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PERFECT MATCHINGS, EIGENVALUES AND EXPANSION

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ABSTRACT. In this note, we prove a sufficient condition for the existence of a perfect matching in a regular graph in terms of its eigenvalues and its expansion constant. We improve a recent result of Brouwer and Haemers.

RÉSUMÉ. Dans cette note, nous prouvons un état suffisant pour l'existence d'un assortiment parfait dans un graphe régulier en termes de ses valeurs propres et son constante d'expansion. Nous améliorons un résultat récent de Brouwer et Haemers.

1. **Preliminaries.** Our graph notation is standard, see West [9]. For a graph G on n vertices, we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of its adjacency matrix. If G is a connected k-regular graph, it is well known that $\lambda_1 = k$ and we let $\lambda(G) = \max_{\lambda_i \neq \pm k} |\lambda_i|$. For $S \subset V(G)$, we let $e(S, S^c)$ denote the number of edges with exactly one endpoint in S. We denote $\text{odd}(G \setminus S)$ the number of odd components of $G \setminus S$. The expansion constant of a graph G is $h(G) = \min \frac{e(S,S^c)}{|S|}$, where the minimum is taken over all $S \subset V(G)$ with $|S| \leq \frac{|V(G)|}{2}$. For a k-regular graph G, it is known (see [1], [2], [8]) that

$$\frac{k-\lambda_2}{2} \leq h(G) \leq \sqrt{2k(k-\lambda_2)}.$$

In this note, we show that if G is a k-regular graph on n vertices, n is even and

$$h(G) > \begin{cases} \dfrac{k-2}{k+1} & \text{if } k \text{ is even} \\ \dfrac{k-2}{k+2} & \text{if } k \text{ is odd} \end{cases} \tag{1}$$

then G has a perfect matching.

Krivelevich and Sudakov [7] show that if G is a k-regular graph and $k - \lambda(G) \geq 2$, then G contains a perfect matching. Recently, Brouwer and Haemers [3] proved the following stronger result.

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THEOREM 1.1. (Brouwer-Haemers, 2005) *A connected k-regular graph on n vertices with eigenvalues $k = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, and n even which satisfies*

$$k - \lambda_3 \geq \begin{cases} 1 - \dfrac{3}{k+1} & \text{if } k \text{ is even} \\ 1 - \dfrac{3}{k+2} & \text{if } k \text{ is odd,} \end{cases} \tag{2}$$

has a perfect matching.

In this note, we slightly improve Brouwer and Haemers' result when k is odd. We prove that if a k-regular graph G satisfies $k - \lambda_3 \geq 1 - \frac{4}{k+2}$, then G has a perfect matching.

The main tool in our proofs is the famous theorem of Tutte (see [9, p. 137]).

THEOREM 1.2. (Tutte, 1947) *A graph G has a perfect matching if and only if*

$$\operatorname{odd}(G \setminus S) \leq |S|$$

for each $S \subset V(G)$.

2. Expansion and perfect matchings.

In this section, we determine a lower bound on the expansion constant of a regular graph that implies the existence of a perfect matching.

THEOREM 2.1. *Let G be a k-regular graph. If*

$$h(G) > \begin{cases} \dfrac{k-2}{k+1} & \text{if } k \text{ is even} \\ \dfrac{k-2}{k+2} & \text{if } k \text{ is odd} \end{cases} \tag{3}$$

then G contains a perfect matching.

PROOF. Assume G has no perfect matching. By Tutte's theorem, it follows that there exists a subset S such that $q = \operatorname{odd}(G \setminus S) > |S| = s$. Let $G_1, \ldots, G_q$ denote the odd components of $G \setminus S$. Denote by n_i and e_i the order and the size of G_i respectively. It is easy to see that $\sum_{i=1}^{q} n_i + s$ is even. Since each n_i is odd, it follows that $q + s$ is even. Because $q > s$, we deduce that $q \geq s + 2$.

For $i \in [q]$, denote by t_i the number of edges with one endpoint in G_i and another in S. Because G is connected, it follows that $t_i \geq 1$ for each $i \in [q]$. Also, since vertices in G_i are adjacent only to vertices in G_i or S, we deduce that $2e_i = kn_i - t_i = k(n_i - 1) + k - t_i$. Because n_i is odd, it follows that $k - t_i$ is even. Thus, t_i and k have the same parity for each i.

