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ON AN OPTIMAL MULTIVARIATE MULTIPERIOD MEAN-VARIANCE PORTFOLIO

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Abstract. We offer a closed-form solution to an unconstrained multiperiod mean-variance problem when the investor's portfolio consists of multiple stocks and bonds and where only fairly general conditions are imposed on these assets. As the reader will see, among the advantages of the proposed solution one finds that it is general enough to allow for the incorporation of time dependence in modelling the relative excess rate of return, as well as dependence, if one so wishes, on exogeneous variables, such as economic factors that might have the property to improve substantially our ability to assess future rate of return.

Résumé. Nous proposons une solution fermée à un problème sans contraintes de moyenne-variance en contexte multipériodique lorsque le portefeuille de l'investisseur est constitué de plusieurs titres risqués et d'un titre sans risque et que des conditions plutôt générales sont imposées à ces titres. Comme le lecteur constatera, parmi les avantages de notre solution, c'est qu'elle est suffisament générale pour permettre une dépendance temporelle dans la modélisation du rendement excédentaire relatif ainsi qu'une dépendance avec des variables exogènes telles que des facteurs économiques qui bénéficieraient de la propriété d'améliorer la capacité d'estimation des rendements futurs.

1. **The multiperiod setting.** We place ourselves in a context where a small investor holds a portfolio consisting of N risky assets and one riskless asset in a frictionless market. We use the term small investor in the sense that the composition of the portfolio held by this investor at any given time does not affect future prices in the market.

Let T be the terminal date and for $t = 0, \ldots, T$ and $n = 0, \ldots, N$ let $\omega_t^{(n)}$ be the fraction or weight of the porfolio allocated to the n-th risky asset just before time t, $R_t^{(n)}$ the rate of return of the n-th risky asset between time $t-1$ and t, r_t the deterministic rate of return of the riskless asset between time $t-1$ and t, then $\tilde{R}_t$ the rate of return the portfolio between time $t-1$ and t can be

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expressed as:

$$\begin{aligned}\tilde{R}_t &= \sum_{n=1}^{N} \omega_t^{(n)} R_t^{(n)} + \Big(1 - \sum_{i=1}^{n} \omega_t^{(n)}\Big) r_t \\ &= r_t + \sum_{n=1}^{N} \omega_t^{(n)} (R_t^{(n)} - r_t).\end{aligned}$$

Furthermore

$$\begin{aligned}(1+r_t)^{-1}(1+\tilde{R}_t) &= 1 + \sum_{n=1}^{N} \omega_t^{(n)} (1+r_t)^{-1} (R_t^{(n)} - r_t) \\ &= 1 + \rho_t \omega_t^*\end{aligned}$$

where the weight ω_t and relative excess rate of return ρ_t are N-dimensional vectors defined respectively by

$$\omega_t = [\omega_t^{(n)}]_{1\le n\le N} \text{ and } \rho_t = [(1+r_t)^{-1}(R_t^{(n)} - r_t)]_{1\le n\le N}.$$

Notice that if X_{t-1} represents an available amount of money at time $t-1$, than the expression $\rho_t^{(n)} = (1+r_t)^{-1}(R_t^{(n)} - r_t) = \frac{(1+R_t^{(n)})X_{t-1} - (1+r_t)X_{t-1}}{(1+r_t)X_{t-1}}$ simply represents the relative gain (or loss) of investing that amount in the n-th stock between time $t-1$ and time t, rather than investing it in the bond market.

For each $n = 1, \ldots, N$, consider $\mathcal{F}_t^{(n)}$ the σ-field generated by $\{\rho_s^{(n)}, 0 \le s \le t\}$, $\mathcal{G}_t$ the σ-field generated by exogeneous variables called signals $\{S_s, 0 \le s \le t\}$, $\mathcal{H}_t^{(n)} = \mathcal{F}_t^{(n)} \vee \mathcal{G}_t$ and $\mathcal{H}_t = \bigvee_{n=1}^{N} \mathcal{H}_t^{(n)}$. Thus $\mathcal{H}_t$ consists of all information disposable to the agent up to time t concerning the history of the relative excess rate of return of all the risky assets as well as economic indicators of the flow of the market. Therefore ω_t is considered to be $\mathcal{H}_{t-1}$-measurable, meaning that the fraction of wealth allocated to the risky asset is determined just before time t and based on information given up to time $t-1$.

