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VALUES OF NÖRLUND EULER POLYNOMIALS AND NÖRLUND BERNOULLI POLYNOMIALS

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Presented by M. Ram Murty, FRSC

ABSTRACT. In this paper, the authors discuss the values of Nörlund Euler polynomials and Nörlund Bernoulli polynomials at rational points, and generalize a result concerning Euler polynomials.

RÉSUMÉ. Dans cet article, nous étudions valeurs des polynômes de Nörlund Euler et polynômes de Nörlund Bernoulli aux nombres rationnels, et nous étendons un résultat des polynômes d'Euler.

1. Introduction.

The Nörlund Euler polynomial $E_n^{(N)}(x|\omega_1,\ldots,\omega_N)$ and the Nörlund Bernoulli polynomials $B_n^{(N)}(x|\omega_1,\ldots,\omega_N)$ are defined (see [2] and [5]):

$$\frac{2^N e^{xt}}{(e^{\omega_1 t}+1)\cdots(e^{\omega_N t}+1)} = \sum_{n=0}^{\infty} E_n^{(N)}(x|\omega_1,\ldots,\omega_N)\frac{t^n}{n!}, \tag{1}$$

$$\frac{\omega_1\cdots\omega_N t^N e^{xt}}{(e^{\omega_1 t}-1)\cdots(e^{\omega_N t}-1)} = \sum_{n=0}^{\infty} B_n^{(N)}(x|\omega_1,\ldots,\omega_N)\frac{t^n}{n!}. \tag{2}$$

Clearly, when $\omega_1 = N = 1$, $E_n^{(N)}(x|\omega_1,\ldots,\omega_N)$ and $B_n^{(N)}(x|\omega_1,\ldots,\omega_N)$ are Euler polynomial $E_n(x)$ and Bernoulli polynomial $B_n(x)$, respectively, where

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{e^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

and $2^n E_n(1/2)$ and $B_n(0)$ are the Euler numbers and Bernoulli numbers, respectively. Euler numbers, Euler polynomials, Bernoulli numbers, and Bernoulli polynomials have important applications in many subjects, especially in function theory and analytic number theory. For various properties of Nörlund Euler polynomials and Nörlund Bernoulli polynomials, see literatures [2] and [4]. In this paper, we will discuss the values of Nörlund Euler polynomials and Nörlund Bernoulli polynomials at rational points. Many authors have shown interest in studying the values of the Euler polynomials and of the Bernoulli polynomials at rational points. For example, for Euler polynomials $E_n(x)$, Fox [3]

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proved that for a rational number r/s, $s^n(E_n(r/s) + (-1)^{rs-1}E_n(0)) \in \mathbf{Z}$ for every $n \geq 0$, and Sury [7] extended this result: $s^nE_n(r/s) \in \mathbf{Z}$ if s is even and $s^n(E_n(r/s) + (-1)^{r-1}E_n(0)) \in \mathbf{Z}$ if s is odd. For Bernoulli polynomials $B_n(x)$, Almkvist and Meurman [1] showed that $s^n(B_n(r/s)-B_n(0)) \in \mathbf{Z}$ for every $n \geq 0$, and a simple proof of this was given in [6]. Inspired by the above results, we want to discuss the values of Nörlund Euler polynomials and Nörlund Bernoulli polynomials at rational points. It seems that no one has studied this problem. Do Nörlund Euler polynomials and Nörlund Bernoulli polynomials give results similar to those of Euler polynomials and Bernoulli polynomials? The purpose of this paper is to investigate values of Nörlund Euler polynomials and Nörlund Bernoulli polynomials. We will establish some conclusions involving Nörlund Euler polynomials and Nörlund Bernoulli polynomials, and generalize a result of Sury [7].

2. Main results.

THEOREM 1. *For any arbitrary rational number r/s, positive integer N and any integer ω, we have for each n that*

$$s^n \sum_{l=0}^{N} (-1)^l \binom{N}{l} B_n^{(N)}((N-l)\omega r/s|\omega,\omega) \in \mathbf{Z}.$$

PROOF. Let

$$c_n = s^n \sum_{l=0}^{N} (-1)^l \binom{N}{l} B_n^{(N)}((N-l)\omega r/s|\omega,\omega).$$

From (2), it follows that

$$\sum_{n\geq 0} c_n \frac{t^n}{n!} = (\omega st)^N \frac{(e^{r\omega t}-1)^N}{(e^{s\omega t}-1)^N}.$$

But, we have

$$\sum_{n\geq 0} \frac{(B_n(r/s)-B_n(o))(\omega st)^n}{n!} = (\omega st)\frac{(e^{r\omega t}-1)}{(e^{s\omega t}-1)}.$$

