C. R. Math. Rep. Acad. Sci. Canada Vol. **28** (3) 2006, pp. 65–84

A GEOMETRIC BOSON-FERMION CORRESPONDENCE

ALISTAIR SAVAGE

Presented by Ed Bierstone, FRSC

ABSTRACT. The fixed points of a natural torus action on the Hilbert schemes of points in $\mathbb{C}^2$ are quiver varieties of type A_∞. The equivariant cohomology of the Hilbert schemes and quiver varieties can be given the structure of bosonic and fermionic Fock spaces respectively. Then the localization theorem, which relates the equivariant cohomology of a space with that of its fixed point set, yields a geometric realization of the important boson-fermion correspondence.

RÉSUMÉ. Les points fixes d'une action canonique d'un tore sur le schéma de Hilbert de $\mathbb{C}^2$ sont des variétés de quiver de type A_∞. On peut donner la cohomologie équivariante des schémas de Hilbert et des variétés de quiver la structure des éspaces de Fock fermionique et bosonique, respectivement. Alors, la théorème de localisation, qui lie la cohomologie équivariante d'une éspace avec la cohomologie équivariante de son ensemble des point fixes, nous permet de donner une réalisation géométrique de la correspondance bosonique-fermionique.

Introduction. Recently there has been substantial interest in geometric constructions in representation theory. Such constructions translate between purely algebraic representation theoretic statements and statements involving geometric objects such as flag varieties, affine Grassmannians, quiver varieties and Hilbert schemes. This often provides one with new geometric techniques to examine various topics in representation theory (such as in the proof of the Kazhdan–Lusztig conjecture) as well as representation theoretic tools to organize and study the structure of various geometric objects.

In this paper, we will focus on two particular geometric constructions. One of these involves varieties associated to quivers. Quivers, which are simply directed graphs, and their representations have a long history (see [21]). Lusztig [14] associated certain varieties to quivers and used these to provide a geometric realization of half of the universal enveloping algebra (or its quantum analogue) of Kac–Moody algebras. Then Nakajima [17, 19] modified these quiver varieties and gave a geometric construction of the representations of these algebras. The underlying vector space of the representation is the homology of the quiver varieties and the action of the Kac–Moody algebra is given by certain *correspondences* in products of these varieties. Nakajima's construction was motivated by his work with Kronheimer on solutions to the anti-self-dual Yang–Mills equations

Received by the editors on September 7, 2006.

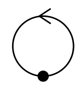
This research was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada

AMS subject classification: Primary: 14C05,17B69; secondary: 55N91.

Keywords: Boson-fermion correspondence, vertex algebras, affine Lie algebras, quivers, quiver varieties, Hilbert schemes, equivariant cohomology.

on ALE gravitational instantons [10].

The second geometric construction we consider in this paper was developed by Nakajima [20] and Grojnowski [7]. It realizes irreducible representations of infinite-dimensional Heisenberg algebras in the (co)homology of the Hilbert schemes of points on surfaces. This is very similar to the quiver variety picture. In fact, one can view the Hilbert schemes as quiver varieties associated to the so-called Jordan quiver.



These geometric realizations have proven to be very useful. In particular, they provide us with remarkable bases for the corresponding algebraic objects (the so-called canonical and semicanonical bases) that have very nice positivity, integrality and compatibility properties. They also have beautiful connections to other areas of mathematics such as the theory of crystals. Their discovery has opened up a fruitful line of research: to extend the geometric viewpoint by developing geometric constructions of other algebraic results. This approach has been successful in several areas including Weyl group actions, representations of the Virasoro algebra, Demazure modules, and Clifford algebras. In the current paper, we continue along this path and provide a geometric construction of the so called *boson-fermion correspondence*.

The boson-fermion correspondence is of fundamental importance in mathematical physics. In physics, the terms *boson* and *fermion* refer to particles of integer and half-integer spin, respectively. *Bosonic* and *fermionic Fock space*, which can be thought of as certain state spaces of bosons and fermions, are mathematically defined to be representations of an infinite-dimensional Heisenberg or oscillator algebra and an infinite-dimensional Clifford algebra (and the Lie algebra sl_∞ or gl_∞), respectively. The boson-fermion correspondence describes a precise relationship between these two spaces. It plays an important role in the theory of vertex operators and the basic representation of affine Lie algebras (see [3, 5] and references therein).

We will use the constructions discussed above to describe a realization of the boson-fermion correspondence using the geometry of Hilbert schemes and quiver varieties. Let X_n denote the Hilbert scheme of n points in $\mathbb{C}^2$ (see Section 4). There is a natural torus action on X_n and one can consider the associated equivariant cohomology on which there is an action of an infinite-dimensional Heisenberg algebra. The generators of the algebra act by “adding or removing points” along the x-axis in $\mathbb{C}^2$. This yields the geometric realization of bosonic Fock space. That of the fermionic Fock space is obtained by considering the quiver varieties corresponding to the basic representation of sl_∞. In this case the varieties are simply points which, as was shown in [6], can be naturally enumerated by Young diagrams.

We will see that the torus fixed points of the Hilbert schemes are naturally identified with the sl_∞ quiver varieties. The localization theorem states that under certain assumptions the equivariant cohomology of a space with the action of a torus T is isomorphic to the equivariant cohomology of its T-fixed points. Thus, the localization theorem yields an isomorphism between the geometric constructions of the bosonic and fermionic Fock spaces and a geometric boson-fermion correspondence. We then see that the bosons correspond to "global" additions of points while the fermions corresponds to "local" operators. That is, the bosons naturally act on the equivariant cohomology of the entire Hilbert scheme while the fermions naturally act at its torus fixed points.

We note that in [13] Li, Qin, and Wang found that the *multi-point trace function*, a generating function of intersection numbers of equivariant Chern characters in spaces isomorphic to the Hilbert schemes mentioned above, is related in a simple way to the characters of the fermionic Fock space when the equivariant cohomology of these spaces (with the natural geometric structure of bosonic Fock space) is identified with fermionic Fock space via the boson-fermion correspondence.

As mentioned above, the geometric constructions of the two representations considered in this paper are not new. The goal here is rather to examine the interplay between them. The important step is the identification of the quiver varieties for sl_∞ with the torus-fixed points of the Hilbert scheme. We believe this point of view to be important for two reasons. First of all, it allows one to identify a fundamental concept in equivariant cohomology, the localization theorem, with an important concept in mathematical physics, the boson-fermion correspondence. Secondly, we expect this idea to lead to other interesting results. For example, if one considers a finite cyclic subgroup Γ of the torus, the Γ-fixed points of the Hilbert scheme can be naturally identified with affine quiver varieties of type A. Thus, we expect that an extension to this setting of the ideas presented here will lead to new geometric interpretations of the vertex operator construction of representations of affine Lie algebras (see [7] for some results in this direction) and perhaps explicit algebraic descriptions in the vertex operator framework of the natural bases coming from the geometric picture. We see the current paper as an important first step in this direction.

The organization of the paper is as follows. In Sections 1 and 2 we review the boson-fermion correspondence and the localization theorem in equivariant cohomology. In Section 3 we describe the geometric construction of fermionic Fock space using quiver varieties and in Section 4 we describe the torus fixed points of the Hilbert scheme and their identification with quiver varieties. Then in Section 5 we recall the geometric realization of bosonic Fock space using Hilbert schemes. Finally, in Section 6 we state the geometric boson-fermion correspondence.

