C. R. Math. Rep. Acad. Sci. Canada Vol. **27**, (2), 2005 pp. 41–47

ZEROS OF A REAL LINEAR RECURRENCE OF DEGREE $N \geq 4$

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Presented by M. Ram Murty, FRSC

Abstract. Let $S = \{a_i\}_{i=0}^{\infty}$ be a real linear recurrence of degree n with companion polynomial $f_S(x)$. Let N_S be the zero-multiplicity for S. Assume that the roots of $f_S(x)$ are simple, real, and nondegenerate. When $n = 3$, Smiley [2] and Picon [3] showed $N_S \leq 3$. When $n = 4$, we establish the sharp bound $N_S \leq 5$. In general n, we prove $N_S \leq 2n - 3$.

Résumé. Soit $S = \{a_i\}_{i=0}^{\infty}$ une suite définie par une relation de récurence linéaire réels de degré n avec polynôme charactéristique $f_S(x)$. Désignons par N_S le zéro-multiplicité de S. Supposons que les racines de $f_S(x)$ soient simples, réelles, et non-dégénérées. Dans le cas $n = 3$, Smiley [2] et Picon [3] ont obtenu le resultat $N_S \leq 3$. Dans le cas $n = 4$, nous démontrons la borne optimale $N_S \leq 5$. Enfin nous démontrons que, étant donné un entier n quelconque, $N_S \leq 2n - 3$.

An n-th degree real linear recurrence is a sequence $S = \{a_i\}_{i=0}^{\infty}$ of real numbers defined by its initial values $a_0, a_1, \ldots, a_{n-1} \in \mathbf{R}$ and by the linear recurrence relation

$$a_j = c_1 a_{j-1} + c_2 a_{j-2} + \cdots + c_n a_{j-n}, \quad \text{for } j \geq n, \tag{1}$$

where $c_1, c_2, \ldots, c_n$ are a fixed set of real numbers and $c_n \neq 0$. If all the terms of S are 0, we call S the zero sequence. To a recurrence S, we associate its companion (or characteristic) polynomial

$$f_S(x) = x^n - c_1 x^{n-1} - \cdots - c_{n-1} x - c_n = \prod_{i=1}^{m} (x - \alpha_i)^{e_i},$$

with roots $\alpha_i \in \mathbf{C}$. The sequence S is called simple if $e_i = 1$ for all i. The sequence S is called nondegenerate if α_i/α_j is not a root of unity for all $1 \leq i < j \leq m$.

Let S be a nonzero sequence. The zero-multiplicity N_S is defined to be the number of solutions $k \geq 0$ to the equation $a_k = 0$. If S is nondegenerate, then N_S is finite by the Skolem–Mahler–Lech Theorem [6]. Ward [11] conjectured the existence of a bound for N_S which depends only on the degree of S. Schmidt [8] proved this conjecture by establishing the bound $N_S \leq \exp\bigl(\exp(\exp(3n \ln n))\bigr)$. Schlickewei and Schmidt [7] subsequently improved the bound to

$$N_S \leq (2n)^{35n^3}.$$

Received by the editors on January 18, 2005.
AMS subject classification: 11B37.

We now examine the best possible bounds N_S for sequences S with low degree. Let S be a nonzero, nondegenerate recurrence sequence of degree three. If we assume $a_i, c_i \in \mathbf{Q}$, Beukers [1] showed that $N_S \leq 6$. If $a_i, c_i \in \mathbf{R}$ and in addition, S is simple and the roots of $f_S(x)$ are real, Smiley [2] showed $N_S \leq 3$. Picon [3] and Kulkarni and Sury [4] subsequently gave alternate proofs of Smiley's theorem and Scott [9] extended the result to all real nondegenerate cubic recurrences such that $f_S(x)$ has real roots. Ward [10] and Picon [3] also determined necessary conditions on k for $a_k = 0$.

In this paper, we generalize the work of [2], [4], [3] and establish an analogue of Smiley's result for simple, nondegenerate recurrences of higher degree. We show:

THEOREM 1. *Let S be a nonzero, simple, nondegenerate real linear recurrence of degree n. If the roots of $f_S(x)$ are real, then $N_S \leq 2n - 3$.*

When $n = 4$, Theorem 1 gives the bound $N_S \leq 5$. For some general classes of sequences S, we can improve this bound.

