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## ON THE SOLUTIONS OF A FAMILY OF QUARTIC THUE EQUATIONS, II

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**ABSTRACT.** In this paper, we completely solve the following family of diophantine equations associated with a family of cyclic quartic number fields:

$$\Phi_n(x, y) = x^4 - n^2x^3y - (n^3 + 2n^2 + 4n + 2)x^2y^2 - n^2xy^3 + y^4 = 1.$$

There is no integral solution except for the trivial ones:

$$(1, 0), (-1, 0), (0, 1), (0, -1).$$

We extend a previous result obtained in 2000. In fact, the new result is achieved by sharpening the previous result, using another technique.

**RÉSUMÉ.** Dans cet article, nous résolvons complètement la famille suivante d'équations Diophantiennes associées à une famille de corps de nombres cycliques de degré 4:

$$\Phi_n(x, y) = x^4 - n^2x^3y - (n^3 + 2n^2 + 4n + 2)x^2y^2 - n^2xy^3 + y^4 = 1.$$

Il n'existe aucune solution entière à l'exception des solutions triviales:

$$(1, 0), (-1, 0), (0, 1), (0, -1).$$

Nous prolongeons un résultat précédent obtenu en 2000. En réalité, nous utilisons une nouvelle technique pour améliorer le précédent résultat.

**1. Introduction.** In a previous paper [10], we proved that the Thue equation

$$(1.1) \quad \Phi_n(x, y) = x^4 - n^2x^3y - (n^3 + 2n^2 + 4n + 2)x^2y^2 - n^2xy^3 + y^4 = 1$$

has only the trivial solutions  $\pm\{(1, 0), (0, 1)\}$  for  $n \leq 5 \times 10^6$  and  $n \geq N = 1.191 \times 10^{19}$ , with  $n, n+2, n^2+4$  square-free. For this, we used two methods: a method proposed by M. Mignotte [7] and the Bilu–Hanrot method [2]. Moreover, in [9], we have conjectured that (1.1) has only the solutions (1.3) for any parameter  $n \geq 1$ .

The aim of this paper is to completely solve this equation using another technique still based on Baker's method. The result that we prove is the following:

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THEOREM 1.1. *For  $n \geq 1$ , the family of parameterized Thue equations*

$$(1.2) \quad \Phi_n(x, y) = x^4 - n^2x^3y - (n^3 + 2n^2 + 4n + 2)x^2y^2 - n^2xy^3 + y^4 = 1$$

*has only the integral solutions*

$$(1.3) \quad \pm\{(1, 0), (0, 1)\};$$

*except for  $n = 1$ , where we have*

$$(1.4) \quad \pm\{(1, -1), (1, 0), (0, 1)\}.$$

In this paper, we introduce new ideas to refine the result obtained in [10]. First, we try to avoid the condition “ $n, n + 2, n^2 + 4$  square-free” by using units that are “sufficiently close” to fundamental units. So in Section 2, we extend the result obtained by O. Lecacheux [6] and L. C. Washington [12]. In Section 3, we study approximation properties of solutions to (1.1). In the same section, we prove that we need only study the solutions of type  $j = 1, 4$ . The most important result in this section is the determination of a lower bound for  $\log(y)$  that helps to significantly reduce the previous bound of the parameter  $n$ . In Section 4, we prove Theorem 1.1 for large values of the parameter  $n$  using a linear form in two logarithms. This helps us to prove that the Thue equation (1.1) has no further solution for  $n$  exceeding some bound  $n_0 \approx 10^6$ . For the remaining Thue equations, we use a computational method and Kash [3]. This is done in Section 4. Most of the computations involve manipulations with asymptotic approximations using Maple. In the following, (I.n.m) refers to formula (n.m) in [10] and Lemma I.n.m refers to Lemma n.m in [10].

**2. Associated number field.** One can check that  $\phi_n(x) = \Phi_n(x, 1)$  is irreducible. We know that  $\phi_n(x)$  has 4 real roots  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ , with:

$$(2.1) \quad \alpha_1 := \frac{n^2 + (n+2)\sqrt{n^2+4} + \sqrt{2n(n+2)(n^2+4) + 2n^2(n+2)\sqrt{n^2+4}}}{4};$$

$$\alpha_2 := \frac{n^2 - (n+2)\sqrt{n^2+4} - \sqrt{2n(n+2)(n^2+4) - 2n^2(n+2)\sqrt{n^2+4}}}{4}.$$

We set  $\alpha := \alpha_1$ ,  $\beta := -\alpha_2$ . The quadratic subfield of  $\mathbb{K}_n = \mathbb{Q}(\alpha)$  is  $k = \mathbb{Q}(\sqrt{n^2+4})$ , and  $\varepsilon = \frac{n+\sqrt{n^2+4}}{2}$  is a unit of  $k$ , therefore of  $\mathbb{K}_n$ . Now we extend Theorem I.2.2 of [10].

