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NON-NEGATIVE SOLUTIONS OF A CONVOLUTION EQUATION

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Abstract. We show that any Lebesgue measurable function $f\colon \mathbb{R} \to [0,\infty)$ satisfying

$$f(x) = \int_0^\infty f(x+y)f(y)\,dy$$

has the form

$$f(x) = 2\lambda e^{-\lambda x}$$

with a $\lambda \in [0,\infty)$.

Résumé. Nous démontrons que toute fonction mesurable au sens de Lebesgue $f\colon \mathbb{R} \to [0,\infty)$ satisfaisant à

$$f(x) = \int_0^\infty f(x+y)f(y)\,dy$$

est de la forme

$$f(x) = 2\lambda e^{-\lambda x}$$

avec un $\lambda \in [0,\infty)$.

1. Referring to [2, Problem 7, p. 194, posed by József Bukszár and presented by János Aczél] we consider the equation

$$f(x) = \int_0^\infty f(x+y)f(y)\,dy. \tag{1}$$

Our result reads as follows.

Theorem. *Any Lebesgue measurable function $f\colon \mathbb{R} \to [0,\infty)$ satisfying* (1) *for $x \in \mathbb{R}$ has the form*

$$f(x) = 2\lambda e^{-\lambda x}$$

with a $\lambda \in [0,\infty)$.

We will prove this theorem in Section 3 based on some of our results concerning

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solutions of

$$\varphi(x) = \int_{\mathbb{R}} \varphi(x+y)\nu\,(dy) \tag{2}$$

and established in [1]. For the reader's convenience we present these results in Section 2 as lemmas. Note however that (1) has also solutions $f\colon \mathbb{R} \to \mathbb{R}$ which change the sign, *e.g.*,

$$f(x) = 4\lambda(1-\lambda x)e^{-\lambda x}$$

with a $\lambda \in (0,\infty)$, and that the paper [3] by L. v. Wolfersdorf and E. Wegert deals with more general equations. We thank Professor Vladimir Mityushev for calling our attention to that paper.

2. Assume ν is a Borel measure on $\mathbb{R}$ with $\nu(\mathbb{R}\setminus\{0\}) > 0$ and every point of $\mathbb{R}$ has a neighbourhood of finite measure ν. Let G denote the additive subgroup of $\mathbb{R}$ generated by the support of ν (*i.e.*, by the set of all points each neighbourhood of which has a positive measure). The following has been proved (among others) in [1] (as Corollary 1.1, a part of Theorem 2.1 and Lemma 2.2, respectively).

Lemma 1. *Any non-negative continuous solution of* (2) *defined on a coset of* G *either vanishes everywhere, or is positive everywhere.*

Lemma 2. *If* (2) *has a non-negative and non-zero continuous solution defined on a coset of* G *then there exists a real number* λ *such that*

$$\int_{\mathbb{R}} e^{\lambda y}\nu\,(dy) = 1. \tag{3}$$

Lemma 3. *If* $\nu(\mathbb{R}) = 1$ *then every non-negative continuous solution of* (2) *defined on a coset of* G *is monotonic.*

3. Assume $f\colon \mathbb{R} \to [0,\infty)$ is Lebesgue measurable and satisfies (1) for $x \in \mathbb{R}$. To get the required form of f we may assume that it is non-zero.

Since

$$f(0) = \int_0^\infty f(y)^2\,dy \tag{4}$$

the square of each of the functions $g, h\colon \mathbb{R} \to [0,\infty)$, defined by

$$g = f\cdot \mathbb{I}_{[0,\infty)}, \quad h(x) = g(-x) \quad \text{for } x \in \mathbb{R},$$

is Lebesgue integrable. Consequently, the convolution $g * h$ is continuous and

$$(g*h)(x) \le \left(\int_{\mathbb{R}} g(y)^2\,dy\right)^{1/2} \cdot \left(\int_{\mathbb{R}} h(y)^2\,dy\right)^{1/2} = \int_0^\infty f(y)^2\,dy = f(0)$$

for $x \in \mathbb{R}$. On the other hand, by (1) we have

$$(g * h)(x) = f(x) \quad \text{for } x \in [0, \infty).$$

This shows that $f|_{[0,\infty)}$ is continuous and

$$f(x) \leq f(0) \quad \text{for } x \in [0, \infty). \tag{5}$$

In particular, there exists a positive δ such that $f(x) \geq \frac{1}{2} f(0)$ for $x \in [0, \delta]$ whence, according to (1),

$$f(x) \geq \frac{1}{2} f(0) \int_0^\delta f(x+y)\, dy = \frac{1}{2} f(0) \int_x^{x+\delta} f(z)\, dz \quad \text{for } x \in \mathbb{R}.$$

This implies that f is Lebesgue integrable on every compact interval.

