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COMPLEX QUOTIENTS BY NONCLOSED GROUPS AND THEIR STRATIFICATIONS

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Abstract. We define the notion of complex stratification by quasifolds and show that such stratified spaces occur as complex quotients by certain nonclosed subgroups of tori associated to convex polytopes. The spaces thus obtained provide a natural generalization to the nonrational case of the notion of toric variety associated with a rational convex polytope.

Résumé. On définit la notion de stratification complexe de quasifolds et on montre que ces espaces stratifiés se réalizent comme quotients complexes par des sousgroupes non fermés de tores, associés aux polytopes convexes. Les espaces ainsi obtenus donnent une généralization naturelle, au cas non rationnel, de la notion de variété torique associée à un polytope convexe rationnel.

Introduction Our aim is to define a geometric object that naturally generalizes to the nonrational setting, the notion of toric variety associated with a rational convex polytope. Let $\mathfrak{d}$ be a real vector space of dimension n and let Δ be a polytope in $\mathfrak{d}^*$, rational with respect to a lattice L, with d faces of codimension 1. Then the toric variety corresponding to Δ is the categorical quotient of a suitable open subset of $\mathbb{C}^d$, modulo the action of a subtorus of $(\mathbb{C}^*)^d$, as shown by Cox [C]. On the other hand, complex toric spaces corresponding to *simple* convex polytopes, not necessarily rational, were constructed as complex quotients in joint work with Elisa Prato [BP1], who had previously given the construction of these space as symplectic quotients [P]. More precisely, to each (simple) convex polytope in $\mathfrak{d}^*$, the moment polytope, there corresponds a unique fan in $\mathfrak{d}$ — generated by the 1-dimensional cones relative to the codimension-1 faces of the polytope — and many choices of the following data: the generators of the 1-dimensional cones of the fan and a quasilattice Q in $\mathfrak{d}$ containing such generators. A quasilattice in the vector space $\mathfrak{d}$ is a $\mathbb{Z}$-submodule of $\mathfrak{d}$ generated by a set of generators of $\mathfrak{d}$ (see [S]). In [BP1], for each choice of data relative to the polytope Δ, we construct an n-dimensional toric space given by the geometric quotient of an open subset $\mathbb{C}^d_\Delta$ of $\mathbb{C}^d$ by the action of a non necessarily closed subgroup $N_\mathbb{C}$ of $(\mathbb{C}^*)^d$. The space thus obtained has the structure of a complex quasifold and is endowed with the holomorphic action of the n-dimensional complex quasitorus $D_\mathbb{C} = \mathfrak{d}_\mathbb{C}/Q$. Quasitori are a first, natural example of quasifolds;

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they have been introduced, together with the notion of quasifold, in Prato's article [P]. Complex quasifolds are topological spaces whose local models are open subsets of $\mathbb{C}^n$ modulo the action of finitely generated groups. Local models are glued together to give rise to a global quasifold structure. Quasifold structures are highly singular: for example the finitely generated groups obtained are in general nonclosed, so that the corresponding spaces are non Hausdorff.