1. The boson-fermion correspondence.

The *boson-fermion correspondence* is an isomorphism between two representations of an infinite-dimensional Heisenberg algebra (or the closely related *oscillator algebra*). These representations are on the bosonic and fermionic Fock spaces. In this section we recall the two representations and the isomorphism between them. For further details we refer the reader to [9, §§ 14.9–14.10].

Define the *oscillator algebra* to be the Lie algebra

$$\mathfrak{s} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} s_m \oplus \mathbb{C} K$$

with commutation relations

$$[\mathfrak{s}, K] = 0, \quad [s_m, s_n] = m \delta_{m,-n} K. \tag{1.1}$$

The subalgebra spanned by s_n, $n \neq 0$, and K is an infinite-dimensional Heisenberg algebra. The oscillator algebra has a natural representation on the *full bosonic Fock space*

$$B = \mathbb{C}[p_1, p_2, \ldots; q, q^{-1}],$$

a polynomial algebra on indeterminates $p_1, p_2, \ldots$ and q, q^{-1}. Physically, this space can be thought of as a certain state space of bosons (particles of integer spin). The indeterminate p_k represents a particle in state k. Note that more than one boson can occupy the same state.

The representation r^B of the oscillator algebra $\mathfrak{s}$ on B is given by

$$r^B(s_m) = m \frac{\partial}{\partial p_m}, \quad r^B(s_{-m}) = p_m \text{ for } m > 0,$$
$$r^B(s_0) = q \frac{\partial}{\partial q}, \quad r^B(K) = 1.$$

The operators s_{-m} and s_m can be thought of as creation and annihilation operators respectively.

We next describe another representation of the oscillator algebra. An infinite expression of the form

$$\underline{i_0} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \cdots,$$

where $i_0, i_1, \ldots$ are integers satisfying

$$i_0 > i_1 > i_2 > \cdots, \; i_n = i_{n-1} - 1 \text{ for } n \gg 0,$$

is called a *semi-infinite monomial*. Let F be the complex vector space with basis consisting of all semi-infinite monomials, and let $H(\cdot, \cdot)$ denote the Hermitian form on F for which this basis is orthonormal. F is called the *full fermionic Fock space*. Physically, it can be thought of as a certain state space of fermions (particles of half-integer spin), with the integers appearing in a semi-infinite monomial labeling the various states. Note that the fermions satisfy the *Pauli exclusion principle*: no two particles can occupy the same state. Let

$$|m\rangle = \underline{m} \wedge \underline{m-1} \wedge \underline{m-2} \wedge \cdots$$

be the *vacuum vector* of charge m. We say that a semi-infinite monomial has *charge* m if it differs from $|m\rangle$ at only a finite number of places. Thus $\varphi = \underline{i_0} \wedge \underline{i_1} \wedge \cdots$ is of charge m if $i_k = m - k$ for $k \gg 0$. Let $F^{(m)}$ denote the linear span of all semi-infinite monomials of charge m. Then we have the charge decomposition

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}.$$

We call $F^{(0)}$ *fermionic Fock space.*

To any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ we associate a semi-infinite monomial $\varphi_\lambda = \underline{i_0} \wedge \underline{i_1} \wedge \cdots$ of charge m, by letting $i_k = (m-k) + \lambda_k$. This gives a bijection between the set of all semi-infinite monomials of a fixed charge m and the set $\mathcal{P}$ of all partitions (finite non-increasing sequences of non-negative integers). We define the *energy* of φ_λ to be $|\lambda| := \sum_i \lambda_i$, the size of the partition λ. Let $F_j^{(m)}$ denote the linear span of all semi-infinite monomials of charge m and energy j. We then have the energy decomposition

$$F^{(m)} = \sum_{j \in \mathbb{Z}} F_j^{(m)}.$$

For $j \in \mathbb{Z}$, define the *wedging* and *contracting* operators ψ_j and ψ_j^* on F by:

$$\psi_j(\underline{i_0} \wedge \underline{i_1} \wedge \cdots) = \begin{cases} 0 & \text{if } j = i_s \text{ for some } s, \\ (-1)^{s+1} \underline{i_0} \wedge \cdots \wedge \underline{i_s} \wedge \underline{j} \wedge \underline{i_{s+1}} \wedge \cdots & \text{if } i_s > j > i_{s+1}. \end{cases}$$

$$\psi_j^*(\underline{i_0} \wedge \underline{i_1} \wedge \cdots) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s, \\ (-1)^s \underline{i_0} \wedge \underline{i_1} \wedge \cdots \wedge \underline{i_{s-1}} \wedge \underline{i_{s+1}} \wedge \cdots & \text{if } j = i_s. \end{cases}$$

These can be thought of as creation and annihilation operators. The operator ψ_j creates a particle in state j while the operator ψ_j^* annihilates a particle in state j. Note that

$$\psi_j(F^{(m)}) \subseteq F^{(m+1)}, \quad \psi_j^*(F^{(m)}) \subseteq F^{(m-1)}.$$

The operators ψ_j and ψ_j^* are called *free fermions.* One can check directly that ψ_j and ψ_j^* are adjoint with respect to the Hermitian form $H(\,\cdot\,,\,\cdot\,)$ and that the following relations hold:

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}, \quad \psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0. \tag{1.2}$$

Thus, the operators ψ_j and ψ_j^* generate a Clifford algebra Cl. It is easily seen that F is an irreducible Cl-module and that

$$\psi_j \, |m\rangle = 0 \text{ for } j \leq m, \quad \psi_j^* \, |m\rangle = 0 \text{ for } j > m.$$

Let gl_∞ denote the Lie algebra of all complex (infinite) matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ such that the number of nonzero a_{ij} is finite, with the usual commutator bracket, and let

$$sl_\infty = \{a \in gl_\infty \mid \operatorname{tr} a = 0\}.$$

Let $E_{ij} \in gl_\infty$ denote the matrix with (i,j)-entry equal to one and all other entries equal to zero.

There is an embedding $gl_\infty \to \mathrm{Cl}$ defined by

$$r(E_{ij}) = \psi_i \psi_j^*$$

and this defines a representation r of gl_∞ on F. It is easy to see that each $F^{(m)}$ is stable under the action of gl_∞ and thus r restricts to a representation r_m of gl_∞ on $F^{(m)}$ for each $m \in \mathbb{Z}$. Note that $r({}^t\bar{a})$ and $r(a)$, for $a \in gl_\infty$, are adjoint operators with respect to the Hermitian form H.