THEOREM 2. *Let $S = \{a_i\}_{i=0}^{\infty}$ be a nonzero simple, nondegenerate real recurrence of degree 4 such that the roots α_i of $f_S(x)$ are real. Number the α_i so that $|\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4|$. Define $\epsilon_i = \alpha_i/|\alpha_i|$. Assume that the following two conditions are satisfied:*

(i) *If exactly two of the ϵ_i are negative, then $\epsilon_1\epsilon_2 = -1$.*
(ii) *If $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$, then $\epsilon_1\epsilon_4 = 1$.*

Then $N_S \leq 4$.

The criteria in Theorem 2 are necessary and the bound of Theorem 1 is sharp. For example, let S be defined by the initial terms $(0, 0, 1, 0)$ and the recursion relation

$$a_n = -3a_{n-1} + 2a_{n-2} + 6a_{n-3} - 4a_{n-4}.$$

The sequence begins $(0, 0, 1, 0, 2, 0, 0, 12, -44, \dots)$ and $N_S = 5$. We have

$$f_S(x) = x^4 + 3x^3 - 2x^2 - 6x + 4 = (x - 1)(x + 2)(x^2 + 2x - 2).$$

The roots of the quadratic are approximately .73205 and -2.73205. Thus, S is simple and nondegenerate and $f_S(x)$ has four real roots. The signs are $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = \epsilon_4 = -1$. Hence condition (i) fails for sequence S.

While the bound $N_S \leq 2n - 3$ of Theorem 1 is sharp for $n = 2, 3, 4$, it is not clear whether it is sharp in general. Computational work when $n = 5$ has not produced a simple, nondegenerate sequence S with $N_S = 7$ such that $f_S(x)$ has real roots.

1. **Properties of S and Δ_n.** We will repeatedly use the following elementary lemma, whose proof we omit.

LEMMA 1. *Let $S = \{a_i\}_{i=0}^{\infty}$ be an n-th degree linear recurrence defined by* (1) *and suppose that $\alpha_1, \dots, \alpha_m$ are the roots of $f_S(x)$. Then*

(a) *For any $k \in \mathbf{Z}$, there is a sequence $T = \{b_i\}_{i=0}^{\infty}$ satisfying the same recurrence relation as S such that $b_i = a_{i+k}$ when* $\min(i, i+k) \geq 0$.
(b) *The sequence $T = \{(-1)^i a_i\}_{i=0}^{\infty}$ is an n-th degree linear recurrence such that $f_T(-x) = (-1)^n f_S(x)$.*
(c) *There is an n-th degree linear recurrence sequence $T = \{b_i\}_{i=0}^{\infty}$ and $j \in \mathbf{Z}$ such that*

   (i) $f_T(x) = -x^n f_S(1/x)/c_n$,

   (ii) $N_T \geq N_S$, *and*

   (iii) $a_i = 0$ *if and only if* $b_{j-i} = 0$ *for all* $j \geq i \geq 0$.

Additionally, in (a), (b), *and* (c), *if S is simple (resp. nondegenerate, $f_S(x)$ has real roots), we can also require T to be simple (resp. nondegenerate, $f_T(x)$ has real roots).*

Let S be a nonzero simple linear recurrence of degree n. Let $\{\alpha_i\}$ be the roots of $f_S(x)$. It is well-known that there are constants $\beta_i \in \mathbf{C}$ such that $a_k = \sum_{i=1}^{n} \beta_i \alpha_i^k$, for all $k \geq 0$. Assume that $r_1, r_2, \dots, r_n$ are distinct non-negative integers such that $a_{r_i} = 0$. Since $(\beta_1, \beta_2, \dots, \beta_n) \neq 0$, we have $\det(\alpha_i^{r_j}) = 0$. Define

$$\Delta_n(x_1, \dots, x_n; s_1, \dots, s_n) = \begin{vmatrix} x_1^{s_1} & x_2^{s_1} & \dots & x_n^{s_1} \\ x_1^{s_2} & x_2^{s_2} & \dots & x_n^{s_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{s_n} & x_2^{s_n} & \dots & x_n^{s_n} \end{vmatrix}.$$

Then S vanishes at $r_1, \dots, r_n$ if and only if

$$\Delta_n(\alpha_1, \dots, \alpha_n; r_1, \dots, r_n) = 0. \tag{2}$$

Sometimes, it is convenient to simplify this. For example, if we let $\omega_i = \alpha_{i+1}/\alpha_1$, $s_i = r_{i+1} - r_1$, for $1 \leq i \leq n-1$, then (2) is equivalent to

$$\Delta_n(1, \omega_1, \dots, \omega_{n-1}; 0, s_1, \dots, s_{n-1}) = 0.$$

We will use the following well-known lemma (see [5, p. 43]) to prove the general bound $N_S \leq 2n-2$.