Let us now consider a *nonsimple* convex polytope, not necessarily rational; once generators of 1-cones and a quasilattice are chosen, we carry on the construction of our space as a complex quotient. Both $\mathbb{C}^d_\Delta$ and $N_\mathbb{C}$ can still be defined: the open set $\mathbb{C}^d_\Delta$ depends only on the combinatorics of the polytope, or, equivalently, of the associated fan, whilst the group $N_\mathbb{C}$ depends on the choice of generators and quasilattice. The first problem we have to deal with is to make sense of the quotient $\mathbb{C}^d_\Delta /\!/ N_\mathbb{C}$: as in the rational nonsimple case, we cannot simply take the geometric quotient, hence in Section 2 we define a suitable notion of quotient. Then in Section 3 we work out its structure. Consider a p-dimensional face F of Δ and a point $\nu \in F$. The polytope Δ, in a neighborhood of ν, is given by the product of F by a cone over an $n-p-1$ dimensional polytope Δ_F. We denote by X_{Δ_F} the complex space associated to Δ_F, together with the induced choice of generators of 1-cones and of quasilattice. The face F is regular if Δ_F is a simplex, singular otherwise. As in the rational case, each p-dimensional face of the polytope corresponds to a p-dimensional orbit of the quasitorus $D_\mathbb{C}$ acting on X_Δ. In particular the interior of the polytope corresponds to the dense open orbit of $D_\mathbb{C}$ in X_Δ. Orbits produce a stratification of the quotient X_Δ that mirrors the structure of the associated polytope Δ. The union of orbits corresponding to regular faces gives the regular set of X_Δ. Orbits corresponding to singular faces are the singular strata. Intuitively it is clear that each stratum has a natural structure of complex quasifold, since it is an orbit (or union of orbits) of the quasitorus $D_\mathbb{C}$. Furthermore, in a neighborhood of a singular orbit, the quotient X_Δ is the product, in a suitable way, of the singular stratum by a complex cone over X_{Δ_F}; it is a twisted product by a finitely generated group. This group is finite in the rational case, when the decomposition is indeed locally trivial and strata are smooth, namely the usual notion of stratification is satisfied (see [GM]). The definitions of complex cone and complex stratification are given in Section 1. We can then state our main result: given a convex polytope Δ, to each set of generators and of quasilattice containing them, there corresponds a complex quotient X_Δ, endowed with an n-dimensional complex stratification by quasifolds and acted on holomorphically by a complex quasitorus with a dense open orbit. Moreover the space X_Δ is homeomorphic to its symplectic counterpart, obtained as symplectic quotient (see [BP2, B1]). Such homeomorphism respects the decompositions and its restriction to each stratum is a diffeomorphism, with respect to which the symplectic and complex structures of strata are compatible. In particular the space X_Δ is compact and its strata are in fact Kähler quasifolds. Remark that, as in the simple case [P, BP1], the space X_Δ as complex space depends only on the set of generators and on the quasilattice, while its symplectic structure depends on the polytope.

These results complete the generalization of the notion of toric spaces associated to nonrational convex polytopes. In particular, they shed light onto the relationship with the theory of classical toric varieties. The geometry and topology of our spaces and the relationship with the properties of the polytope are natural questions related to our work. A first step towards a better understanding of these different aspects, which will be pursued in the sequel, is to define and investigate cohomological invariants of our spaces. In this note we give the general idea of our construction and results, leaving the details of the proofs to a forthcoming paper.[1]

1. **Complex Stratifications by Quasifolds** For the detailed definitions of real symplectic quasifold, quasitorus and related notions we refer the reader to [P]; for the complex version of these notions, see [BP1]. Roughly speaking, as we have already observed, a complex quasifold of dimension n is locally modelled on the topological quotient of an open subset of $\mathbb{C}^n$ by the action of a finitely generated group. A basic example of real quasifold is Prato's quasicircle $D^1_\alpha = \mathbb{R}/\mathbb{Z} + \alpha\mathbb{Z}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. This also gives an example of quasitorus [P]. Notice that if α is taken in $\mathbb{Q}$, then D^1_α is either an orbifold or S^1. The complexification $(D^1_\alpha)_\mathbb{C}$ of D^1_α is the complex quasitorus given by $\mathbb{C}/\mathbb{Z} + \alpha\mathbb{Z}$.