One can check that gl_∞ acts by derivations on F. That is, for $a = (a_{ij}) \in gl_\infty$,

$$r(a)(\underline{i_0} \wedge \underline{i_1} \wedge \cdots) = (a \cdot \underline{i_0}) \wedge \underline{i_1} \wedge \cdots + \underline{i_0} \wedge (a \cdot \underline{i_1}) \wedge \cdots, \tag{1.3}$$

where we view $\underline{j}$ as the vector with j-th component one and all other components zero. Thus $a \cdot \underline{j} = \sum_i a_{ij} \underline{i}$. One can then use the usual rules of exterior algebra to express the right-hand side of (1.3) in terms of semi-infinite monomials. Thus F is an infinite generalization of the usual exterior algebra. For this reason, r is often called the *infinite wedge representation.*

The representations r_m are irreducible. Also,

$$r_m(E_{ij})\,|m\rangle = \begin{cases} 0 & \text{if } i < j \text{ or } i = j > m, \\ |m\rangle & \text{if } j \le m. \end{cases}$$

Thus, as an sl_∞-module, $F^{(m)}$ is isomorphic to the irreducible integral representation $L(\omega_m)$ of highest weight ω_m, where ω_m is the fundamental weight defined by $\langle \omega_m, \alpha_j^\vee \rangle = \delta_{mj}$. Here the $\alpha_j^\vee$ are the simple coroots and $\langle \cdot\,,\,\cdot \rangle$ is the usual pairing between weights and coweights. The representation r_0 of gl_∞ (or sl_∞) on $F^{(0)}$ is called the *basic representation.*

We can define an $\mathfrak{s}$-module structure on $F^{(m)}$ by introducing the *free bosons*

$$\alpha_n = \sum_{j\in\mathbb{Z}} \psi_j \psi_{j+n}^*, \; n \in \mathbb{Z}\backslash\{0\},$$

$$\alpha_0 = \sum_{j>0} \psi_j \psi_j^* - \sum_{j\le 0} \psi_j^* \psi_j.$$

Note that while the sums involved in the above definitions are infinite, all but a finite number of them act as zero on any semi-infinite monomial, and so the operations are well-defined.

Proposition 1.1. ([9, Proposition 14.9]) *The map $s_n \mapsto \alpha_n$, $n \in \mathbb{Z}$, $K \mapsto \mathrm{Id}$ defines an $\mathfrak{s}$-module structure on $F^{(m)}$, and $F^{(m)}$ is irreducible for all $m \in \mathbb{Z}$.*

Note that

$$\alpha_0|_{F^{(m)}} = m\mathbf{1}$$

and that α_m and α_{-m} are adjoint operators.

It follows from Proposition 1.1 and a uniqueness property for representations of $\mathfrak{s}$ (see [9, Corollary 9.13]) that there is a unique isomorphism of $\mathfrak{s}$-modules

$$\sigma\colon F \xrightarrow{\cong} B,$$

such that $\sigma(|m\rangle) = q^m$. Note that

$$\sigma(F^{(m)}) = B^{(m)} := q^m \mathbb{C}[p_1, p_2, \dots].$$

We will denote this restriction of σ to $F^{(m)}$ by σ_m. Also, we call $B^{(0)}$ *bosonic Fock space*.

For a partition $\lambda \in \mathcal{P}$, we recall the definition of the *Schur polynomial* S_λ. First, one defines the elementary Schur polynomials S_n by

$$S_n = 0 \text{ for } n < 0, \quad S_0 = 1,$$

$$S_n = \sum_{\mu \vdash n} \frac{1}{z_\lambda} p_1^{m_1(\mu)} p_2^{m_2(\mu)} \dots \text{ for } n > 0.$$

Here, $\mu \vdash n$ means that μ is a partition of size n. Recall that $m_i(\mu)$ is the number of parts of μ equal to i. Then for $\lambda = (\lambda_1 \le \lambda_2 \le \cdots) \in \mathcal{P}$ define

$$S_\lambda = \det(S_{\lambda_i + j - i})_{1 \le i,j \le |\lambda|}.$$

Note that to translate to the definition of Schur polynomials in the ring of symmetric functions in variables $x_1, x_2, \dots$ (see [15]), one should replace p_i with the i-th power sum.

If $\varphi_\lambda \in F^{(m)}$ is a semi-infinite monomial, then (see [9, Theorem 14.10])

$$\sigma(\varphi_\lambda) = q^m S_\lambda.$$

The process we have just outlined, constructing bosons in terms of fermions acting on full fermionic Fock space, is called *bosonization*. There exists an opposite procedure, *fermionization*, which consists of constructing fermions in terms of bosons acting on full bosonic Fock space. This task is somewhat more complicated, involving vertex operator algebras. Since we do not need fermionization in the current paper, we will not describe the procedure here, but instead refer the reader to [9, §§ 14.9-14.10].

2. Equivariant cohomology and localization. In this section, we recall the definition of equivariant cohomology and the localization theorem. This will be our main tool in relating the geometric realizations of the bosonic and fermionic Fock spaces, thus yielding a geometric boson-fermion correspondence. We concentrate on the case where the group is the torus, and follow the presentation in [2].

Let $T = \mathbb{C}^*$ be the one-dimensional torus. For an algebraic variety X equipped with a T-action, let $H_T^*(X)$ denote the equivariant cohomology ring of X with complex coefficients. We recall the definition. Let $BT = \mathbb{CP}^\infty$ and ET be the tautological bundle on $\mathbb{CP}^\infty$. Set $X_T = X \times_T ET$. This is a bundle over BT with fiber X. Then, by definition,

$$H_T^*(X) = H^*(X_T),$$

where $H^*(X_T)$ is the ordinary cohomology of X_T.

Recall that we have flat equivariant pullbacks and proper equivariant pushforwards in equivariant cohomology. If pt is the space consisting of a single point with the trivial T-action, then $H_T^*(\mathrm{pt}) = \mathbb{C}[t]$ where t is an element of degree 2. Thus, by the pullback via $M \to \mathrm{pt}$, $H_T^*(X)$ has the structure of a $\mathbb{C}[t]$-module. For a proper T-equivariant morphism $f\colon Y \to X$ of algebraic varieties, we have a Gysin map $f_!\colon H_T^*(Y) \to H_T^*(X)$. If Y is a T-equivariant codimension-k closed subvariety of X and $i\colon Y \hookrightarrow X$ is the inclusion map, we define

$$[Y] = i_!(1_Y) \in H_T^{2k}(X),$$

where $1_Y \in H_T^0(Y)$ is the unit in $H_T^*(Y)$

An *equivariant vector bundle* is a vector bundle E over X such that the action of T on X lifts to an action of E which is linear on the fibers. Then E_T is a vector bundle over X_T and the *equivariant Chern classes* $c_k^T(E) \in H_T^*(X)$ are defined to be the ordinary Chern classes $c_k(E_T)$. If E has rank r, then the top Chern class $c_r^T(E)$ is called the *equivariant Euler class* of E and is denoted $e_T(E) \in H_T^*(X)$.

Very important in our discussion will be the *localization theorem* which we now describe. Suppose that X is smooth and has a T-action. Then the fixed point locus X^T is a union of smooth connected components Z_j. Let $i_j\colon Z_j \hookrightarrow X$ be the inclusion and let N_j denote the normal bundle of Z_j in X. Then N_j is an equivariant vector bundle and thus has an equivariant Euler class

$$e_T(N_j) \in H_T^*(Z_j).$$

The equivariant inclusion $i_j\colon Z_j \to X$ induces the pullback map $i_j^*\colon H_T^*(X) \to H_T^*(Z_j)$. The Gysin map

$$i_{j!}\colon H_T^*(Z_j) \to H_T^*(X)$$

has the property that for any $\alpha \in H_T^*(Z_j)$,

$$i_j^* \circ i_{j!}(\alpha) = \alpha \cup e_T(N_j).$$

Let $\mathbb{C}(t)$ be the field of fractions of $\mathbb{C}[t]$ and form the *localization* $H^*_T(Z_j)\otimes\mathbb{C}(t)$. Then $e_T(N_j)$ is invertible in $H^*_T(Z_j)\otimes\mathbb{C}(t)$. The following proposition is referred to as the *localization theorem*.