LEMMA 2. *Let $\{\omega_i\}_{i=1}^{n}$, $\{s_i\}_{i=1}^{n}$ be two increasing sequences of positive real numbers. Then $\Delta_n(\omega_1, \dots, \omega_n; s_1, \dots, s_n) > 0$.*

PROPOSITION 1. *Let $S = \{a_i\}_{i=0}^{\infty}$ be a nonzero simple, nondegenerate real linear recurrence of degree n and assume that the roots of $f_S(x)$ are real. Let $\{r_1, \ldots, r_n\}$ be a set of distinct non-negative integers with the same parity. If $a_{r_1} = a_{r_2} = \cdots = a_{r_{n-1}} = 0$, then $a_{r_n} \neq 0$.*

PROOF. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $f_S(x)$. Assume that $a_{r_i} = 0$ for $i = 1, \ldots, n$. We can renumber the r_i so that r_1 is the smallest. Let $s_i = r_{i+1} - r_1$, for $i = 1, \ldots, n-1$. Then from (2), we have

$$\Delta = \Delta_n(\alpha_1, \ldots, \alpha_n; 0, s_1, \ldots, s_{n-1}) = 0.$$

But as the s_i are even integers, we also have

$$\Delta = \Delta_n(\alpha_1^2, \ldots, \alpha_n^2; 0, s_1/2, \ldots, s_{n-1}/2) \neq 0,$$

by Lemma 2. Hence we have a contradiction and $a_{r_n} \neq 0$. ■

COROLLARY 1. *Let S be a nonzero simple, nondegenerate real linear recurrence of degree n. If the roots of $f_S(x)$ are real, then $N_S \leq 2n - 2$.*

2. General bound: $N_S \leq 2n - 3$.

DEFINITION 1. A function $f(s)$ is of *exponential type* with order n if there is a decreasing sequence of n distinct positive real numbers α_i such that $f(s) = \sum_{i=1}^{n} A_i \alpha_i^s$, for some nonzero real numbers A_i.

Since $\{\alpha_i^s\}$ is a linearly independent set of functions, the order of $f(s)$ is well-defined. To every function $f(s)$ of exponential type of order n, we associate a signature $\epsilon_f \in \{\pm 1\}^n$. If $f(s)$ is given as in Definition 1, we define $\epsilon_f = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, where $\epsilon_i = \pm 1$ has the same sign as A_i.

LEMMA 3. *Let $f(s) = \sum_{i=1}^{n} A_i \alpha_i^s$ be a function of exponential type of order n with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. For each j such that $1 \leq j \leq n$, the function*

$$g_j(s) = \alpha_j^s \frac{d}{ds} \frac{f(s)}{\alpha_j^s} = \sum_{\substack{1 \leq i \leq n \\ i \neq j}} A_i \ln\Big(\frac{\alpha_i}{\alpha_j}\Big) \alpha_i^s$$

is a function of exponential type of order $n-1$ with signature $(\epsilon_1', \ldots, \epsilon_{n-1}')$, where $\epsilon_i' = \epsilon_i$ if $i < j$, and $\epsilon_i' = -\epsilon_{i+1}$ if $i \geq j$.

PROPOSITION 2. *Let $f(s)$ be a function of exponential type of order $n > 1$ with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. If there is an i such that $\epsilon_i = \epsilon_{i+1}$, then there are at most $n-2$ distinct real numbers z such that $f(z) = 0$.*

PROOF. We use induction. When $n = 2$, the hypothesis implies that $f(s)$ is always nonzero. Hence the proposition holds. Now assume that the theorem holds for functions of order $n - 1 \geq 2$. Given $f(s)$ in the proposition, choose $j \neq i, i + 1$ such that $1 \leq j \leq n$. By Lemma 3, $g_j(s)$ is a function of exponential type with order $n - 1$. The signature of $g_j(s)$ inherits the signature property of the hypothesis. Thus $g_j(s)$ vanishes at at most $n - 3$ distinct real numbers. By Rolle's theorem, if $f(s)$ vanishes at k distinct real numbers, $g_j(s)$ will vanish at at least $k - 1$ distinct real numbers. Hence $k - 1 \leq n - 3$ and $k \leq n - 2$ and the proposition is proved. ■

We can now prove the sharp bound $N_S \leq 2n - 3$.