The notions of decomposition and stratification of a space X were given in [B1]. We allow the pieces of a decomposition to be quasifolds and we require then the usual properties of a decomposition. The dimension of the maximal piece is by definition the dimension of X, say n. A smooth map (resp. an isomorphism) between decomposed spaces is a continuous map (resp. homeomorphism) that respects the decompositions and is smooth (resp. a diffeomorphism) when restricted to pieces. A stratification is a decomposition that locally, near to each point t of an r-dimensional stratum $\mathcal{T}$, is given by a twisted product of the kind $\tilde{B} \times C(L)/\Gamma$, where $\tilde{B}/\Gamma$ is a local model of $\mathcal{T}$ around t such that the finitely generated group Γ acts freely on $\tilde{B}$; L is an $(n-r-1)$-dimensional compact space, called the *link* of t, decomposed by quasifolds and endowed with an action of Γ that preserves the decomposition; $C(L)$ is a real cone over L. Then the decomposition of the link L is required to satisfy the above condition. Recursively we end up, after a finite number of steps, with links that are compact quasifolds.

We then give the notion of complex structure of the stratified space. Our requirements are very strong and are not usually satisfied by complex stratified spaces, for example the local trivialization is usually far from being holomorphic. However they are verified by toric varieties as well as by our toric spaces. More precisely, we require that for each link L there exists a compact space Y decomposed by quasifolds and a smooth surjective map $s\colon L \to Y$, with fibers diffeomorphic to a fixed 1-parameter subgroup S of a real torus; each piece of the decomposed spaces X, $C(L)$'s and Y's is endowed with a complex structure

[1] F. Battaglia, *Compactification of complex quasitori and nonrational convex polytopes*, in preparation.

and the natural projection $C(L)\setminus\{\text{cone pt}\}\to Y$, induced by s, when restricted to each piece, is holomorphic, with fibers biholomorphic to the complexification of S. Moreover, the space X is locally biholomorphic to the product $\widetilde{B}\times C(L)/\Gamma$, that is, the identification mapping is not only a diffeomorphism but a biholomorphism when restricted to pieces. We call $C(L)$ a complex cone over Y.

2. **Complex Quotients by Nonclosed Groups** Our goal here is to generalize the construction given by Cox: we need to produce, in association with Δ, an open subset of $\mathbb{C}^d$ and a subgroup of the torus $(\mathbb{C}^*)^d$. We first take care of the open subset. Consider the open faces of Δ. They can be described as follows. Write the polytope as intersection of half spaces: $\Delta=\bigcap_{j=1}^d\{\mu\in\mathfrak{d}^*\mid\langle\mu,X_j\rangle\geq\lambda_j\}$ for inward pointing vectors $X_1,\dots,X_d$ in $\mathfrak{d}$ and the real numbers $\lambda_1,\dots,\lambda_d$ determined by our choice of X_i. Notice that the vectors X_i are generators of the 1-dimensional cones of the fan in $\mathfrak{d}$ dual to the polytope. For each face F there exists a subset $I_F\subset\{1,\dots,d\}$ such that $F=\{\mu\in\Delta\mid\langle\mu,X_j\rangle=\lambda_j$ if and only if $j\in I_F\}$. The n-dimensional open face of Δ corresponds to the empty subset. Define the open set $\widehat{V}_F=\{(z_1,\dots,z_d)\in\mathbb{C}^d\mid z_j\neq0 \text{ if } j\notin I_F\}$ and denote by $\mathbb{C}^d_\Delta$ the open subset of $\mathbb{C}^d$ given by $\mathbb{C}^d_\Delta=\bigcup_{F\in\Delta}\widehat{V}_F$. Notice that in the definition of the open subset $\mathbb{C}^d_\Delta$, only the combinatorics of the polytope intervenes. Moreover, the open subset $\mathbb{C}^d_\Delta$ coincides with the one defined in [C] for the rational case. It is in the definition of the group acting on $\mathbb{C}^d_\Delta$ that nonrationality intervenes. In order to define the group $N_\mathbb{C}$ we adopt the following procedure, which is an extension of the procedure introduced by Delzant [D], extended to the nonrational case first in [P] and then in [BP1, B1]. Let us fix a quasilattice Q in the space $\mathfrak{d}$ containing the elements X_j (for example $\mathrm{Span}_\mathbb{Z}\{X_1,\dots,X_d\}$). Consider the surjective linear mappings $\pi\colon\mathbb{R}^d\to\mathfrak{d}$ (resp. $\pi_\mathbb{C}\colon\mathbb{C}^d\to\mathfrak{d}_\mathbb{C}$) defined by $\pi(e_j)=X_j$ (resp. $\pi_\mathbb{C}(e_j)=X_j$), with $\{e_1,\dots,e_d\}$ the standard basis. Consider the quasitorus $\mathfrak{d}/Q$ and its complexification $\mathfrak{d}_\mathbb{C}/Q$. The mappings π and $\pi_\mathbb{C}$ induce the group homomorphisms $\Pi\colon(S^1)^d=\mathbb{R}^d/\mathbb{Z}^d\to\mathfrak{d}/Q$ and $\Pi_\mathbb{C}\colon(\mathbb{C}^*)^d=\mathbb{C}^d/\mathbb{Z}^d\to\mathfrak{d}_\mathbb{C}/Q$, respectively. We define N (resp. $N_\mathbb{C}$) to be the kernel of the mapping Π (resp. $\Pi_\mathbb{C}$). The group N has dimension $(d-n)$. Let $\mathfrak{n}=\operatorname{Ker}\pi$ be the Lie algebra of N. Then for the complexified group $N_\mathbb{C}$, the polar decomposition holds, namely