PROPOSITION 2.1. ([2, Proposition 9.1.2],[1]) *There is an isomorphism*

$$H^*_T(X)\otimes\mathbb{C}(t) \xrightarrow{\simeq} \bigoplus_j H^*_T(Z_j)\otimes\mathbb{C}(t)$$

given by $\alpha \mapsto (i_j^*(\alpha)/e_T(N_j))_j$. *The inverse map is given by*

$$(\alpha_j)_j \mapsto \sum_j i_{j!}(\alpha_j).$$

In particular, for any $\alpha \in H^*_T(X)\otimes\mathbb{C}(t)$, *we have*

$$\alpha = \sum_j i_{j!}\Big(\frac{i_j^*(\alpha)}{e_T(N_j)}\Big).$$

3. Geometric realization of fermionic Fock space. In this section we describe a geometric realization of fermionic Fock space using the quiver varieties of Nakajima and the results of [6]. We only introduce here the special case of quiver varieties corresponding to the basic representation $L(\omega_0)$ of the Lie algebra sl_∞ of type A_∞. In this case, the quiver varieties are simply points. However, the reader should keep in mind the fact that the construction generalizes to irreducible integrable representations of symmetric Kac–Moody algebras (see [17, 19]). Note that we use a different stability condition than the one used in [17, 19] and so our definitions differ slightly from the ones that appear there. One can translate between the two stability conditions by taking transposes of the maps appearing in the definitions of the quiver varieties. See [18] for a discussion of various choices of stability condition. Another difference in our presentation below is that we use equivariant cohomology rather than ordinary cohomology (or Borel–Moore homology). Since the varieties involved are points, this change is minor. However, we will need this formulation to connect the quiver variety picture to the geometric construction of the bosonic Fock space described later.

Let $V=\bigoplus_{k\in\mathbb{Z}} V_k$ be a finite dimensional complex $\mathbb{Z}$-graded vector space of graded dimension $\mathbf{v}=(\dim V_k)_{k\in\mathbb{Z}}$. Then we define $\mathbf{M}(\mathbf{v})$ to be the set of all triples $(B_1,B_{-1},\tilde{v})$ where $\tilde{v}\in V_0$ and B_1 and B_{-1} are endomorphisms of the graded vector space V of degrees 1 and -1, respectively, satisfying

$$[B_1,B_{-1}] = B_1B_{-1}-B_{-1}B_1 = 0.$$

Now, let

$$G_{\mathbf{v}} = \prod_{k\in I} GL(V_k)$$

b_{00}	b_{10}	b_{20}	b_{30}
b_{01}	b_{11}	b_{21}	b_{31}
b_{02}	b_{12}	b_{22}	
b_{03}			

Figure 1: The labels of the boxes of the Young diagram $D_{(4,4,3,1)}$.

be the group of grading-preserving automorphisms of V. Then we have a natural action of $G_{\mathbf{v}}$ on $\mathbf{M}(\mathbf{v})$ given by

$$g \cdot (B_1, B_{-1}, \tilde{v}) = (gB_1g^{-1}, gB_{-1}g^{-1}, g(\tilde{v})).$$

We say that a graded subspace S of V is (B_1, B_{-1})-*invariant* if $B_1(S) \subset S$ and $B_{-1}(S) \subset S$. We say that a point $(B_1, B_{-1}, \tilde{v})$ of $\mathbf{M}(\mathbf{v})$ is *stable* if any (B_1, B_{-1})-invariant graded subspace S of V containing $\tilde{v}$ is equal to all of V.

We let $\mathbf{M}(\mathbf{v})^s$ denote the set of all stable points. It is known (see [19, Lemma 3.10]) that the stabilizer in $G_{\mathbf{v}}$ of any point in $\mathbf{M}(\mathbf{v})^s$ is trivial, and we define the *quiver variety*,

$$\mathfrak{M}(\mathbf{v}) = \mathbf{M}(\mathbf{v})^s / G_{\mathbf{v}}.$$

This is a geometric quotient (it can also be viewed as a symplectic quotient, although we do not discuss the symplectic structure here). For $(B_1, B_{-1}, \tilde{v}) \in \mathbf{M}(\mathbf{v})^s$, we denote the corresponding orbit in $\mathfrak{M}(\mathbf{v})$ by $[B_1, B_{-1}, \tilde{v}]$.

For a partition $\lambda \in \mathcal{P}$, let D_λ denote the Young diagram corresponding to λ. We view D_λ as a collection of boxes, with $b_{jk} \in D_\lambda$ denoting the box in the j-th column and k-th row where we start numbering from zero. For example, the labels of the boxes of the Young diagram $D_{(4,4,3,1)}$ are as in Figure 1.

We define the *residue* of a box b_{jk} to be $j - k$. Define V_k^λ to be the $\mathbb{C}$-span of the boxes in D_λ of residue k. Then $\dim V^\lambda = \mathbf{v}^\lambda$ where v_k^λ is the number of boxes in D_λ of residue k. Define an element $(B_1^\lambda, B_{-1}^\lambda, \tilde{v}^\lambda)$ of $M(\mathbf{v})$ by

$$\begin{aligned} B_1^\lambda(b_{j,k}) &= b_{j+1,k},\ \ j,k \in \mathbb{Z}_{\geq 0}, \\ B_{-1}^\lambda(b_{j,k}) &= b_{j,k+1},\ \ j,k \in \mathbb{Z}_{\geq 0}, \\ \tilde{v}^\lambda &= b_{00}, \end{aligned}$$

where $b_{jk} = 0$ if $b_{jk} \notin D_\lambda$. We picture $(B_1^\lambda, B_{-1}^\lambda, \tilde{v}^\lambda)$ as in Figure 2.

In was shown in [6] that $\mathfrak{M}(\mathbf{v})$ is empty unless $\mathbf{v} = \mathbf{v}^\lambda$ for some $\lambda \in \mathcal{P}$. Also, $\mathfrak{M}(\mathbf{v}^\lambda) = [B_1^\lambda, B_{-1}^\lambda, \tilde{v}^\lambda]$ is a single point. This follows from the fact that, in general, the quiver variety $\mathfrak{M}(\mathbf{v})$ associated to a Kac–Moody algebra with symmetric Cartan matrix C is connected (see [19, Theorem 6.2]) and its dimension is given by (see [19, Corollary 3.12])

$$\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}) = 2v_0 - \mathbf{v} \cdot C\mathbf{v}.$$

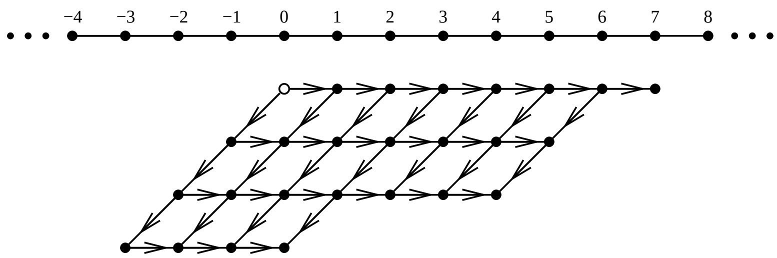


Figure 2: A pictorial representation of $(B_1^\lambda, B_{-1}^\lambda, \tilde{v}^\lambda)$ for $\lambda = (8,7,7,4)$. The top line is the Dynkin graph of type A_∞. The vertices below represent the boxes $b_{jk} \in D_\lambda$, while the arrows represent the actions of B_1^λ and B_{-1}^λ. The vector space V_k^λ is spanned by the vertices directly beneath the vertex k in the Dynkin diagram and the vector $\tilde{v}^\lambda = b_{00}$ is indicated by a hollow vertex.