LEMMA 2.1. *Let us consider  $\mathcal{O} = \mathbb{Z}[\alpha, \beta, \varepsilon]$ , and  $\langle -1, \alpha, \beta, \varepsilon \rangle$  a subgroup of the unit group. For  $n \geq 33$ , we have*

$$(2.2) \quad I := [\mathcal{O}^\times : \langle -1, \alpha, \beta, \varepsilon \rangle] < 49.27 \log^3(n).$$

PROOF. We would like to determine an upper bound for the index of  $\langle -1, \alpha, \beta, \varepsilon \rangle$  in the unit group  $\mathcal{O}^\times$  of  $\mathbb{K}_n$  by estimating the regulators of the two groups. Let  $R$  be the regulator of  $\langle -1, \alpha, \beta, \varepsilon \rangle$ :

$$(2.3) \quad R = 2 \log(\varepsilon)(\log^2(\alpha) + \log^2(\beta)).$$

Using asymptotic expressions, one can check that

$$(2.4) \quad 10 \log^3(n) \leq R \leq 10.11 \log^3(n)$$

for  $n \geq 33$ . So  $R > 0$  and  $\alpha, \beta, \varepsilon$  are independent units. Using Theorem B in [4], the regulator of  $\mathbb{K}_n$  can be bounded by  $\text{Reg } \mathbb{Z}_{\mathbb{K}_n} \geq 0.2052$ . From [8, p. 361] we find the following bound for the index

$$(2.5) \quad I = [\mathcal{O}^\times : \langle -1, \alpha, \beta, \varepsilon \rangle] \leq \frac{R}{\text{Reg } \mathbb{Z}_{\mathbb{K}_n}} \leq \frac{10.11 \log^3(n)}{0.2052} \leq 49.27 \log^3(n). \quad \blacksquare$$

Therefore, we can avoid the condition “ $n, n + 2, n^2 + 4$  square-free”.

**3. Approximation properties of solutions.** Let  $(x, y) \in \mathbb{Z}^2$  be a solution to (1.1). We define the type  $j$  of a solution to (1.1) if

$$|x - \alpha_j y| = \min_{1 \leq i \leq 4} \{ |x - \alpha_i y| \}.$$

The following result will help us to reduce our study to the cases  $j = 1, 4$ .

LEMMA 3.1. *If  $(x, y)$  is a solution of type  $j$ , then  $(y, x)$  is a solution of type  $j + 2$ .*

PROOF. One can see that  $\prod_{i=1}^4 |\frac{x}{y} - \alpha_i| = \frac{1}{y^4} \leq \frac{1}{16}$ , for  $y \geq 2$ . This means that the rational  $\frac{x}{y}$  is close to one of the zeros of  $\phi_n$ . For  $i \neq j$  we have  $y|\alpha_i - \alpha_j| \leq 2|x - \alpha_i y|$ , then

$$(3.1) \quad |x - \alpha_j y| = \frac{1}{\prod_{i \neq j} |x - \alpha_i y|} \leq \frac{8}{y^3} \cdot \frac{1}{\prod_{i \neq j} |\alpha_i - \alpha_j|} = \frac{8}{\phi'_n(\alpha_j) y^3}.$$

As  $|\alpha_i - \alpha_j| \geq n$  except for  $|\alpha_3 - \alpha_4| \geq \frac{9}{10n}$  when  $n \geq 3$ , we obtain  $|\frac{x}{y} - \alpha_i| < \frac{1}{2y^2}$ . This means that  $\frac{x}{y}$  is a convergent to  $\alpha_j$ . Also we have

$$|x| \leq |x - \alpha_j y| + |\alpha_j y| \leq |\alpha_j y| + \frac{1}{y^2} \leq 2|\alpha_j y|.$$