$$N_\mathbb{C}=NA, \tag{1}$$

where $A=\exp(i\mathfrak{n})$. If Q is an honest lattice, then N is a compact real torus.

For a given polytope, a choice of normals and of quasilattice Q is said to be *rational* if Q is a true lattice. There are polytopes that do not admit rational choices and that are not combinatorially equivalent to rational polytopes [G]. If the polytope is rational in a lattice L, the categorical quotient $\mathbb{C}^d_\Delta/\!/N_\mathbb{C}$ constructed by Cox [C], can be thought of as the quotient of $\mathbb{C}^d_\Delta$ by the following equivalence relation: two points in $\mathbb{C}^d_\Delta$ are equivalent if the closures of the $N_\mathbb{C}$-orbits through these points have nonempty intersection. In our context this definition does not lead anywhere, since the group N itself is nonclosed. Therefore we have to distinguish two different ways in which orbits are nonclosed. The first is given by the fact that N is nonclosed; this is peculiar to the nonrational setting, and produces the quasifold structure of strata. The other is due to the fact that, as in the rational case, there are *nonclosed* A-orbits. There is absolutely no difference in behavior when it comes to A-orbits. As an example consider $e^{2\pi at}$, with $t \in \mathbb{R}$ and a a real constant. Let $z \in \mathbb{C}$. Obviously the orbit $e^{2\pi at}z$ is not influenced by having a rational or not. Therefore we are led to consider A-orbits. Let $\underline{z} \in \mathbb{C}^d_\Delta$. We say that the A-orbit $A\underline{z}$ is closed if it is closed in $\mathbb{C}^d_\Delta$. Let J be any set of indices in $\{1,\dots,d\}$. We let

$$\mathbb{K}^J = \{\underline{z} \in \mathbb{C}^d \mid z_j \in \mathbb{K} \text{ if } j \in J, z_j = 0 \text{ if } j \notin J\},$$

where $\mathbb{K}$ can be either $\mathbb{C}$ or $\mathbb{C}^*$. Consider the $(\mathbb{C}^*)^d$-orbit

$$(\mathbb{C}^*)^{I^c_F} = \{(z_1,\dots,z_d) \in \mathbb{C}^d \mid z_j = 0 \text{ if and only if } j \in I_F\}.$$