PROOF. [Proof of Theorem 1] We use proof by contradiction. Assume that there are $2n - 2$ non-negative integers $r = r_1, \ldots, r_{2n-2}$ such that $a_r = 0$. By Lemma 2, we can assume that $r_1, \ldots, r_{n-1}$ is an increasing sequence of even integers and $r_n, \ldots, r_{2n-2}$ is an increasing sequence of odd integers. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f_S(x)$. By Lemma 1(b), we can assume that at most $[n/2]$ of the roots are negative. After renumbering we can assume that $|\alpha_1|, \ldots, |\alpha_n|$ is an increasing sequence. Define $\omega_i = |\alpha_i|$ and $\epsilon_i = \alpha_i/\omega_i$ for $1 \leq i \leq n$. Define $f(s)$ by

$$f(s) = \begin{vmatrix} \omega_1^{r_1} & \omega_2^{r_1} & \ldots & \omega_n^{r_1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{r_{n-1}} & \omega_2^{r_{n-1}} & \ldots & \omega_n^{r_{n-1}} \\ \epsilon_1\omega_1^s & \epsilon_2\omega_2^s & \ldots & \epsilon_n\omega_n^s \end{vmatrix}.$$

Expanding $f(s)$, we have

$$f(s) = \epsilon_n\beta_n\omega_n^s - \epsilon_{n-1}\beta_{n-1}\omega_{n-1}^s + \cdots + (-1)^{n-1}\epsilon_1\beta_1\omega_1^s,$$

where β_i are positive real numbers by Lemma 2. The signature of $f(s)$ is $\epsilon_f = (\epsilon_n, -\epsilon_{n-1}, \ldots, (-1)^{n-1}\epsilon_1)$. By (2), $f(r_n) = f(r_{n+1}) = \cdots = f(r_{2n-2}) = 0$. Suppose that all the roots are positive. Then $\epsilon_i = 1$ for all i. By Lemma 2, $f(r_n) \neq 0$ and we obtain a contradiction. Hence we can assume that $\epsilon_i = -1$ for at least one value of i. Since $\epsilon_i = 1$ for at least one i, there exists j with $0 < j < n - 1$ such that $\epsilon_j = -\epsilon_{j+1}$. Then, by Proposition 2, $f(s)$ can vanish at at most $n-2$ distinct real numbers, which contradicts the fact that $f(s)$ vanishes at the $n - 1$ distinct numbers $s = r_n, \ldots, r_{2n-2}$. Hence, the theorem is proved. ■

3. Proof of Theorem 2. We now prove Theorem 2. The proof relies on Rolle's Theorem and is similar to the methods of [3] and the proof of Theorem 1. Let $\alpha_1, \ldots, \alpha_4$ be the roots of $f_S(x)$, numbered so that $0 < |\alpha_1| < |\alpha_2| < |\alpha_3| < |\alpha_4|$. Without loss of generality, we can assume that at most two of the α_i are negative. Lemma 2 shows $N_S < 4$ if the α_i are all positive. Assume now that $a_r = 0$ for five non-negative integers $r = r_1, \ldots, r_5$. By Proposition 1 and Lemma 1(a), we can assume that $r_1 < r_2 < r_3$ are even integers, and that $r_4 < r_5$ are odd integers. Both (i) and (ii) of the hypothesis still hold after these simplifications. We now consider two cases. In both cases, we obtain a contradiction, thus proving the theorem.