For C the Cartan matrix of type A_∞ (*i.e.*, $C_{ij} = 2\delta_{ij} - \delta_{i-1,j} - \delta_{i+1,j}$), one can see from this formula that $\dim M(\mathbf{v}^\lambda) = 0$ for $\lambda \in \mathcal{P}$. This is, of course, not true in general. For other types, quiver varieties can have higher dimension.

Let the torus $T = \mathbb{C}^*$ act trivially on the quiver varieties. Since the equivariant cohomology of a point $H_T^*(pt)$ is $\mathbb{C}[t]$, we have the following.

$$H_T^*(\mathfrak{M}(\mathbf{v})) \cong \begin{cases} \mathbb{C}[t] & \text{if } \mathbf{v} = \mathbf{v}^\lambda \text{ for } \lambda \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

For $k \in I$, define the *Hecke correspondence* $\mathfrak{B}_k(\mathbf{v})$ to be the variety of all $(B_1, B_{-1}, \tilde{v}, S)$ (modulo the $G_{\mathbf{v}}$-action) such that $(B_1, B_{-1}, \tilde{v}) \in \mathbf{M}(\mathbf{v})^s$ and S is a (B_1, B_{-1})-invariant subspace such that $\dim S = \mathbf{e}^k$ where $\mathbf{e}^k$ has k-component equal to one and all other components equal to zero. We consider the $G_{\mathbf{v}}$-orbit through $(B_1, B_{-1}, \tilde{v}, S)$ as a point in $\mathfrak{M}(\mathbf{v} - \mathbf{e}^k) \times \mathfrak{M}(\mathbf{v})$ by factoring by the subspace S in the first factor. Note that from the explicit description of the $M(\mathbf{v})$ given above, we see that $\mathfrak{B}_k(\mathbf{v})$ is empty unless $\mathbf{v} = \mathbf{v}^\lambda$ for some $\lambda \in \mathcal{P}$ such that there exists a $\mu \in \mathcal{P}$ with D_μ differing from D_λ by the removal of a single box of residue k. Then

$$\mathfrak{B}_k(\mathbf{v}^\lambda) = ([B_1^\mu, B_{-1}^\mu, \tilde{v}^\mu], [B_1^\lambda, B_{-1}^\lambda, \tilde{v}^\lambda]).$$

Let $\omega\colon \mathfrak{M}(\mathbf{v}^1) \times \mathfrak{M}(\mathbf{v}^2) \to \mathfrak{M}(\mathbf{v}^2) \times \mathfrak{M}(\mathbf{v}^1)$ be the map that interchanges the two factors. We then define two operators E_k and F_k that act on $\bigoplus_{\mathbf{v}} H_T^*(\mathfrak{M}(\mathbf{v}))$ as follows.

$$E_k c = t^{-1} \cup p_{1!}\big(p_2^* c \cap [\mathfrak{B}_k(\mathbf{v})]\big], \quad c \in H_T^*(\mathfrak{M}(\mathbf{v})), \tag{3.1}$$

$$F_k c = t \cup p_{2!}\big[(p_1^* c \cap [\omega(\mathfrak{B}_k(\mathbf{v} + \mathbf{e}^k, \mathbf{w}))]\big), \quad c \in H_T^*(\mathfrak{M}(\mathbf{v})). \tag{3.2}$$

Here we have used the projection maps

$$\mathfrak{M}(\mathbf{v}_1) \xleftarrow{p_1} \mathfrak{M}(\mathbf{v}_1) \times \mathfrak{M}(\mathbf{v}_2) \xrightarrow{p_2} \mathfrak{M}(\mathbf{v}_2).$$

Note that the operators E_k and F_k preserve the subspace

$$\mathbb{H}^F := \bigoplus_{\lambda \in \mathcal{P}} H_T^{2|\lambda|}(\mathfrak{M}(\mathbf{v}^\lambda)).$$

We also define

$$\mathbb{H}_n^F = \bigoplus_{\lambda \vdash n} H_T^{2n}(\mathfrak{M}(\mathbf{v}^\lambda)) = H_T^{2n}\Big(\bigsqcup_{\lambda \vdash n} \mathfrak{M}(\mathbf{v}^\lambda)\Big).$$

Note that since the $\mathfrak{M}(\mathbf{v}^\lambda)$ are all points, we have

$$H_T^{2k}(\mathfrak{M}(\mathbf{v}^\lambda)) = t^k \cup H_T^0(\mathfrak{M}(\mathbf{v}^\lambda)), \quad k \in Z_{\geq 0}, \lambda \in \mathcal{P}.$$

In particular, we see from this that E_k as defined above is indeed an operator on $\mathbb{H}^F$ (*a priori*, it is an operator on $\mathbb{H}^F \otimes \mathbb{C}(t)$).

THEOREM 3.1. *The operators E_k and F_k satisfy the relations of the Chevalley generators of sl_∞ and thus define an action of sl_∞ on $\mathbb{H}^F$. Under this action, $\mathbb{H}^F$ is the basic representation. The class $[\mathfrak{M}(\mathbf{0})]$ is a highest weight vector and $H^{2|\lambda|}(\mathfrak{M}(\mathbf{v}^\lambda))$ is the weight space of weight $\omega_0 - \sum v_k^\lambda \alpha_k$, where α_k are the simple roots of sl_∞.*

PROOF. This follows immediately from the results of [19]. Our modifications, as noted at the beginning of this section, are minor. ■

4. The Hilbert scheme, torus action and fixed points.

In this section we introduce the Hilbert scheme of n points in $\mathbb{C}^2$ and recall some results regarding the fixed points of the natural torus action. We shall see that these fixed points are naturally identified with the quiver varieties of Section 3. This precise relationship between the spaces involved in the geometric constructions of the bosonic and fermionic Fock spaces will allow us to use the localization theorem to yield a geometric boson-fermion correspondence. For other results relating quiver varieties and Hilbert schemes, we refer the reader to [8, 11, 23, 25].

Consider the space of n (unordered) points in $\mathbb{C}^2$. This is a singular space, the singularities occurring when points collide. The Hilbert scheme of n points in $\mathbb{C}^2$ is a resolution of singularities of this space. It resolves the singularities by "retaining information about how points collided". It is thus a very natural space and has been used to construct representations of infinite dimensional Heisenberg algebras and Virasoro algebras. We describe here the former.

Precisely, the Hilbert scheme X_n parameterizes 0-dimensional closed subschemes of $\mathbb{C}^2$ of length n. So if x and y are the standard coordinate functions on $\mathbb{C}^2$,

$$X_n = \{I \mid I \text{ is an ideal of } \mathbb{C}[x,y],\ \dim \mathbb{C}[x,y]/I = n\}.$$

Consider the action of the one-dimensional torus $T = \mathbb{C}^*$ on $\mathbb{C}^2$ by

$$z \cdot (x,y) = (zx, z^{-1}y), \quad z \in T.$$

The only fixed point is the origin, which we will denote by u.