2. The main results. The primary objective is to develop a strategy that, given a desired expected discounted cumulative rate of return

$$\Pi_{s=1}^{T}(1+r_s)^{-1}(1+\widetilde{R}_s) - 1$$

of the portfolio at terminal date T, minimizes the variance of this global rate of return. For a survey of this well-known problem we refer the reader to [3].

For the following results define $\sum_{i\epsilon\phi} \alpha_i = 0$ and $\Pi_{i\epsilon\phi}\alpha_i = 1$ for sums and products over the empty set. Furthermore, define recursively for $t = T-1$, $T-2, \ldots, 2, 1, 0$, the scalar

$$\tau_t = \mu_t^* V_t^{-1} \mu_t \tag{1}$$

where $\mu_t^* = \big[\mathbb{E}[(1-\sum_{s=t+1}^{T-1}\tau_s)\rho_{t+1}^{(n)}|\mathcal{H}_t]\big]_{1\le n\le N}$ is an N-dimensional vector and we assume that $V_t = \big[\mathbb{E}[(1-\sum_{s=t+1}^{T-1}\tau_s)\rho_{t+1}^{(n)}\rho_{t+1}^{(m)}|\mathcal{H}_t]\big]_{1\le n,m\le N}$ is an $N\times N$ invertible matrix.

We will show in the following that a portfolio with the weight matrix associated to the risky assets defined by

$$\omega_t = -\Big(1+\frac{\lambda_t}{2\Pi_{s=1}^{t-1}(1+r_s)^{-1}(1+\widetilde{R}_s)}\Big)\mu_{t-1}^* V_{t-1}^{-1} \tag{2}$$

when $\Pi_{s=1}^{T-1}(1+\widetilde{R}_s)>0$, has all the given properties to be an optimal solution to the unconstrained multiperiod mean-variance portfolio problem if we make a judicious choice of the constant λ_t. Proofs of all the following results stated here are available upon request.

First, we shall exhibit sufficient conditions to safeguard against parameters τ_t, ω_t and λ_t being ill-defined.

Lemma 2.1. *With the conventions above, there holds almost surely, for every $t\in\{0,1,\ldots,T-2\}$*

$$0\le\tau_t\le\mathbb{E}\Big[1-\sum_{s=t+1}^{T-1}\tau_s\ \Big|\ \mathcal{H}_t\Big]\le 1.$$

Under the hypothesis $\mathbb{E}[\tau_0]>0$, the parameters appearing in formulas (1) and (2) are well-defined.

As in the univariate case treated in [4], by establishing explicit expressions for the conditional variance and covariance of the total rate of return of the portfolio from a given time up to the terminal date we derive a solution to a general multiperiod mean-variance problem.

Proposition 2.2. *Let $\mathbb{E}[\Pi_{t=1}^{T}(1+r_t)^{-1}(1+\widetilde{R}_t)]=1+c>1$ be fixed and assume $\mathbb{E}[\tau_0]>0$. Then the weights ω_t, $t=1,\ldots,T$ defined by (2) by setting*

$$\lambda_t = -2\Big[\frac{c}{\sum_{s=0}^{T-1}\mathbb{E}(\tau_s)}+1\Big]\in\mathbb{R} \tag{3}$$

minimize the variance $\mathbb{VAR}[\Pi_{t=1}^{T}(1+\widetilde{R}_t)]$ and this minimal variance is given by

$$\mathbb{VAR}[\Pi_{t=1}^{T}(1+r_t)^{-1}(1+\widetilde{R}_t)] = c^2\Big[\frac{1}{\mathbb{E}[\sum_{s=0}^{T-1}\tau_s]}-1\Big].$$

The following shows that a solution to the multiperiod mean-variance problem gives rise to a natural extension of the classical capital asset pricing model in a multiperiod setting.

Corollary 2.3. *Let P be a portfolio with weights ω_t, $t=1,\ldots,T$ satisfying the conditions of Proposition 2.2, then for every other portfolio Q we have*

$$\mathbb{E}[\Pi_{t=1}^{T}(1+r_t)^{-1}(1+\widetilde{R}_t^{Q})-1] = \beta_t\big(\mathbb{E}[\Pi_{t=1}^{T}(1+r_t)^{-1}(1+\widetilde{R}_t^{P})-1]\big)$$

where

$$\beta_t = \frac{\mathbb{COV}(\Pi_{t=1}^T (1+r_t)^{-1}(1+\widetilde{R}_t^P), \Pi_{t=1}^T (1+r_t)^{-1}(1+\widetilde{R}_t^Q))}{\mathbb{VAR}(\Pi_{t=1}^T (1+r_t)^{-2}(1+\widetilde{R}_t^P))}.$$

It is worth mentioning that results by [1] represent a special case of Proposition 2.2 where the vectors $\rho_t = [(1+r_t)^{-1}(R_t^{(n)} - r_t)]_{1\le n\le N}$ are assumed to be independent, a very strong restriction to put on models in practice. Furthermore, our proposition extends some univariate results of Schweizer [2], although he considered a more general problem by allowing options in his portfolio.