Case 1: There is only one negative root α_i. By part (ii) of the hypothesis, we can conclude that either α_2 or α_3 is negative. For $i = 1, 2, 3$, we define $\omega_i > 0$ and $\epsilon_i = \pm 1$ by the equation $\omega_i = \epsilon_i \alpha_{i+1}/\alpha_1$. By our assumptions, $\epsilon_3 = 1$ and $\{\epsilon_1, \epsilon_2\} = \{1, -1\}$. Hence $\epsilon_1 = -\epsilon_2$. We define $s_i = r_{i+1} - r_1$ for $i = 1, 2, 3, 4$, and define

$$f(s) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_1^{s_1} & \omega_2^{s_1} & \omega_3^{s_1} \\ 1 & \omega_1^{s_2} & \omega_2^{s_2} & \omega_3^{s_2} \\ 1 & \epsilon_1\omega_1^{s} & \epsilon_2\omega_2^{s} & \omega_3^{s} \end{vmatrix}. \tag{3}$$

By (2), $f(s_3) = f(s_4) = 0$. Expanding $f(s)$, we have

$$f(s) = A_{1,2}\omega_3^s - \epsilon_2(A_{1,3}\omega_2^s + A_{2,3}\omega_1^s) - B,$$

where

$$A_{i,j} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega_i^{s_1} & \omega_j^{s_1} \\ 1 & \omega_i^{s_2} & \omega_j^{s_2} \end{vmatrix}, \quad B = \begin{vmatrix} 1 & 1 & 1 \\ \omega_1^{s_1} & \omega_2^{s_1} & \omega_3^{s_1} \\ \omega_1^{s_2} & \omega_2^{s_2} & \omega_3^{s_2} \end{vmatrix}.$$

By Lemma 2, $A_{1,2}$, $A_{1,3}$, $A_{2,3}$, B are all positive. Whether $\epsilon_2 = 1$ or -1, $f(s)$ has at most one zero, contradicting $f(s_3) = f(s_4) = 0$.

Case 2: There are exactly two negative roots α_j, α_k. By Lemma 1(b), we can assume that $k = 4$. Part (i) of the hypothesis still holds and thus j is 1 or 2. Let ϵ be the sign of α_1. Define

$$f(s) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_1^{s_1} & \omega_2^{s_1} & \omega_3^{s_1} \\ 1 & \omega_1^{s_2} & \omega_2^{s_2} & \omega_3^{s_2} \\ \epsilon & -\epsilon\omega_1^{s} & \omega_2^{s} & -\omega_3^{s} \end{vmatrix},$$

where $\omega_i = \alpha_{i+1}/|\alpha_1|$, $s_i = r_{i+1} - r_1$, for $i = 1, 2, 3, 4$. By logic similar to that which derived (2), $f(s_3) = f(s_4) = 0$. Expanding $f(s)$, we have

$$f(s) = -\omega_3^s A_{1,2} - \omega_2^s A_{1,3} - \epsilon\omega_1^s A_{2,3} - \epsilon B,$$

where $A_{i,j}$, B are positive numbers defined as above. Regardless of ϵ's sign, $f(s)$ has at most one zero, contradicting $f(s_3) = f(s_4) = 0$. ■

References

1. F. Beukers, *The zero multiplicity of ternary recurrences.* Compositio Math. (2) **77** (1991), 165–177.
2. M. F. Smiley, *On the zeros of a cubic recurrence.* Amer. Math. Monthly **63** (1956), 171–172.
3. P. A. Picon, *Sur certaines suites rècurrentes cubiques ayant deux ou trois termes nuls.* Discrete Mathematics **21** (1978), 285–296.
4. M. Kulkarni and B. Sury, *On the vanishing of cubic recurrences.* C. R. Math. Acad. Sci. Soc. R. Can. (2) **24** (2002), 72–76.
5. G. Pólya and G. Szegö, *Problems and Theorems in Analysis.* Vol. II, translated by C. E. Billigheimer, Springer-Verlag, 1976.
6. C. Lech, *A note on recurring series.* Ark. Math. **2** (1953), 417–421.
7. H. P. Schlickewei and W. M. Schmidt, *The number of solutions of polynomial-exponential equations.* Compositio Math. (2) **120** (2000), 193–225.
8. W. M. Schmidt, *The zero multiplicity of linear recurrence sequences.* Acta. Math. **182** (1999), 243–282.
9. S. J. Scott, *On the number of zeros of a cubic recurrence.* Amer. Math. Monthly **67** (1960), 169–170.
10. M. Ward, *On the number of vanishing terms in an integral cubic recurrence.* Amer. Math. Monthly **62** (1955), 155–160.
11. ———, *Some diophantine problems connected with linear recurrences.* Report of the Institute in the Theory of Numbers, Univ. of Colorado, 1959, pp. 250–257.

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