Now, by the theorem of Almkvist and Meurman ([1] or [6]), the numbers $a_d = s^d(B_d(r/s) - B_d(0)) \in \mathbf{Z}$. A comparison of coefficients of t^n shows that

$$c_n = \sum_{l_1+\cdots+l_N=n} (\omega)^n a_{l_1} a_{l_2} \cdots a_{l_N} \frac{n!}{l_1!\cdots l_N!}.$$

Thus, $c_n \in \mathbf{Z}$ for all n. ■

It is obvious that Theorem 1 generalizes the result of Almkvist and Meurman. Now, we give the special case of Theorem 1.
Setting $G_n = 2^n(B_n^{(2)}(4|4,4) - 2B_n^{(2)}(2|4,4) + B_n^{(2)}(0|4,4))$, we have

$$G_0 = 0, \quad G_1 = 0, \quad G_2 = 32, \quad G_3 = -396, \quad \text{and}$$

$$G_n = -\sum_{i=0}^{n-1} 2^{n-i} G_i \binom{n}{i} 4^{n-1-i} - 2\sum_{i=0}^{n-1} G_i \binom{n}{i} 4^{n-1-i}, \quad n \geq 4.$$

THEOREM 2. *For integers N, r, s, and ω_i $(1 \leq i \leq N)$,*

$$s^n E_n^{(N)}(r/s|\omega_1, \ldots, \omega_N) \in \mathbf{Z}$$

if s is even and $r \equiv 0 \pmod{N}$.

PROOF. It follows from (1) that

$$\sum_{n=0}^{\infty} E_n^{(N)}(x|\omega_1, \ldots, \omega_N)\frac{t^n}{n!} \tag{3}$$
$$= \sum_{n=0}^{\infty} E_n(x/(\omega_1 N))\frac{(\omega_1 t)^n}{n!} \cdots \sum_{n=0}^{\infty} E_n(x/(\omega_N N))\frac{(\omega_N t)^n}{n!}.$$

Comparing the coefficients of t^n on both sides of (3), we get

$$E_n^{(N)}(x|\omega_1, \ldots, \omega_N) = \sum_{i_1+\cdots+i_N=n} \frac{n!\, E_{i_1}(x/(\omega_1 N))\omega_1^{i_1} \cdots Ei_N(x/(\omega_N N))\omega_N^{i_N}}{i_1! \cdots i_N!}.$$

Therefore, the following identity is valid:

$$s^n E_n^{(N)}(r/s|\omega_1, \ldots, \omega_N)$$
$$= \sum_{i_1+\cdots+i_N=n} \frac{n!\, (s\omega_i)^{i_1} E_{i_1}(r/(s\omega_1 N)) \cdots (s\omega_N)^{i_n} Ei_N(rx/(s\omega_N N))}{i_1! \cdots i_N!}.$$

In order to prove Theorem 2, it suffices to show that $(s\omega_k)^{i_k} E_{i_k}(r/(s\omega_k N)) \in \mathbf{Z}$ $(1 \leq k \leq N)$ if s is even and $r \equiv 0 \pmod{N}$.
But, if s is even and $r/N \in \mathbf{Z}$, this follows from Sury [7] quoted above

$$(s\omega_k)^{i_k} E_{i_k}(r/(s\omega_k N)) = (s\omega_k)^{i_k} E_{i_k}((r/N)/(s\omega_k)). \qquad \blacksquare$$

This extends a result of Sury [7].

Now, we compute the special case of Theorem 2 according to the values of r, s, N, and ω_i ($i = 1, 2, \dots, N$). In Theorem 2, put $r = 2$, $s = 4$, $N = 2$, $\omega_1 = 1$, $\omega_2 = 3$, and $A_n = 4^n E_n^{(2)}(1/2|1,3)$. After calculation, we have

$$A_0 = 1, \quad A_1 = -6, \quad \text{and} \quad A_2 = -4.$$

By (1), we obtain

$$4e^{2t} = (e^{16t} + e^{4t} + e^{12t} + 1) \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}.$$

This means that $\{A_n\}$ satisfies the following relation

$$A_n = 2^n - \sum_{i=0}^{n-1} A_i \binom{n}{i} 4^{2n-1-2i} - \sum_{i=0}^{n-1} A_i \binom{n}{i} 4^{n-1-i} - \sum_{i=0}^{n-1} 3^{n-i} \binom{n}{i} 4^{n-1-i}, \quad n \geq 3.$$

THEOREM 3. *For any arbitrary integers r, s, and ω_i ($1 \leq i \leq N$ and $\omega_i > 0$), $s^n E_n^{(N)}(r/s|\omega_1, \dots, \omega_N) \in \mathbf{Z}$ if $s \equiv 0 \pmod{2^N}$ or $\omega_i \equiv 0$ ($\operatorname{mod} 2^N$, $1 \leq i \leq N$).*

PROOF. We prove Theorem 3 by induction on n.
Let $b_n = s^n E_n^{(N)}(r/s|\omega_1, \dots, \omega_N)$, then we have