Therefore we obtain

$$(3.2) \quad \left| \frac{1}{\alpha_j} - \frac{x}{y} \right| = \frac{|x - \alpha_j y|}{|\alpha_j x|} \leq \frac{1}{2|\alpha_j x y|} < \frac{1}{|x y|} \leq \frac{1}{4}.$$

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This means that

$$\left| \frac{x}{\alpha_j} - y \right| = \min_{1 \leq i \leq 4} \{ |x - \alpha_i y| \}.$$

As

$$(3.3) \quad \alpha_3 = \frac{1}{\alpha_1}, \quad \alpha_4 = \frac{1}{\alpha_2}, \quad \text{i.e.,} \quad \frac{1}{\alpha_i} = \alpha_{i+2},$$

therefore it suffices to consider the cases  $j = 1$  and  $j = 4$ . ■

One can check that  $\gamma_i := x - \alpha_i y$  are units in  $\mathcal{O}^\times$ . The following lemma will be very useful for the proof of Theorem 1.1:

LEMMA 3.2. *Let  $n \geq 1$  and  $(x, y)$  be a solution to (1.1) of type  $j$  such that  $y \geq 2$ . Then*

$$(3.4) \quad |\gamma_j| \leq k_j \frac{1}{y^3}, \quad \text{where } k_j = \begin{cases} \frac{8}{n^6} & \text{if } j = 1, \\ \frac{8}{n^2} & \text{if } j = 4, \end{cases}$$

$$(3.5) \quad \log |\gamma_i| = \log(y) + \log |\alpha_i - \alpha_j| + \begin{cases} \frac{1}{2n^8} & \text{if } j = 1, \\ \frac{1}{2n^4} & \text{if } j = 4. \end{cases}$$

PROOF. For  $i \neq j$  we have

$$y|\alpha_i - \alpha_j| \leq 2|\gamma_i|,$$

then

$$(3.6) \quad |\gamma_j| = \frac{1}{\prod_{i \neq j} |\gamma_i|} \leq \frac{8}{y^3} \cdot \frac{1}{\prod_{i \neq j} |\alpha_i - \alpha_j|}.$$

Since

$$\prod_{i \neq 1} |\alpha_i - \alpha_j| \geq \begin{cases} n^6 & \text{if } j = 1, \\ n^2 & \text{if } j = 4, \end{cases}$$

we obtain (3.4). Moreover, we know that

$$(3.7) \quad \frac{|\gamma_i|}{y|\alpha_i - \alpha_j|} = \left| 1 + \frac{\gamma_j}{y(\alpha_i - \alpha_j)} \right|.$$

Then, taking the log of the previous expression and using expressions of  $\alpha_i$ , (3.4), we obtain (3.5). This completes the proof. ■

The next result is very important for the proof of Theorem 1.1. We consider here the  $L$ -notation defined as follows. Let  $c$  be a real number and assume that  $f(x)$ ,  $g(x)$ ,  $h(x)$  are real functions such that  $h(x) > 0$  for  $x > c$ . We will write

$$f(x) = g(x) + L_c(h(x))$$

for

$$g(x) - h(x) \leq f(x) \leq g(x) + h(x).$$

LEMMA 3.3. *Let  $(x, y)$  be a solution to (1.1) with  $y \geq 2$  and  $n \geq 33$ . Then*

$$(3.8) \quad \log y \geq 2.485n \log(n).$$

PROOF. If  $(x, y)$  is a solution to (1.1), then using (I.3.2), the expressions of  $u_2, u_3$  contained in the proof of Lemma I.4.3, p. 843, the formulas (3.4) and (3.5) to obtain for  $j = 1$

$$(3.9) \quad \begin{cases} \frac{Ru_2}{I} = \left(-8 \log(n) - \frac{4}{n} - \frac{2}{n^2} + L_{33}\left(\frac{16}{3n^3}\right)\right) \log(y) - 16 \log^2(n) \\ \quad + 4 \log(2) \log(n) + \frac{2 \log(2) - 9 \log(n)}{n} + \frac{2 \log(2) - 11 \log(n)}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right), \\ \frac{Ru_3}{I} = \left(-4 \log(n) - \frac{4}{n} - \frac{2}{n^2} + L_{33}\left(\frac{8}{3n^3}\right)\right) \log(y) - 8 \log^2(n) \\ \quad + 2 \log(2) \log(n) + \frac{2 \log(2) - 11 \log(n)}{n} + \frac{2 \log(2) - 7 \log(n) - 2}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right), \end{cases}$$