Theorem 2.1. *Let $\underline{z} \in \mathbb{C}^d_\Delta$. Then the A-orbit $A\underline{z}$ through $\underline{z}$ is closed if and only if there exists a face F such that $\underline{z} \in (\mathbb{C}^*)^{I^c_F}$. Moreover, if $A\underline{z}$ is nonclosed, then it contains one and only one closed A-orbit.*

Theorem 2.1, which holds in the rational setting too, allows us to define on the open set $\mathbb{C}^d_\Delta$ the following equivalence relation: two points $\underline{z}$ and $\underline{w}$ are equivalent if and only if

$$\big(N(\overline{A\underline{z}})\big) \cap (\overline{A\underline{w}}) \neq \varnothing,$$

where the closure is meant in $\mathbb{C}^d_\Delta$. We define the space X_Δ to be the quotient of $\mathbb{C}^d_\Delta$ by the equivalence relation just defined, we denote the quotient by $X_\Delta = \mathbb{C}^d_\Delta /\!/ N_\mathbb{C}$. Notice that if the polytope is simple, then $\mathbb{C}^d_\Delta = \bigcup_{F\in\Delta}(\mathbb{C}^*)^{I^c_F}$ and the quotient X_Δ is just the geometric quotient, whilst if the polytope is nonsimple and rational, the quotient X_Δ is the known categorical quotient.

3. The Stratification The decomposition in pieces of the quotient X_Δ reflects the geometry of the polytope Δ. A partial order on the set of all faces of Δ is defined by setting $F \leq F'$ if $F \subseteq \overline{F'}$. The polytope Δ is the disjoint union of its faces. Let F be a p-dimensional face and let $r_F = \text{card}(I_F)$, clearly $r_F \geq n-p$. When the equality holds the face F is said to be *regular*, otherwise *singular*. Consider $\mathfrak{d}_F = \text{Span}\{X_j \mid j \in I_F\}$, the natural injection $j_F\colon \mathfrak{d}_F \hookrightarrow \mathfrak{d}$ and the subset $\Sigma^\diamond_F = \bigcap_{j\in I_F}\{\xi \in \mathfrak{d}^* \mid \langle \xi, X_j\rangle \geq \lambda_j\}$. Then $j^*_F(\Sigma^\diamond_F) = \Sigma_F$ is a polyhedral cone with vertex $j^*_F(F)$. By cutting this cone with an affine hyperplane transversal to its codimension 1 faces, we obtain an $(n-p-1)$-dimensional polytope that we call Δ_F, which of course depends on the choice of the hyperplane. Each q-dimensional face G of Δ greater than F gives a face in Δ_F of dimension $q-p-1$, singular if and only if G is singular in Δ.

Near to F the polytope Δ is the product of F by a cone over Δ_F, and this is exactly the stratified structure of the toric space that we have constructed. The maximal stratum is $\mathcal{T}_{\max} = \bigcup_{F\,\mathrm{reg}}((\mathbb{C}^*)^{I_F^c})/N_{\mathbb{C}}$, whilst there is a piece $\mathcal{T}_F$ for each singular face F of Δ, which can be identified with the orbit space $\mathcal{T}_F = (\mathbb{C}^*)^{I_F^c}/N_{\mathbb{C}}$. Theorem 2.1 implies that the space X_Δ is given by the union of the strata just defined. The structure of X_Δ as decomposed space and the properties that characterize X_Δ as toric space associated to Δ are given in the following statement:

THEOREM 3.1. *The subset $\mathcal{T}_F$ of X_Δ corresponding to each p-dimensional singular face of Δ is a p-dimensional complex quasifold. The open subset $\mathcal{T}_{\max}$ is an n-dimensional complex quasifold. These subsets give a decomposition by complex quasifolds of X_Δ. Moreover there is a continuous action of $D_{\mathbb{C}}$ on X_Δ, with the dense open orbit $(\mathbb{C}^*)^d/N_{\mathbb{C}}$. Such action is holomorphic when restricted to pieces.*

The maximal stratum is said to be the regular stratum, it is the analogue of the open set of rationally smooth points in the rational case, whilst the strata corresponding to singular faces are singular (see Lemma 3.2).