The T-action on $\mathbb{C}^2$ induces a T-action on X_n. The support of a T-fixed point in X_n is u since it must be a fixed point of $\mathbb{C}^2$. In order to explicitly describe the T-fixed points and relate them to the quiver varieties described above, we give the following alternate description of the Hilbert schemes (see [20, Theorem 1.9]).

$$X_n \cong \{(B^1, B^2, \tilde{v}) \bigm| [B^1, B^2] = 0,\ (B^1, B^2, \tilde{v}) \text{ is stable}\}/GL(V), \tag{4.1}$$

where $B^j \in \operatorname{End}(V)$, $\tilde{v} \in V = \mathbb{C}^n$, and we say that $(B^1, B^2, \tilde{v})$ is stable if there exists no proper subspace $S \subsetneq V$ such that $B^j(S) \subset S$ for $j = 1, 2$, and $\tilde{v} \in S$. The action of $GL(V)$ is given by

$$g \cdot (B^1, B^2, \tilde{v}) = (gB^1g^{-1}, gB^2g^{-1}, g(\tilde{v})), \quad g \in GL(V).$$

In this description, the action of T on X_n is given by

$$z \cdot [B^1, B^2, \tilde{v}] = [zB^1, z^{-1}B^2, \tilde{v}], \quad z \in T.$$

Here $[B^1, B^2, \tilde{v}]$ denotes the $GL(V)$-orbit through $(B^1, B^2, \tilde{v})$.

From another description of the Hilbert scheme (see [20, Theorem 3.24]), one can see that $[B^1, B^2, \tilde{v}] \in X_n$ is a fixed point if and only if there exists a homomorphism $\lambda \colon T \to U(\mathbb{C}^n)$ such that

$$\begin{aligned} zB^1 &= \lambda(z)^{-1} B^1 \lambda(z), \\ z^{-1}B^2 &= \lambda(z)^{-1} B^2 \lambda(z), \\ \tilde{v} &= \lambda(z)^{-1}(\tilde{v}). \end{aligned}$$

Thus, if $[B^1, B^2, \tilde{v}]$ is a fixed point, we have a weight decomposition of V with respect to $\lambda(z)$ given by

$$V = \bigoplus_k V_k,$$

where $V_k = \{v \in V \mid \lambda(z) \cdot v = z^{-k}v\}$. It follows that the only non-zero components of B^1 and B^2 are

$$B^1 \colon V_k \to V_{k+1}, \tag{4.2}$$

$$B^2 \colon V_k \to V_{k-1}, \tag{4.3}$$

and

$$\tilde{v} \in V_0. \tag{4.4}$$

So B^1 and B^2 are endomorphisms of the graded vector space V of degrees 1 and -1, respectively. Conversely, one can see that any triple $(B^1, B^2, \tilde{v})$ satisfying conditions (4.2)–(4.4) for some λ is a T-fixed point. This precisely matches the description of the points of the A_∞ quiver varieties given in Section 3. We simply set $B_1 = B^1$ and $B_{-1} = B^2$, and then the T-fixed points of the Hilbert scheme are exactly the quiver varieties of type A_∞. In particular, the fixed point set consists of isolated points. Recall that the total dimension of the vector space V appearing in the definition of the quiver variety $\mathfrak{M}(\mathbf{v}^\lambda)$ is $|\lambda|$, the size of the partition λ. Thus there is a natural identification

$$X_n^T = \bigsqcup_{\lambda \vdash n} \mathfrak{M}(\mathbf{v}^\lambda),$$

and hence a natural identification

$$\bigoplus_n H_T^{2n}(X_n^T) = \mathbb{H}^F.$$

5. **Geometric realization of bosonic Fock space.** In this section we describe the geometric realization of bosonic Fock space in the equivariant cohomology of the Hilbert schemes of points in $\mathbb{C}^2$. We refer the reader to [7, 12, 16, 20, 24] for more details.

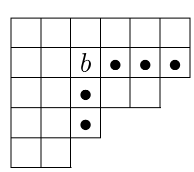
Let Σ be the x-axis in $\mathbb{C}^2$ and recall that u is the origin. As a T-module, we have $T_u\Sigma = \theta^{-1}$ where the θ is the one-dimensional standard T-module. By the localization theorem,

$$[\Sigma] = -t^{-1}[u]. \tag{5.1}$$

The tangent space of X_n at the T-fixed point $\mathfrak{M}(\mathbf{v}^\lambda)$ is T-equivariantly isomorphic to (see [20, Proposition 5.8] and [4, 16])

$$T_{\mathfrak{M}(\mathbf{v}^\lambda)} X_n = \bigoplus_{b \in D_\lambda} (\theta^{\mathrm{hook}(b)} \oplus \theta^{-\mathrm{hook}(b)})$$

where hook(b) is the hook length of the box b. Recall that the hook length of a box is the number of boxes directly to its right plus the number of boxes directly below it plus one (for the box itself). For example, suppose b is the indicated box in the following Young diagram.



Then $\mathrm{hook}(b) = 6$ (the number of boxes containing bullets plus one for the box containing b). Thus

$$e_T(T_{\mathfrak{M}(\mathbf{v}^\lambda)} X_n) = (-1)^n h(\lambda)^2 t^{2n}, \tag{5.2}$$

where

$$h(\lambda) = \prod_{b \in D_\lambda} \mathrm{hook}(b).$$

Note that $[\mathfrak{M}(\mathbf{v}^\lambda)] \in H_T^{4n}(X_n)$. Define

$$[\lambda] = \frac{(-1)^n}{h(\lambda)} t^{-n} [\mathfrak{M}(\mathbf{v}^\lambda)].$$

The odd Betti numbers of X_n are equal to zero and $H^k(X_n) = 0$ for $k > 2n$. We have

$$H_T^{2k}(X_n) = t^{k-n} \cup H_T^{2n}(X_n), \quad k \geq n.$$

Define

$$\mathbb{H}_n^B = H_T^{2n}(X_n), \quad \mathbb{H}^B = \bigoplus_{n=0}^{\infty} \mathbb{H}_n^B.$$

We now introduce a bilinear form on the equivariant cohomology. Let

$$\iota \colon X_n^T \hookrightarrow X_n$$

denote the inclusion of the torus fixed points. Then we have the Gysin map

$$\iota_! \colon H_T^*(X_n^T) \to H_T^*(X_n).$$

We denote the induced map $H_T^*(X_n^T)' \to H_T^*(X_n)'$ also by $\iota_!$, where

$$H_T^*(X_n^T)' = H_T^*(X_n^T) \otimes \mathbb{C}(t),$$
$$H_T^*(X_n)' = H_T^*(X_n) \otimes \mathbb{C}(t)$$

are the localizations. This is an isomorphism by the localization theorem.

Define the bilinear form $\langle \cdot , \cdot \rangle : H_T^*(X_n)' \times H_T^*(X_n)' \to \mathbb{C}(t)$ by

$$\langle \alpha, \beta \rangle = (-1)^n p_! \iota_!^{-1} (\alpha \cup \beta),$$

where $p \colon X_n^T \to \mathrm{pt}$ is the projection to a point. This induces a bilinear form on

$$\mathbb{H}' = \bigoplus_{n=0}^{\infty} H_T^*(X_n)'.$$

For $\lambda, \mu \vdash n$, we have that (see [24, Lemme 2] and [12, Equation 2.21])

$$\langle [\lambda], [\mu] \rangle = \delta_{\lambda,\mu}.$$

Now, by the localization theorem, the classes $[\lambda]$, $\lambda \vdash n$, form a linear basis of the $\mathbb{H}_n^B$. Thus, the restriction to $\mathbb{H}_n^B$ of the bilinear form $\langle \cdot\,,\,\cdot \rangle$ is non-degenerate. This induces a non-degenerate bilinear form $\langle \cdot\,,\,\cdot \rangle : \mathbb{H}^B \times \mathbb{H}^B \to \mathbb{C}$.