To prove that f is continuous fix arbitrarily $x_0 \in (-\infty, 0]$ and an $a \in (-x_0, \infty)$. Putting

$$F = f \cdot \mathbb{I}_{[-a, 2(x_0+a)]}, \qquad c = \max\{f(0), f(0)^{1/2}\},$$

and using (1), (5), Hölder's inequality and (4), for every $\xi \in [-(x_0+a), x_0+a]$ we have

$$\begin{aligned}
&|f(x_0+\xi) - f(x_0)| \\
&\quad \leq \int_0^\infty |f(x_0+\xi+y) - f(x_0+y)| f(y)\, dy \\
&\quad \leq f(0) \int_0^a |f(x_0+\xi+y) - f(x_0+y)|\, dy \\
&\qquad\quad + \left(\int_a^\infty |f(x_0+\xi+y) - f(x_0+y)|^2\, dy \right)^{1/2} \cdot f(0)^{1/2} \\
&\quad \leq c \left(\int_{x_0}^{x_0+a} |F(z+\xi) - F(z)|\, dz + \left(\int_{x_0+a}^\infty |g(z+\xi) - g(z)|^2\, dz \right)^{1/2} \right).
\end{aligned}$$

Due to the Lebesgue integrability of F and of the square of g the last sum tends to zero when ξ does and the continuity of f follows.

Now define a Borel measure ν on $\mathbb{R}$ by

$$\nu(B) = \int_{B \cap [0,\infty)} f(y)\, dy.$$

Since $f|_{[0,\infty)}$ is continuous and non-zero, ν has the properties stated at the beginning of Section 2 and the additive subgroup of $\mathbb{R}$ generated by the support of ν coincides with the whole group $\mathbb{R}$. Moreover, f is a continuous solution of

(2). According to Lemma 1, the function f is positive everywhere and Lemma 2 provides a real number λ satisfying (3). Consider another Borel measure μ on $\mathbb{R}$ given by

$$\mu(B) = \int_B e^{\lambda y} \nu\,(dy)$$

and a function $\varphi\colon \mathbb{R} \to (0,\infty)$ defined by

$$\varphi(x) = f(x)e^{-\lambda x}.$$

Clearly, $\mu(B)$ vanishes exactly when $\nu(B)$ does, (3) means that $\mu(\mathbb{R}) = 1$, and

$$\varphi(x) = \int_{\mathbb{R}} \varphi(x+y)\mu\,(dy) = \int_{[0,\infty)} \varphi(x+y)\mu\,(dy),$$

i.e.,

$$\int_{[0,\infty)} \big(\varphi(x+y) - \varphi(x)\big)\mu\,(dy) = 0 \quad \text{for } x \in \mathbb{R}. \tag{6}$$

In particular, μ shares the properties stated at the beginning of Section 2 with ν, the additive subgroup of $\mathbb{R}$ generated by the support of μ coincides with $\mathbb{R}$ and, according to Lemma 3, φ is monotonic. Hence and from (6) for any fixed $x \in \mathbb{R}$ we have

$$\varphi(x+y) = \varphi(x) \tag{7}$$

for μ–a.e., *i.e.*, ν–a.e., $y \geq 0$, and, since f is positive everywhere, (7) holds for a.e. (with respect to the Lebesgue measure) $y \geq 0$. Due to the continuity of φ equality (7) holds for every $y \geq 0$ and $x \in \mathbb{R}$. This means that φ is constant:

$$\varphi(x) = \varphi(0) = f(0) \quad \text{for } x \in \mathbb{R},$$

or, in other words,

$$f(x) = f(0)e^{\lambda x} \quad \text{for } x \in \mathbb{R}.$$

This jointly with (1) gives $f(0) = -2\lambda$ and ends the proof.

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