N}_0$ is α_x-amenable with a unique α_x-mean.*
(iii) *If $h(n)$ is unbounded, then $\mathbb{N}_0$ is not α_x-amenable for $x \in (-1,1)$.*
(iv) *If $h(n)$ is bounded, then $\mathbb{N}_0$ is α_x-amenable for $x \in (-1,1)$.*

PROOF. (i) This is shown in [12, Theorem 7].

(ii) Let $\mathcal{S}$ be as in part (i). If $A \cap \,]1,\infty[\, \neq \varnothing$, then $x_1 := \sup A \in A$ corresponds to a positive character of $\mathbb{N}_0$ [4, Theorem 5.3]. But this contradicts the fact that a positive character in $\mathcal{S}$ cannot be isolated [16, Theorem 2.1], hence $A \subset \,]-\infty,-1[$. By [12, Theorem 18, p. 36] we have

$$\lim_{n\to\infty} \frac{h(n+1)}{h(n)} \Big| \frac{p_{n+1}(x)}{p_n(x)} \Big| = C \lim_{n\to\infty} \Big| \frac{p_{n+1}(x)}{p_n(x)} \Big| = \big(|x| + (x^2-1)^{1/2}\big)^{-1} < 1,$$

whenever $x \in A$. This shows that α_x belongs to $\ell^1(\mathbb{N}_0)$. Hence, by Theorem 3.1, $\mathbb{N}_0$ is α_x-amenable with a unique α_x-mean.

(iii) and (iv) are shown in [8, Theorems 4.10–11]. □

REMARK 4.2.

(i) Theorem 4.1 reveals that the α-amenability of K in general depends on the asymptotic behavior of the Haar measure and α.

(ii) Observe that in Theorem 4.1 (iii) if $x \in (-1,1)$ then the functionals m_{α_x} are distinct.

CONJECTURE 4.3. *Let K be a α-amenable hypergroup. Then K has either a unique α-mean or the cardinality of the set of α-means is at most 2^{2^d}, where d is the smallest cardinality of a cover of K by compact sets.*

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