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A NOTE ON THE DIOPHANTINE EQUATION $X^2 - DY^4 = 1$ WITH PRIME DISCRIMINANT

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Abstract. Ljunggren proved that for a nonsquare positive integer d, the quartic Diophantine equation $X^2 - dY^4 = 1$ has at most two solutions in positive integers, and gave precise information on the location of these solutions in the case that two such solutions actually do exist. Inspired by recent work of P. Samuel, we show that in the case that $d > 3$ is prime, there is at most one positive integer solution to $X^2 - dY^4 = 1$, and that it arises from the fundamental solution of the Pell equation $X^2 - dY^2 = 1$.

Résumé. Ljunggren a montré que pour un nombre entier positif de nonsquare d, l'équation $X^2 - dY^4 = 1$ a au plus deux solutions dans des nombres entiers positifs, et a fourni l'information précise sur l'endroit de ces solutions dans le cas que deux telles solutions réellement existent. Inspirer par les travaux récents de P. Samuel, nous montrons cela dans le cas que $d > 3$ est une nombre premier, il y a au plus une solution positive de nombre entier $X^2 - dY^4 = 1$, et qu'elle résulte de la solution fondamentale de l'équation de Pell $X^2 - dY^2 = 1$.

1. Introduction.

Ljunggren [5] proved that the Diophantine equation

$$X^2 - dY^4 = 1 \tag{1}$$

has at most two solutions in positive integers, and gave precise information on the location of the solutions when two solutions exist. This general theorem has recently been improved substantially in [3]. Specifically, in that paper the assumption of the existence of two solutions has been removed, and a conclusion similar to that in Ljunggren's result has been proved.

We first define some notation that will be used throughout the paper. For a positive nonsquare integer d, we denote by $\epsilon_d = T + U\sqrt{d}$ the minimal unit in $\mathbf{Z}[\sqrt{d}]$ of norm 1, and for $k \geq 1$, we define $T_k + U_k\sqrt{d} = (T + U\sqrt{d})^k$.

Theorem A (Togbe, Voutier, Walsh 2004).

(1) *There are at most two positive integer solutions* (x, y) *to equation* (1)*. If two solutions* $y_1 < y_2$ *exist, then* $y_1^2 = U_1$, $y_2^2 = U_2$, *except only if* $d = 1785$ *or* $d = 16 \cdot 1785$, *in which case* $y_1^2 = U_1$, $y_2^2 = U_4$.

(2) *If only one positive integer solution* (x, y) *exists to equation* (1), *then* $y^2 = U_l$, *where* $U_1 = lv^2$ *for some squarefree integer* l, *and either* $l = 1$, $l = 2$, *or* $l = p$ *for some prime* $p \equiv 3 \pmod 4$.

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P. Samuel [9] has proved a number of interesting related results in the case that d is prime, or twice a prime, but for equation (1), with d a prime, these results fall short of what is the best possible result. Therefore, the purpose of the present paper is to establish a sharp result on the solutions of equation (1) in the case that d is prime, and hence improve upon Theorem 5.2 in [9]. Serious obstacles stand in the way of proving a similar result in the case that d is twice a prime, and so we do not deal with that case.

THEOREM 1. *If p is a prime number, and x and y are positive integers satisfying $x^2 - py^4 = 1$, then $x + y^2\sqrt{p} = \epsilon_p$, except for $p = 3$, in which case $7 + 2^2\sqrt{3} = \epsilon_3^2$ is the only solution in positive integers to $x^2 - 3y^4 = 1$.*

We remark that this result is sharp in the sense that there are primes p for which the minimal solution to $X^2 - pY^2 = 1$ is of the form $x + y^2\sqrt{p}$. In particular, if p is any prime of the form $x^4 \pm 2$, then the minimal solution to $X^2 - pY^2 = 1$ is $(x^4 \pm 1) + x^2\sqrt{x^4 \pm 2}$. This evidently suggests that there are infinitely many such primes.

2. Preliminary results.

In this section we will collect those results which will be needed in the course of proving Theorem 1, although the following lemma is used implicitly in the proof of Theorem 1, and it is stated here to provide the underlying framework of the proof.

LEMMA 1. *Let $d > 1$ be a squarefree integer, and let $\epsilon_d = T + U\sqrt{d}$ denote the minimal unit (> 1) in $\mathbf{Q}(\sqrt{d})$. Then*

$$\epsilon_d = \tau^2,$$

where

$$\tau = \frac{a\sqrt{m} + b\sqrt{n}}{\sqrt{c}},$$

$c \in \{1, 2\}$, a, b are positive integers for which $U = 2ab/c$, m, n are positive integers for which $d = mn$, m is not a square if $c = 1$, and $a^2m - b^2n = c$.

PROOF. This is well known; for example, see Nagell [8].

