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THE KY FAN FIXED POINT THEOREM ON STAR-SHAPED DOMAINS

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ABSTRACT. We extend the Ky Fan fixed point theorem for a set-valued map with convex non-empty values and open fibers defined on a star-shaped compact subset in a topological vector space. The proof is elementary and based on a matching theorem of Ky Fan for open covers of convex sets.

RÉSUMÉ. Nous généralisons le théorème de point fixe de Ky Fan pour une application à valeurs d'ensembles convexes non-vides et à images inverses ouvertes définie sur un domaine compact et étoilé dans un espace vectoriel topologique. La preuve est élémentaire et se base sur un théorème de "matching" de Ky Fan pour les recouvrements ouverts d'ensembles convexes.

1. **Introduction.** The fixed point theorem of Ky Fan (see [1] and [2]) asserts that:

THEOREM 1. *Given a non-empty compact convex subset X of a topological vector space E and a set-valued map $A\colon X \longrightarrow 2^X$ satisfying the conditions:*

(i) *$A(x)$ is convex and non-empty for all $x \in X$, and*
(ii) *$A^{-1}(y)$ is open in X for every $y \in X$.*

Then A has a fixed point, i.e., a point $x_0 \in A(x_0)$.

This theorem is fundamental in convex analysis. It is precisely the formulation of the fixed point property for relations with convex images and open fibers contained in the cartesian product of a compact convex set X by itself: such relations must intersect the diagonal in $X \times X$. This result can be used as the basis for the study of variational inequalities, complementarity problems, minimax inequalities, equilibria for generalized games, best approximations problems, *etc.* Theorem 1 turns out to be equivalent to a host of deep results in functional analysis such as the KKM principle, the Ky Fan infsup inequality, and the Brouwer fixed point theorem (see, *e.g.*, [1], [2], [3] and references therein).

It is important to point out that both the convexity and the compactness of the domain X play a crucial role in the proof of Theorem 1. Indeed, the proof relies on the fact that a set-valued map A from a *compact* topological space X into a

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convex subset Y of a topological vector space that verifies hypotheses (i) and (ii) in Theorem 1 admits a continuous single valued selection s (*i.e.*, $s(x) \in A(x)$, $\forall x \in X$) with values in a finite convex polytope (*i.e.*, $s(X) \subset C :=$ the convex hull of a finite subset of Y). In case $X = Y$ is *compact and convex*, the Brouwer fixed point theorem, applied to the restriction of the selection s to the polytope C, yields the existence of a fixed point $x_0 = s(x_0) \in A(x_0)$ for both the selection and the set-valued map.

The purpose of this note is to extend this fixed point property to the case where X is star-shaped instead of convex.

Vector spaces are assumed to be real and topologies are Hausdorff.

A set-valued map satisfying hypotheses (i) and (ii) of Theorem 1 is called a *Ky Fan map*.

2. The main result. Recall that a subset X of a vector space E is *star-shaped at a point* $\bar{x} \in X$ if and only if for any other point $x \in X$, the line segment $[\bar{x}, x]$ remains in X; the point $\bar{x}$ is said to be a *center* of X (X is convex if and only if it is star-shaped in each of its points). It is readily seen that X is star-shaped at $\bar{x} \in X$ if and only if

$$X \cap C^{+}_{E \setminus X}(\bar{x}) = \varnothing \tag{1}$$

where

$$C^{+}_{E \setminus X}(\bar{x}) := \{y \in E : \exists x \in E \setminus X, \exists t > 1, \text{ such that } y = \bar{x} + t(x - \bar{x})\}$$

is the truncated cone with vertex at $\bar{x}$ and based on $E \setminus X$. Indeed, assume without loss of generality that $\bar{x} = 0_E$ and let $X \cap C^{+}_{E \setminus X}(0_E) = \varnothing$. If X is not star-shaped at 0_E, then there exist $(x, t) \in X \times (0, 1)$ with $y = tx \notin X$, *i.e.*, $y \in E \setminus X$ and $x = \frac{1}{t} y \in C^{+}_{E \setminus X}(0_E)$, a contradiction. The converse is obvious.

The main ingredient in proving our main result is the following matching theorem for open covers of convex sets.