and for  $j = 4$

$$(3.10) \quad \begin{cases} \frac{Ru_2}{I} = \left(4 \log(n) + \frac{4}{n} + \frac{2}{n^2} + L_{33}\left(\frac{8}{3n^3}\right)\right) \log(y) + 9 \log^2(n) \\ \quad - \log(5) \log(n) + \frac{9 \log(n) - \log(5)}{n} + \frac{11 \log(n) - \log(5) + 4}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right), \\ \frac{Ru_3}{I} = \left(-8 \log(n) - \frac{4}{n} - \frac{2}{n^2} + L_{33}\left(\frac{16}{3n^3}\right)\right) \log(y) - 3 \log^2(n) \\ \quad + 2 \log(5) \log(n) + \frac{\log(5) - \log(n)}{n} + \frac{5 \log(n) + \log(5)}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right). \end{cases}$$

For each  $j$ , we define the following linear combinations of the  $u_k$  such that we eliminate, if it is possible, the terms in  $\log(n) \log(y)$ ,  $\log^2(n)$ , and  $\log(n)$  appearing in the expressions of the  $\frac{Ru_2}{I}$  and  $\frac{Ru_3}{I}$ :

$$(3.11) \quad \frac{Rv_j}{I} = \begin{cases} \frac{Ru_2}{I} - \frac{2Ru_3}{I} = \left(\frac{4}{n} + \frac{2}{n^2} + L_{33}\left(\frac{32}{3n^3}\right)\right) \log(y) \\ \quad + \frac{2 \log(2) - 9 \log(n)}{n} + \frac{2 \log(2) - 11 \log(n)}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right) & \text{if } j = 1, \\ \frac{2Ru_2}{I} + \frac{Ru_3}{I} = \left(\frac{4}{n} + \frac{2}{n^2} + L_{33}\left(\frac{32}{3n^3}\right)\right) \log(y) + 15 \log^2(n) \\ \quad + \frac{17 \log(n) - \log(5)}{n} + \frac{27 \log(n) - \log(5) + 8}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right) & \text{if } j = 4. \end{cases}$$

As  $y \geq 2$  and  $\frac{v_j}{I}$  is an integer, we have  $\frac{Rv_j}{I} \geq R$ . Therefore, (3.11) helps to obtain (3.8).  $\blacksquare$

**4. Proof of the main theorem for  $n$  large.** We consider the linear forms in logarithms defined by

$$(4.1) \quad \tilde{\Lambda}_j = \tilde{A}_j \log(\alpha) + \tilde{B}_j \log(\beta) + \log(\lambda_j) = \log |1 + \tau_j|,$$

where  $j = 1, 4$ , with

$$(4.2) \quad \tilde{A}_1 = \frac{2u_3}{I}, \quad \tilde{B}_1 = -\frac{2u_2}{I}, \quad \lambda_1 = \frac{\alpha + \beta}{\alpha + 1/\beta}, \quad \tau_1 = \frac{\gamma_1}{\gamma_2} \left(\frac{\beta^2 - 1}{\alpha\beta + 1}\right),$$

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and

$$(4.3) \quad \tilde{A}_4 = \frac{2u_2}{I}, \quad \tilde{B}_4 = \frac{2u_3}{I}, \quad \lambda_4 = \frac{1/\alpha + 1/\beta}{\alpha + 1/\beta}, \quad \tau_4 = -\frac{\gamma_4}{\gamma_3} \left( \frac{\alpha - 1/\alpha}{\alpha + 1/\beta} \right).$$

One can verify that  $\tilde{\Lambda}_j \neq 0$ . From (4.1), (4.2), and (4.3), we have

$$\log |\tilde{\Lambda}_j| = \log |\log |1 + \tau_j|| \leq \log |\tau_j|.$$

Now using (3.4) and (3.5), one can obtain the following upper bound for  $\log |\tilde{\Lambda}_j|$ :

$$(4.4) \quad \log |\tilde{\Lambda}_j| \leq -4 \log(y) + \log(8) - a_j \log(n) - \frac{2}{n},$$

with  $a_j = 4$  if  $j = 1$  and  $a_j = 3$  if  $j = 4$ .