Now consider the cone Σ_F in the space $\mathfrak{d}_F^*$, with induced normal vectors $X_j \in \mathfrak{d}_F$, $j \in I_F$, and quasilattice $Q \cap \mathfrak{d}_F$. Then let $X_0 = \sum_{j=1}^d s_j X_j$, with $s_j \in (0,1)$ suitably chosen for $j \in I_F$ and $s_j = 0$ for $j \notin I_F$. We consider X_0 in $\mathfrak{d}_F$ and we denote by $\mathrm{ann}(X_0)$ the annihilator of X_0. We can view Δ_F as a convex polytope lying in the linear hyperplane $\mathrm{ann}(X_0) \overset{k_F}{\hookrightarrow} \mathfrak{d}_F^*$, with induced normal vectors $k_F^*(X_j)$ in $\mathrm{ann}(X_0)^* \simeq \mathfrak{d}_F/\langle X_0\rangle$, $j \in I_F$, and quasilattice $k_F^*(Q \cap \mathfrak{d}_F)$. The spaces corresponding to the convex sets Σ_F and Δ_F are $X_{\Sigma_F} = \mathbb{C}^{I_F}/\!/N_{\mathbb{C}}^F$ and $X_{\Delta_F} = \mathbb{C}^{I_F}_{\Delta_F}/\!/(N_0^F)_{\mathbb{C}}$, where the $(r_F - n + p)$-dimensional group $N_{\mathbb{C}}^F$ is given by $N_{\mathbb{C}} \cap (\mathbb{C}^*)^{I_F}$ and $(N_0^F)_{\mathbb{C}}/N_{\mathbb{C}}^F \simeq \exp(\mathfrak{s} + i\mathfrak{s})$, with $\mathfrak{s} = \mathrm{Span}\{(s_1, \ldots, s_d)\}$. We first prove that the stratification satisfies the local triviality condition.

LEMMA 3.2. *Let F be a singular face, then the singular stratum $\mathcal{T}_F$ can be identified with $(\mathbb{C}^*)^p/\Gamma_F$, where Γ_F is a finitely generated subgroup of $(\mathbb{C}^*)^p$ acting on X_{Σ_F} and freely on $(\mathbb{C}^*)^p$. There is a mapping from $(\mathbb{C}^*)^p \times X_{\Sigma_F}/\Gamma_F$ onto the open subset $(\mathbb{C}^{I_F} \times (\mathbb{C}^*)^{I_F{}^c})/\!/N_{\mathbb{C}}$ of X_Δ, which is a homeomorphism and a biholomorphism when restricted to the pieces of the respective decompositions.*

Now, in order to complete the proof that the space X_Δ is a stratified space, it remains to show that:

($*$) for each singular face F in Δ there is a link L_F which satisfies the definition. Moreover, the cone X_{Σ_F}, the link L_F and the toric space $Y_F = X_{\Delta_F}$ have the properties required for a stratification to be complex.