3. **Example.** In this section, we shall exhibit a rich class of models for the relative excess rate of return $\rho_t^{(n)} = (1+r_t)^{-1}(R_t^{(n)} - r_t)$ of the n-th risky asset at time t. These models were selected in order to satisfy certain basic criteria, namely simplicity of use, computational efficiency and interpretability. All of them offer a simplified symbolic representation for the optimal values of the parameters τ_t, ω_t and λ_t defining the solution to the multiperiod mean-variance portfolio, given by equations (1), (2) and (3). Many will be seen to afford computational implementations with tractability throughout the calculation process, thereby ensuring accurate results in real time, even when parameter estimation is required.

Our example is a very general class of models driven by *stationary independent multiplicative market impulses* (hereafter called the SIMMI class). To describe this class, we first need a sequence of independent N-dimensional random vectors $\{\xi_t : t \ge 1\}$, where ξ_t represents the random fluctuations (the noise) of the relevant part of the market at time t. We only require that the sequence $\{\xi_t : t \ge 1\}$ be identically distributed, independent of the whole past of the market and finally that the $N \times N$ matrices $\mathbb{E}[\xi_t^* \xi_t]$ be invertible for every $t = 0, 1, \dots, T-1$. We assume that ρ_t is given by some $\mathcal{H}_{t-1}$-measurable real-valued random $N \times N$ invertible matrix Ψ_{t-1} which is perturbed by real-valued market impulse N-dimensional vectors ξ_t, in the following multiplicative form: for every $t \ge 1$, we have

$$\rho_t = \xi_t \Psi_{t-1}.$$

Thus the SIMMI class is characterized by the fact that the noise source acts by dilating or compressing the signal, rather than translating it, as occurs in additive models, thereby ensuring (by way of a scaling effect) that the fluctuations of the excess rate around its mean value are heteroscedastic in all but the most trivial special cases.

The sequence of relative excess rates of return $\{\rho_t : t \ge 1\}$ will not in general form a stationary stochastic process, only the sequence of market impulses $\{\xi_t : t \ge 1\}$ will exhibit this property, since it is built in. Because of these restrictions, the parameters $\{\tau_t : t = 0, 1, \dots, T-1\}$ in (1) now form the first few terms of a geometric progression: if we denote

$$\theta = \tau_{T-1} = \mathbb{E}(\xi_t)(\mathbb{E}[\xi_t^* \xi_t])^{-1}\mathbb{E}(\xi_t^*),$$

then clearly we have $\tau_{T-2} = \theta(1-\theta)$, and inductively,

$$\tau_t = \theta(1-\theta)^{T-t-1} \quad \text{for } t = 0, 1, 2, \ldots, T-1.$$

Furthermore, expression (2) becomes

$$\lambda_t = -2\Big[\frac{c}{1-(1-\theta)^T} + 1\Big].$$

The optimal weights (2) can now be rewritten as

$$\omega_t = -\Big((1+r) + \frac{\lambda_t}{2(1+r)^{T-t}\Pi_{s=1}^{t-1}(1+\widetilde{R}_s)}\Big)\mathbb{E}(\xi_t)(\mathbb{E}[\xi_t^*\xi_t])^{-1}(\Psi_{t-1}^*)^{-1},$$

a known function of $\mathbb{E}(\xi_t)$, $\mathbb{E}[\xi_t^*\xi_t]$ and Ψ_{t-1}. Therefore, computation of the optimal weights can be effected explicitly with the estimation of only two parameters ($\mathbb{E}(\xi_t^*)$ and $\mathbb{E}[\xi_t^*\xi_t]$), once estimates of the true market signals $\{\Psi_s : s < t\}$ for all preceding times, have been extracted from the noisy data $\{\rho_s : s < t\}$ through nonlinear regression or some other statistical technique.

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