To determine a lower bound for the linear form in logarithms  $\tilde{\Lambda}_j$ , we use the result due to Laurent–Mignotte–Nesterenko (see [5, Corollaire 2, p. 288]). In order to apply this result, we rewrite  $\tilde{\Lambda}_j$  to obtain

$$(4.5) \quad 5\tilde{\Lambda}_j = U_j \log\left(\frac{\beta^2}{\alpha}\right) + \log(\lambda_j^5 \beta^{\pm W_j}),$$

with the sign “−” if  $j = 1$ , the sign “+” if  $j = 4$ , and

$$(4.6) \quad U_1 = -5\tilde{A}_1, \quad W_1 = -10\tilde{A}_1 - 5\tilde{B}_1, \quad U_4 = -5\tilde{A}_4, \quad W_4 = 10\tilde{A}_4 + 5\tilde{B}_4.$$

Using the asymptotic expressions of the  $\frac{u_i}{I}$ , see (3.9) and (3.10), we obtain:

$$(4.7) \quad \begin{cases} U_j = \left( \frac{4}{\log(n)} - \frac{4}{5n \log(n)^2} - \frac{2}{5n^2 \log(n)^2} + L_{33}\left(\frac{0.1}{n^3}\right) \right) \log(y) + b_{j1} \\ \quad + \frac{b_{j2}}{\log(n)} + \frac{b_{j3}}{5n \log(n)} + \frac{b_{j4}}{5n \log(n)^2} + L_{33}\left(\frac{0.1}{n^2}\right), \\ W_j = \left( \frac{4}{n \log(n)^2} + \frac{2}{n^2 \log(n)^2} - \frac{24}{5n^2 \log(n)^3} + L_{33}\left(\frac{0.1}{n^3}\right) \right) \log(y) + b_{j5} \\ \quad + \frac{b_{j6}}{n \log(n)} + \frac{b_{j7}}{n \log(n)^2} + L_{33}\left(\frac{0.1}{n^2}\right), \end{cases}$$

with the  $b_{ji}$  given by the following table:

Case	$b_{j1}$	$b_{j2}$	$b_{j3}$	$b_{j4}$	$b_{j5}$	$b_{j6}$	$b_{j7}$
$j = 1$	8	$-2 \log(2)$	7	$2 \log(2)$	0	13	$-2 \log(2)$
$j = 4$	9	$-\log(5)$	-9	$\log(5)$	15	-1	$-\log(5)$

Table 1: Choice of  $b_{ji}$  depending on  $j$ .

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We take  $D = 4$ . Considering the computations in the proof of Lemma I.4.4, p. 843, we have

$$h\left(\frac{\beta^2}{\alpha}\right) \leq h(\alpha) + 2h(\beta) = \frac{3}{4} \log(\alpha\beta) \leq \frac{9}{4} \log(n) + \frac{2}{n},$$

and

$$h(\lambda^5 \beta^{\pm W_j}) \leq 5h(\lambda) + W_j h(\beta) = \frac{5 + W_j}{4} \log(\alpha\beta);$$

for  $j = 1, 4$ . Therefore, we consider the following values for  $h_1, h_2, b'$ :

$$(4.8) \quad \begin{cases} h_1 = \frac{9}{4} \log(n) + \frac{2}{n}, \\ h_2 = \left(\frac{3}{n \log(n)} + \frac{3}{2n^2 \log(n)} - \frac{8}{5n^2 \log(n)}\right) \log(y) + c_{j1} \log(n) + \frac{c_{j2}}{4n}, \\ b' = c_{j3} n \log(n), \end{cases}$$

with

$$c_{11} = \frac{15}{4}, \quad c_{12} = 49, \quad c_{13} = 0.664, \quad c_{41} = 15, \quad c_{42} = 37, \quad c_{43} = 0.166$$

for  $n \geq 50$ . Therefore, one can quickly observe that

$$(4.9) \quad \log |5\tilde{\Lambda}_j| \geq -6231.04(\log(b') + 0.14)^2 h_1 h_2.$$

Combining (4.4) and (4.9), and Lemma 3.3, we have the following result:

LEMMA 4.1. *For  $n \geq 3309770$ , the equation (1.1) has no solutions, except the trivial solutions.*

Let us specify that for  $j = 1$ , we have  $n \geq 3309770$  and for  $j = 4$ , we have  $n \geq 3045748$ .