General results, for example on bundles, are often not readily applicable to our spaces because of their topology, therefore, although a description of the spaces in statement ($*$) can be given within our set up (see the first row of diagram (2)), we make use of the interplay with the symplectic quotients in order to give a neat description of the link L_F and of X_{Σ_F} as a real cone over it. Hence, before going on to describe the cone X_{Σ_F}, let us briefly recall from [B1] the symplectic construction. Let $\Psi_\Delta^{-1}\colon \mathbb{C}^d \to \mathfrak{n}^*$ be the moment mapping with respect to the N-action such that $\Psi_\Delta(0) = \sum_{j=1}^d \lambda_j \iota^*(e_j^*)$, where $\iota\colon \mathfrak{n} \to \mathbb{R}^d$ is the inclusion. The quotient $M_\Delta = \Psi_\Delta^{-1}(0)/N$ is endowed with a symplectic stratification by quasifolds and is acted on effectively by the quasitorus $D = \mathfrak{d}/Q$. Moreover, there is a continuous mapping $\Phi\colon M_\Delta \to \mathfrak{d}^*$ whose restriction to each stratum, with its symplectic structure, is a moment mapping with respect to the action of D; the image of Φ is exactly Δ. The inclusion $\Psi_\Delta^{-1}(0) \hookrightarrow \mathbb{C}^d$ induces a continuous mapping $\chi_\Delta\colon \Psi_\Delta^{-1}(0)/N \to \mathbb{C}^d_\Delta/\!/N_\mathbb{C}$. It was proved in [BP1] that when the polytope is *simple*, the mapping χ_Δ is a diffeomorphism inducing a Kähler structure on X_Δ. For any convex polytope, it remains to prove the following.

($\diamond$) The mapping χ_Δ identifies the symplectic and complex quotients as stratified spaces.

Statements ($*$) and ($\diamond$) are strictly intertwined and they can be proved together by induction on the depth of the polytope Δ, which is the maximum length that a chain of singular faces can attain in Δ. The key diagram is the following:

$$
\begin{tikzcd}
\mathbb{C}^{I_F}/\!/(N^F_\mathbb{C})\setminus\{[0]\} \arrow[r, "q_2"] \arrow[rr, bend left, "s"] \arrow[dr, phantom, "2"] & \mathbb{C}^{I_F}_{\Delta_F}/\!/N^F_\mathbb{C}\exp(i\mathfrak{s}) \arrow[r, "q_1"] \arrow[dr, phantom, "1"] & \mathbb{C}^{I_F}_{\Delta_F}/\!/N^F_{0\,\mathbb{C}} \\
(\Psi^{-1}_{\Sigma_F}(0)/N^F)\setminus\{[0]\} \arrow[u, "\chi^2_F"] \arrow[r, "p_2"] \arrow[rr, bend right, "s'"] & (\Psi_{\Delta_F})^{-1}(0)/N^F \arrow[u, "\chi^1_F"] \arrow[r, "p_1"] & (\Psi_{\Delta_F})^{-1}(0)/N^F_0 \arrow[u, "\chi^0_F"]
\end{tikzcd}
\tag{2}
$$

The mappings χ^j_F are all diffeomorphisms of decomposed spaces, both diagrams commute, the projections q_2, p_2 allow us to identify the space X_{Σ_F} with a real cone over the link L_F, which is the space in the mid column. The projections p_1, q_1 are fibrations of the link L_F over the compact Kähler space X_{Δ_F}, with fibre $\exp(\mathfrak{s})$, the projections s, s' are fibrations of the complex space $X_{\Sigma_F}\setminus\{\text{cone pt}\}$ over the compact space X_{Δ_F}, with fibre $\exp(\mathfrak{s}+i\mathfrak{s})$. All mappings are natural, preserve the decompositions and the structure of the strata. We finally state our main result.

Theorem 3.3. *Let $\mathfrak{d}$ be a vector space of dimension n, and let $\Delta \subset \mathfrak{d}^*$ be a convex polytope. Choose inward-pointing normals to the facets of Δ, $X_1, \ldots, X_d \in \mathfrak{d}$, and let Q be a quasilattice containing them. The corresponding quotient $X_\Delta = \mathbb{C}^d_\Delta/\!/N_\mathbb{C}$ is a complex stratified space. The mapping $\chi_\Delta\colon \Psi_\Delta^{-1}(0)/N \to \mathbb{C}^d_\Delta/N_\mathbb{C}$ is an equivariant homeomorphism whose restriction* *to each stratum is a diffeomorphism of quasifolds, with respect to which the symplectic and complex structure are compatible, so that strata have the structure of Kähler quasifolds.*

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