THEOREM 2. (Ky Fan, [3, Theorem 3]) *Let V be a convex subset of a topological vector space E, U be a non-empty subset of V, and let $\{O(x) : x \in U\}$ be an open cover of V. Assume that there exists a subset U_0 of U such that $V \setminus \bigcup_{x \in U_0} O(x)$ is compact or empty and that U_0 is contained in a convex compact subset C of V, then there exists a finite subset $\{x_1, \ldots, x_n\}$ of U such that:*

$$\mathrm{Conv}\{x_1, \ldots, x_n\} \cap \bigcap_{i=1}^{n} O(x_i) \neq \varnothing. \tag{2}$$

REMARK 1. It is worth pointing out that Theorem 2 is a generalization of the KKM principle from which it derives.

We are ready to formulate and prove the main result of this note.

THEOREM 3. *Let X be a non-empty star-shaped compact subset of a topological vector space E and let $A\colon X \longrightarrow 2^X$ be a Ky Fan map. Then A has a fixed point.*

PROOF. Since X is compact, A admits a continuous selection $s\colon X \longrightarrow E$ with values in a convex polytope $C \subset E$. We can assume with no loss of generality that the center $\bar{x}$ of X belongs to C (we could otherwise replace C by the convex hull of C and $\bar{x}$, also a convex polytope).

Let $U := s(X) \cup U_0$ where $U_0 := \{\bar{x}\}$ and let $\{O(x) : x \in U\}$ be the family of open subsets of C defined by:

$$O(x) := A^{-1}(x) \cap C,\ \forall x \in s(X) \quad \text{and} \quad O(\bar{x}) := C \setminus X.$$

Note that if $y \in C \cap X$ then $x = s(y) \in A(y) \Leftrightarrow y \in A^{-1}(x) \Rightarrow y \in O(x)$; also, $y \in C \setminus X \Leftrightarrow y \in O(\bar{x})$, that is, $\{O(x) : x \in U\}$ is an open cover of C.

Obviously, U_0 being a singleton is compact and convex and $C \setminus O(\bar{x}) = C \cap X$ is compact. Theorem 2 with $V = C$ implies the existence of a finite set $\{x_1, \dots, x_n\}$ of U such that (2) holds. Given an element $x_0 \in \operatorname{Conv}\{x_1, \dots, x_n\} \cap \bigcap_{i=1}^n O(x_i)$, two cases are possible:

CASE 1. $\bar{x} \notin \{x_1, \dots, x_n\}$.

Thus, $x_0 \in O(x_i) \subseteq A^{-1}(x_i) \Rightarrow x_i \in A(x_0)$, $i = 1, \dots, n$. Since $A(x_0)$ is a convex set, $x_0 \in A(x_0)$ is a fixed point for A.

CASE 2. $\bar{x} \in \{x_1, \dots, x_n\}$.

In this case, $\bar{x} = x_{i_0}$ for some $i_0 \in \{1, \dots, n\}$. Thus, $x_0 \in C \setminus X$ and $x_0 \in A^{-1}(x_i)$ for $i \neq i_0$. Moreover, $x_0 = \sum_{i=1}^n \lambda_i x_i = \sum_{i=1, i\neq i_o}^n \lambda_i x_i + \lambda_{i_0}\bar{x}$ is a convex combination. Clearly, $\lambda = 1 - \lambda_{i_0} = \sum_{i=1, i\neq i_o}^n \lambda_i > 0$ for otherwise $\lambda_{i_0} = 1$, *i.e.*, $x_0 = \bar{x} \in C \setminus X$ contradicting the fact that $\bar{x} \in X$. Dividing by λ we obtain:

$$\frac{1}{\lambda}x_0 = \frac{1}{\lambda}\sum_{i=1, i\neq i_o}^n \lambda_i x_i + \frac{1-\lambda}{\lambda}\bar{x} \Leftrightarrow \frac{1}{\lambda}x_0 + \frac{\lambda - 1}{\lambda}\bar{x} = z$$

$$\text{where } z := \frac{1}{\lambda}\sum_{i=1, i\neq i_o}^n \lambda_i x_i \text{ is a convex combination.}$$

Also, since $A(x_0)$ is convex and $x_i \in A(x_0)$ for all $i \neq i_0$, then $z \in A(x_0) \subset X$. On the other hand $z = \bar{x} + t(x_0 - \bar{x})$ with $t = \frac{1}{\lambda} > 1$, *i.e.*, $z \in C^+_{E \setminus X}(\bar{x})$ contradicting (1). ■

REMARK 2. The theorem also holds for so-called Φ^*-maps in the sense of [2].

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