We now discuss the action of the oscillator algebra $\mathfrak{s}$ on the cohomology of the Hilbert schemes. Define

$$\Sigma_{n,i} = \{(I_1, I_2) \in X_{n+i} \times X_n \mid I_1 \subset I_2,\, \mathrm{Supp}(I_2/I_1) = \{z\},\, z \in \Sigma\}.$$

We have the two natural projections

$$X_{n+i} \xleftarrow{\pi_1} X_{n+i} \times X_n \xrightarrow{\pi_2} X_n.$$

The restriction of π_1 to $\Sigma_{n,i}$ is proper. Thus we can form the linear operator $\mathfrak{p}_{-i} \in \operatorname{End} \mathbb{H}'$ (see [24]) by

$$\mathfrak{p}_{-i}(\alpha) = \pi_{1!}(\pi_2^* \alpha \cup [\Sigma_{n,i}]), \quad \alpha \in H_T^*(X_n)'.$$

We then define $\mathfrak{p}_i \in \operatorname{End} \mathbb{H}'$ to be the adjoint operator to $\mathfrak{p}_{-i}$ with respect to the bilinear form $\langle \cdot\,,\,\cdot \rangle$ on $\mathbb{H}'$. One can show that

$$\mathfrak{p}_i(\alpha) = (-1)^i \pi'_{2!}(\iota \times \mathrm{Id})_!^{-1}(\pi_1^* \alpha \cup [\Sigma_{n-i,i}]), \quad \alpha \in H_T^*(X_n)',$$

where π'_2 is the natural projection $X_n^T \times X_{n-i} \to X_{n-i}$. We also set $\mathfrak{p}_0 = 0$.

For $i > 0$, the restriction $\mathfrak{p}_{-i}$ to $\mathbb{H}^B$ yields a linear operator in $\operatorname{End} \mathbb{H}^B$. The restriction of $\mathfrak{p}_i$ to $\mathbb{H}$ is the adjoint operator to $\mathfrak{p}_{-i}$ with respect to the non-degenerate bilinear form $\langle \cdot\,,\,\cdot \rangle : \mathbb{H}^B \otimes_{\mathbb{C}} \mathbb{H}^B \to \mathbb{C}$, which is the restriction of the bilinear form $\langle \cdot\,,\,\cdot \rangle$ on $\mathbb{H}'$. Thus the restriction of $\mathfrak{p}_i$ to $\mathbb{H}^B$ is an operator in $\operatorname{End} \mathbb{H}^B$, again denoted by $\mathfrak{p}_i$.

Proposition 5.1. ([24, Lemme 1]) *The operators $\mathfrak{p}_k$, $k \in \mathbb{Z}$, acting on $\mathbb{H}^B$ satisfy the following Heisenberg commutation relations:*

$$[\mathfrak{p}_k, \mathfrak{p}_l] = k\delta_{k,-l} \,\mathrm{Id}\,.$$

In particular,

$$s_k \mapsto \mathfrak{p}_k,\ k \in \mathbb{Z},$$
$$K \mapsto \mathrm{Id},$$

defines an action of the oscillator algebra $\mathfrak{s}$ on $\mathbb{H}^B$. Moreover, $\mathbb{H}^B$ is isomorphic to the bosonic Fock space $B^{(0)}$. The unit $1 \in H_T^0(X_0)$ of $H_T^*(X_0)$ corresponds to the highest weight vector $1 \in B^{(0)}$.

6. **A geometric boson-fermion correspondence.** We now have all the tools necessary to describe the geometric construction of the boson-fermion correspondence. We have seen how the quiver varieties used in the geometric realization of fermionic Fock space can be naturally viewed as the set of torus fixed points of the Hilbert scheme used in the geometric realization of the bosonic Fock space. We are then in a position to invoke the localization theorem which gives an isomorphism between the equivariant cohomology of the Hilbert scheme and its fixed point set thus yielding our geometric version of the boson-fermion correspondence.

Define a map $\eta\colon \mathbb{H}^F \to \mathbb{H}^B$ by

$$\eta(\alpha) = \frac{(-1)^{|\lambda|}}{h(\lambda)} t^{-|\lambda|} i_{\lambda !}(\alpha) \in \mathbb{H}^B_{|\lambda|}, \quad \alpha \in H_T^{2|\lambda|}(\mathfrak{M}(\mathbf{v}^\lambda)).$$

The factor of $(-1)^{|\lambda|} t^{-|\lambda|}/h(\lambda)$, which arises from the equivariant Euler class at the fixed point, ensures that η is an isometry if we endow $\mathbb{H}^F$ with the bilinear form for which the $t^{|\lambda|} \cup 1_{\mathfrak{M}(\mathbf{v}^\lambda)}$ form an orthonormal basis. It then follows from the localization theorem and (5.2) that η is an isomorphism with inverse given by

$$\beta \mapsto \Big(\frac{1}{h(\lambda)} t^{-n} i_\lambda^*(\beta) \Big)_{\lambda \vdash n} \in \mathbb{H}^F_n, \quad \beta \in \mathbb{H}^B_n.$$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$, define

$$\begin{aligned} z_\lambda &= \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!, \\ \mathfrak{p}^\lambda &= \prod_{i \geq 1} \mathfrak{p}_{-i}^{m_i(\lambda)}, \\ \mathfrak{p}_\lambda &= \mathfrak{p}^\lambda \cdot 1 \in \mathbb{H}^B_n,\ \lambda \vdash n. \end{aligned}$$

It follows from Proposition 5.1 and the fact that $\mathfrak{p}_{-i}$ and $\mathfrak{p}_i$ are adjoint operators that $\langle \mathfrak{p}_\lambda, \mathfrak{p}_\mu \rangle = \delta_{\lambda,\mu} z_\lambda$.

Theorem 6.1.

(i) *There exists an isomorphism $\phi\colon \mathbb{H}^B \to B^{(0)}$ of $\mathfrak{s}$-modules preserving bilinear forms such that*

$$\phi(\mathfrak{p}_\lambda) = p_\lambda, \quad \phi([\lambda]) = S_\lambda.$$

(ii) *There exists an isomorphism of sl_∞-modules $\tau\colon F^{(0)} \to \mathbb{H}^F$ such that*

$$\tau(\varphi_\lambda) = t^{|\lambda|} \cup 1_{\mathfrak{M}(\mathbf{v}^\lambda)}.$$

(iii) *We have $\phi \circ \eta \circ \tau = \sigma_0$.*

Proof. Part (i) is proven in [24, Proposition 2]. Part (ii) follows from a comparison of the explicit action sl_∞ on the indicated bases in both spaces (see [6, Proposition 5.3]). Part (iii) follows from the fact that both $\phi \circ \eta \circ \tau$ and σ_0 are linear isomorphisms sending a semi-infinite monomial φ_λ to S_λ. ∎

We can then define geometric bosons on the geometric fermionic Fock space as follows. For $\alpha \in \mathbb{H}^F$,

$$s_k(\alpha) = \eta^{-1} \circ \mathfrak{p}_k \circ \eta(\alpha), \quad K = \mathrm{Id}\,.$$

This defines a representation of $\mathfrak{s}$ on $\mathbb{H}^F$ and $\mathbb{H}^F \cong B^{(0)}$ as $\mathfrak{s}$-modules.