**5. Completion of the proof of the main theorem.** The aim of this section is to complete the proof of the main theorem. As a first step, we use linear forms in logarithms once again in order to obtain an upper bound for  $\log(y)$ .

LEMMA 5.1. *For  $50 \leq n \leq 3309770$ , we have*

$$(5.1) \quad \log(y) \leq 6.439 \cdot 10^{19} \log(n) \quad \text{if } j = 1, 4.$$

PROOF. From (4.5), we consider

$$(5.2) \quad 5\tilde{\Lambda}_j = U_j \log\left(\frac{\beta^2}{\alpha}\right) \pm W_j \log(\beta) + \log(\lambda_j^5).$$

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Using the asymptotic expansions of  $W_j$  for  $j = 1, 4$ , see (4.7), we obtain

$$(5.3) \quad W_j := \frac{4.1 \log y}{\log n}.$$

Applying the Baker–Wüstholz theorem, (see [1, p. 20]), to  $\tilde{\Lambda}_j$  as defined by (5.2), using the estimate  $W_j$  given by (5.3), and combining the lower bound with (4.4) results in

$$-4 \log y \geq \log |5\tilde{\Lambda}_j| \geq -C(3, 4)h_1h_2h_3 \log(W_j).$$

In fact, here we take  $n = 3$ ,  $d = 4$ , and

$$(5.4) \quad h_1 = \frac{9.01}{4} \log(n), \quad h_2 = \frac{3.01}{4} \log(n), \quad h_3 = \frac{15.01}{4} \log(n).$$

Consequently, considering that  $50 \leq n \leq 3309770$ , we obtain

$$\frac{\frac{4.1 \log(y)}{\log(n)}}{\log\left(\frac{4.1 \log(y)}{\log(n)}\right)} \leq 5.615 \cdot 10^{18}.$$

Therefore we have

$$(5.5) \quad \log(y) \leq 6.439 \cdot 10^{19} \log(n).$$

This completes the proof of the lemma. ■

We use Lemma 5.1 and the expression for  $W_j$  given by (5.3) to obtain

$$(5.6) \quad 2W_j + 2 \leq \frac{8.2}{n \log^2(n)} \log y \leq \frac{5.28 \cdot 10^{20}}{n \log(n)}; \quad \text{for } j = 1, 4.$$

In order to apply Lemma 5.3 in [11], let us divide (5.2) by  $\log(\frac{\beta^2}{\alpha})$  and use (4.4) to obtain

$$(5.7) \quad |U_j \pm W_j \delta_{j2} + \delta_{j1}| < 10^{-6094},$$

where

$$\delta_{j1} = \frac{\log(\lambda_j^5)}{\log(\frac{\beta^2}{\alpha})}, \quad \delta_{j2} = \frac{\log(\beta)}{\log(\frac{\beta^2}{\alpha})}; \quad \text{for } j = 1, 4 \text{ and } n \geq 50.$$

Hence, for  $j = 1, 4$  and  $50 \leq n \leq 3309770$ , we calculate approximations  $\tilde{\delta}_{j1}$  and  $\tilde{\delta}_{j2}$  such that  $|\delta_{j1} - \tilde{\delta}_{j1}| < Q^{-2}$  and  $|\delta_{j2} - \tilde{\delta}_{j2}| < Q^{-2}$ . In fact, we start with  $Q = 10^m$ ; if it is not successful we try successively  $10^{m+1}, 10^{m+2}, 10^{m+3}, \dots$  until we obtain the desired results. In general, we did the computations with a

precision of 100 digits. For all pairs  $(j, n)$ , we compute the successive convergents of  $\tilde{\delta}_{j2}$  until we find a convergent  $p/q$  of  $\tilde{\delta}_{j2}$  with  $q < Q$  such that

$$q\|q\tilde{\delta}_{j1}\| > \frac{5.28 \times 10^{20}}{n \log(n)}.$$

This contradicts (5.6).

The program was developed in Maple 9.5 and executed on a Pentium 4 with 3.92 GHz running under Linux 7.2. We ran the program starting with  $Q = 10^{22}$ . First we consider  $j = 1, 4$  (together) and  $50 \leq n \leq 3045748$ , and then  $j = 1$  and  $3045748 \leq n \leq 3309770$ . It took on average 1.75 seconds for each value of  $n$ .

For  $1 \leq n \leq 50$ , we use Kash [3] to obtain the solutions in Theorem 1.1. This completes the proof of the theorem.

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