The sum of the degrees of the vertices in S is at least the number of edges between S and $\bigcup_{i=1}^{q} G_i$. Thus, $ks \geq \sum_{i=1}^{q} t_i$. Since $q \geq s + 2$, it follows that there are at least three t_i's such that $t_i < k$. This implies there are at least three t_i's satisfying $t_i \le k-2$. If $t_i \le k-2$, then $n_i > 1$. Assume $t_i \le k-2$ for $i \in [3]$. If $t_i \le k-2$, then $n_i(n_i-1) \ge 2e_i = kn_i - t_i \ge kn_i - k + 2$. Thus, $n_i \ge k + \frac{2}{n_i-1}$. Hence, $n_i \ge k+1$. Now, since k and n_i are both odd, we obtain $n_i \ge k+2$.

Now $n_1 + n_2 + n_3 < n$ implies that there is at least one i such that $n_i < \frac{n}{2}$. Assume $n_1 < \frac{n}{2}$. Then $\frac{t_1}{n_1} \le \frac{k-2}{k+1}$ if k is even and $\frac{t_1}{n_1} \le \frac{k-2}{k+2}$ if k is odd. This contradicts the assumption made on the expansion constant and thus, proves the theorem. ■

In the next section, we provide examples that show that this result is best possible.

3. A slight improvement of Theorem 1.1.

In this section, we present the proof of our improvement of Theorem 1.1.

THEOREM 3.1. *If G is a k-regular graph, k is odd and*

$$k - \lambda_3 \ge 1 - \frac{4}{k+2} \tag{4}$$

then G has a perfect matching.

PROOF. Assume that G has no perfect matching. As in the proof of Theorem 2.1, one shows that there exists S such that $G \setminus S$ as at least three components of odd order G_i, $i \in [3]$ such that $t_i \le k-2$ and $n_i \ge k+2$.

Thus, the average degree $\overline{d_i}$ of G_i satisfies the following inequality

$$\overline{d_i} = \frac{2e_i}{n_i} = k - \frac{t_i}{n_i} \ge k - \frac{k-2}{k+2}. \tag{5}$$

Let μ_i denote the largest eigenvalue of G_i for $i \in [3]$. Suppose $\mu_1 \ge \mu_2 \ge \mu_3$. Then, by interlacing in $G_1 \cup G_2 \cup G_3$ (see [6, Chapter 9]), it follows that $\lambda_3 \ge \mu_3$. Now, $\mu_3 \ge \overline{d_3}$ with equality if and only if G_3 is regular. But G_3 is not regular since $k > \overline{d_3} = k - \frac{t_3}{n_3} > k-1$.

Thus, $\lambda_3 \ge \mu_3 > k - \frac{k-2}{k+2}$. This is a contradiction with (4) and proves the theorem. ■

For k even, Brouwer and Haemers [3] construct examples of k-regular graphs with $\lambda_3 = k - 1 + \frac{3}{k+1} + \mathrm{O}(k^{-2})$ that contain no perfect matchings. This is done by taking k copies of G', where G' is K_{k+1} from which a matching of size $\frac{k-2}{2}$ is deleted. Add $k-2$ new vertices and join each of these vertices to a vertex of degree $k-1$ in each G'.

For k odd, Brouwer and Haemers [3] just mention that their construction is similar to the previous one, but slightly more complicated.

We describe below a construction for k odd. Let H' be a graph on $k+2$ vertices whose complement is the union of two disjoint edges and a cycle of length $k-2$. Take k copies of H'. Add $k-2$ new vertices and join each of these vertices to a vertex of degree $k-1$ in each H'. This is a connected k-regular graph that has k^2+3k-2 vertices and no perfect matching. Also,

$$\lambda_3 = \lambda_1(H') = \frac{k-3+\sqrt{k^2+2k+17}}{2} = k-1+\frac{4}{k+2}+\mathrm{O}(k^{-2}).$$

This shows that for n even and k odd, there are k-regular graphs with no perfect matching, for which λ_3 gets arbitrarily close to the value from Theorem 3.1. These examples also show that Theorem 2.1 is best possible.

Note that the existence of perfect matchings in a k-regular graph does not even imply $k-\lambda_2 > \epsilon > 0$. The Cayley graph of $\mathbb{Z}_{2n}$ with generating set $S=\{\pm 1, n\}$ is a 3-regular graph that contains 3 disjoint perfect matchings and satisfies $\lambda_2 = 2\cos\frac{2\pi}{n}+1$. The difference $k-\lambda_2 = 2-2\cos\frac{2\pi}{n}$ tends to 0 as n gets large.

On the other hand, it follows from a theorem of Weyl that by adding perfect matchings to regular graphs, the eigenvalues of the new graphs increase by at most 1. This was used in [5] to construct new families of regular graphs with small non-trivial eigenvalues. See also [4] for related results regarding the eigenvalues of regular graphs.

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