We can also define an action of sl_∞ on the geometric bosonic Fock space by a similar procedure. For $\beta \in \mathbb{H}^F$ and $k \in \mathbb{Z}$,

$$\begin{aligned} E_k &= \eta \circ E_k \circ \eta^{-1}(\beta), \\ F_k &= \eta \circ F_k \circ \eta^{-1}(\beta). \end{aligned}$$

We thus see that in terms of the geometry of the Hilbert scheme, the bosons correspondence to global operators while the fermions correspond to local operators.

We should also note that the energy decomposition of the fermionic Fock space has a nice geometric interpretation. It corresponds to grouping the quiver varieties according to the Hilbert scheme of which they are a fixed point. More precisely,

$$\tau(F_n^{(0)}) = \mathbb{H}_n^F = H_T^{2n}(X_n^T).$$

Note that we have described the geometric boson-fermion correspondence as a relationship between the basic representation of sl_∞ on fermionic Fock space $F^{(0)}$ and an irreducible representation of the Heisenberg (or oscillator) algebra. We could also define the geometric action of the fermions themselves on the full fermionic Fock space F. To do this we would have to introduce the quiver varieties corresponding to the irreducible representations $L(\omega_k)$ for $k \in \mathbb{Z}$. These are identical to those for $L(\omega_0)$, except that one shifts the grading on the vector space V. Then the fermions would be operators between the cohomology of these different quiver varieties (see [22] for a similar construction of Clifford algebras). However, the sl_∞ action seems to be simpler geometrically and so we have emphasized its role.

We should also comment on why the particular T-action we have considered was chosen. There is a natural action of the two-dimensional torus $T^2 = (\mathbb{C}^*)^2$ on $\mathbb{C}^2$ given by

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y), \quad (z_1, z_2) \in T^2$$

and this induces a T^2 action on the Hilbert scheme X_n. The T-action that we have considered arises from the embedding $T \hookrightarrow T^2$ given by $z \mapsto (z, z^{-1})$. A more general embedding of T into T^2 can certainly be considered. This was examined in [12, 16] and one finds that the fixed point contributions to the equivariant cohomology correspond to Jack polynomials which generalize the Schur polynomials above. Furthermore, one could consider the entire T^2 action and one would obtain polynomials with an enlarged coefficient ring containing a parameter which specialize to the Jack polynomials. However, in these more general settings, one loses the natural identification of the quiver varieties of type A_∞ with the set of fixed points. Under the embedding $z \mapsto (z, z^{-1})$, the grading on the vector space V in the definition of the quiver variety appears naturally as described in Section 4.

Acknowledgements. The author would like to thank I. B. Frenkel for many invaluable comments and suggestions. He would also like to thank M. Harada, A. Knutson, M. Haiman, and W. Wang for answering his questions on equivariant cohomology, and M. Lau for useful discussions. Finally, he would like to thank the American Institute of Mathematics and the organizers of the ARCC Workshop on Generalized Kostka Polynomials, where some of the ideas in this paper were developed.

References

1. M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology.* Topology **23** (1984), no. 1, 1–28.

2. D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry.* Mathematical Surveys and Monographs 68, American Mathematical Society, Providence, RI, 1999.

3. E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, *Transformation groups for soliton equations. Euclidean Lie algebras and reduction of the KP hierarchy.* Publ. Res. Inst. Math. Sci. **18** (1982), no. 3, 1077–1110.

4. G. Ellingsrud and S. A. Strømme, *On the homology of the Hilbert scheme of points in the plane.* Invent. Math. **87** (1987), no. 2, 343–352.

5. I. B. Frenkel, *Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory.* J. Funct. Anal. **44** (1981), no. 3, 259–327.

6. I. B. Frenkel and A. Savage, *Bases of representations of type A affine Lie algebras via quiver varieties and statistical mechanics.* Int. Math. Res. Not. **2003**, no. 28, 1521–1547.

7. I. Grojnowski, *Instantons and affine algebras. I. The Hilbert scheme and vertex operators.* Math. Res. Lett. **3** (1996), no. 2, 275–291.

8. M. Haiman, *Hilbert schemes as quiver varieties.* http://www.math.berkeley.edu/~hausel/quiver/seminar10.html.

9. V. G. Kac, *Infinite-dimensional Lie algebras.* Third edition. Cambridge University Press, Cambridge, 1990.

10. P. B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons.* Math. Ann. **288** (1990), no. 2, 263–307.

11. A. Kuznetsov, *Quiver varieties and Hilbert schemes.* arXiv:math.AG/0111092.

12. W.-P. Li, Z. Qin, and W. Wang, *The cohomology rings of Hilbert schemes via Jack polynomials.* arXiv:math.AG/0411255.

13. W.-P. Li, Z. Qin, and W. Wang, *Hilbert schemes, integrable hierarchies, and Gromov-Witten theory.* Int. Math. Res. Not. **2004** no. 40, 2085–2104.

14. G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras.* J. Amer. Math. Soc. **4** (1991), no. 2, 365–421.

15. I. G. Macdonald, *Symmetric functions and Hall polynomials.* Second edition. Oxford University Press, New York, 1995.

16. H. Nakajima, *Jack polynomials and Hilbert schemes of points on surfaces.* arXiv:math.AG/9610021.

17. ______, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras.* Duke Math. J. **76** (1994), no. 2, 365–416.

18. ______, *Varieties associated with quivers.* In: Representation Theory of Algebras and Related Topics, CMS Conf. Proc. 19, American Mathematical Society, Providence, RI, 1996, pp. 139–157.

19. ______, *Quiver varieties and Kac-Moody algebras.* Duke Math. J. **91** (1998), no. 3, 515–560.

20. ______, *Lectures on Hilbert Schemes of Points on Surfaces.* University Lecture Series 18, American Mathematical Society, Providence, RI, 1999.

21. A. Savage, *Finite-dimensional algebras and quivers.* In: Encylopedia of Mathematical Physics, Vol. 2, (J.-P. Franoise, G. Naber, and Tsou S.T., eds.) Elsevier, Oxford, 2006, pp. 313–320.

22. ______, *A geometric realization of spin representations and Young diagrams from quiver varieties.* J. Algebra **297** (2006), no. 1, 186–207.

23. M. Varagnolo and E. Vasserot, *On the K-theory of the cyclic quiver variety.* Internat. Math. Res. Notices **1999**, no. 18, 1005–1028.

24. E. Vasserot, *Sur l'anneau de cohomologie du schéma de Hilbert de* $\mathbf{C}^2$. C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 1, 7–12.

25. W. Wang, *Algebraic structures behind Hilbert schemes and wreath products.* In: Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory, Contemp. Math. 297, American Mathematical Society, Providence, RI, 2002, pp. 271–295..

University of Ottawa
Ottawa, Ontario
K1N 6N5
email: *alistair.savage@uottawa.ca*