LEMMA 2. *Let a and b be odd positive integers such that $aX^2 - bY^2 = 2$ is solvable in odd integers X and Y. Let $\tau_{a,b} = \frac{V\sqrt{a}+U\sqrt{b}}{\sqrt{2}}$ denote its minimal solution with V and U odd positive integers, and*

$$\tau_{a,b}^{2k+1} = \frac{V_{2k+1}\sqrt{a} + U_{2k+1}\sqrt{b}}{\sqrt{2}} \quad (k \geq 0).$$

If (x, y) is a positive integer solution of the quartic equation $aX^2 - bY^4 = 2$, then either $y^2 = U_1$ or $y^2 = U_3$.

Proof. This has recently been proved in [6], improving upon previous work of Ljunggren.

The following is a beautiful generalization of the aforementioned result of Ljunggren on the equation $x^2-2y^4=-1$. The extensive details of the proof are in [2], or alternatively in [10], as this result was proved independently by Yuan.

Lemma 3. (Chen–Voutier and Yuan) *Let $d>3$ be a squarefree integer such that the Pell equation $X^2-dY^2=-1$ is solvable in positive integers, and let $\tau=v+u\sqrt{d}$ denote its minimal solution. The only possible integer solution to the equation $X^2-dY^4=-1$ is $(X,Y)=(v,\sqrt{u})$.*

Lemma 4. *The equations $x^2-2y^4=1$ and $x^4-2y^2=1$ have no solutions in positive integers, the only positive integer solution to the equation $x^4-2y^2=-1$ is $(x,y)=(1,1)$, and the only positive integer solutions to the equation $x^2-2y^4=-1$ are $(x,y)=(1,1),(239,13)$.*

These are all trivial except for the last equation, which Ljunggren first solved in [4].

3. **Proof of Theorem 1.**

The case $p=2$ is dealt with by Lemma 4. Let p be an odd prime, and let x and y be positive integers satisfying $x^2-py^4=1$. The proof of Theorem 1 falls into two cases, depending on the parity of x. Consider first the case that x is even, then $\gcd(x+1,x-1)=1$, and the factorization $(x+1)(x-1)=py^4$ implies that there are coprime odd positive integers u and v for which $(x+1,x-1)=(pu^4,v^4)$ or (v^4,pu^4). Thus, we have that $v^4-pu^4=\pm 2$. If $v^4-pu^4=2$, then from Lemma 2, it follows that

$$\frac{v^2+u^2\sqrt{p}}{\sqrt{2}}=\tau_{1,p}^t,$$

with $t=1$ or $t=3$, while if $v^4-pu^4=-2$, then again from Lemma 2, it follows that

$$\frac{v^2+u^2\sqrt{p}}{\sqrt{2}}=\tau_{p,1}^t,$$

with $t=1$ or $t=3$. In both cases, the possibility $t=3$ is ruled out by noticing that the coefficient V_3 from Lemma 2 can be written in terms of V_1 as $V_3=2aV_1^3-3V_1=V_1(2aV_1^2-3)$, which we claim can never be a square. If it were, then since $\gcd(V_1,2aV_1^2-3)=1$ or 3, it would follow that either V_1 and $2aV_1^2-3$ are both squares, or both three times a square. Since a and V_1 are odd, we see that $2aV_1^2-3\equiv 3 \pmod 4$, hence is never a square. Suppose now that g and h are integers for which $V_1=3g^2$ and $2aV_1^2-3=3h^2$. Then one deduces that $6ag^4=1+h^2$, which is also not possible since $1+h^2$ cannot be divisible by 3. Therefore, $t=1$, forcing $x+y^2\sqrt{p}$ to be the minimal solution to $X^2-pY^2=1$.

If x is odd, then the factorization $(x+1)(x-1)=py^4$ leads to the following four possibilities, for positive integers u,v:

(i) $x+1=2u^4, x-1=8pv^4$, which implies that $u^4-4pv^4=1$;

(ii) $x+1=8u^4, x-1=2pv^4$, which implies that $4u^4-pv^4=1$;

(iii) $x+1=2pu^4, x-1=8v^4$, which implies that $pu^4-4v^4=1$;

(iv) $x+1=8pu^4, x-1=2v^4$, which implies that $4pu^4-v^4=1$.

In case (i), we notice that either $u^2-1=2w^4$ or $u^2+1=2w^4$ for some positive integer w. By Lemma 4, it follows that $u=1$, in which case $v=0$, or $u=239$, in which case the equation $u^4-4pv^4=1$ does not hold for any prime p.

In case (ii), we see that either $2u^2-1=w^4$ or $2u^2+1=w^4$ for some positive integer w, and so again by Lemma 4, it follows that $u=1$, in which case $p=3$, $x=7$, leading to the only counterexample $7+4\sqrt{3}=\epsilon_3^2$.

In cases (iii) and (iv), the result follows immediately from Lemma 3.

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