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \frac{2^N e^{rt}}{(e^{\omega_1 ts} + 1) \cdots (e^{\omega_N ts} + 1)}.$$

Then

$$(e^{\omega_1 ts} + 1)(e^{\omega_2 ts} + 1) \cdots (e^{\omega_N ts} + 1) \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = 2^N \sum_{n=0}^{\infty} \frac{r^n t^n}{n!}$$

that is,

$$\left(\sum_{i_1 \geq 1} \frac{(\omega_1 ts)^{i_1}}{i_1!} + 2\right)\left(\sum_{i_2 \geq 1} \frac{(\omega_2 ts)^{i_2}}{i_2!} + 2\right) \cdots \left(\sum_{i_N \geq 1} \frac{(\omega_N ts)^{i_N}}{i_N!} + 2\right) \sum_{n \geq 0} \frac{b_n t^n}{n!} = 2^N \sum_{n \geq 0} \frac{r^n t^n}{n!}. \tag{4}$$

Clearly, $b_0 = 1$. For a general $n > 0$, the coefficient t^n on the left side of (4) is a sum of $2^N \frac{b_n}{n!}$ and terms of the form

$$\frac{(\omega_1 s)^{i_1}}{i_1!} \frac{(\omega_2 s)^{i_2}}{i_2!} \cdots \frac{(\omega_N s)^{i_N}}{i_N!} \frac{b_m}{m!}$$

where $i_1 + \cdots + i_N + m = n$ and some $i_j > 0$.

The coefficient of t^n on the right side of (4) is $2^N \frac{b_n}{n!}$.

Therefore, the hypothesis that either some ω_i or s is a multiple of 2^N implies from equating the coefficients of t^n that b_n is an integral linear combination of $b_0, \ldots, b_{n-1}$. This proves by induction on n that $b_n \in \mathbf{Z}$ for all n. ■

THEOREM 4. *For any arbitrary integers r, s, and ω_i ($1 \le i \le N$ and $\omega_i > 0$), $s^n(E_n^{(N)}(r/s|\omega_1, \ldots, \omega_N) + (-1)^{r-1}E_n^{(N)}(0|\omega_1, \ldots, \omega_N)) \in \mathbf{Z}$ if $s \equiv 0 \pmod{2^N}$ or $\omega_i \equiv 0$ (mod 2^N, $1 \le i \le N$).*

The proof of Theorem 4 is similar to that of Theorem 3 and therefore is omitted here. ■

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ON A MODEL FOR STOCK PRICE MOVEMENTS AND THE POWER-VARIANCE FAMILY

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RÉSUMÉ. On utilise une classe de processus de Lévy générée par la famille de puissance-variance. On approxime les fluctuations du cours des valeurs financières par une transformation exponentielle de la différence entre deux processus de cette classe. Cette approche repose sur les travaux de [CGMY]. Le processus des prix des actions considéré peut reproduire des sauts positifs et négatifs observables. Nous révélons l'existence de points critiques en clarifiant la nature de l'événement d'amélioration ou de détérioration du prix des actions. Ce modèle est en accord avec l'observation empirique de gains logarithmiques montrant une asymptote "semi-lourde". Notre choix d'une mesure de risque-neutre dépend des transformées d'Esscher. Un tel choix explique le phénomène d'une asymptote supérieure de la distribution des gains logarithmiques plus mince que l'asymptote inférieure à laquelle l'effet de levier contribue. Nos résultats impliquent que le mouvement brownien géométrique et la formule de Black-Scholes correspondent à l'équilibre. Notre modèle est en accord avec les observations empiriques que la volatilité des actions est une fonction décroissante du prix. Ceci démontre une bonne capacité de notre modèle à expliquer des fluctuations aléatoires du prix des actions surcréditées et des options.

Recently, the following class of stochastic processes was used for describing the stock price movements:

$$S(t) = S(0) \cdot \exp\{X(t) - A \cdot t\}. \tag{1}$$

Here, $S(0) > 0$ is the initial price of an equity, constant $A \in \mathbf{R}^1$ is given by (6), and $X(t)$ is a certain real-valued stochastic process. Assume that $X(t)$ is a driftless Lévy process with no Gaussian component.

We investigate the properties of the geometric Lévy process defined by (1)–(6). The marginal distributions are characterized as those of the difference between two random variables from the power-variance family (see (24)–(25)). This process may perform noticeable up- and downward jumps. Theorem 1 provides the exact asymptotics of the probabilities of large deviations and reveals the existence of two critical points in their formation. This can be useful for studying various value-at-risk measures. Theorem 2 provides a new formula for option pricing. The concluding Theorem 3 demonstrates that in the case when the magnitude of the jumps becomes negligible, process (1)–(6) and the rational price of

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a European call option given in Theorem 2 converge to the geometric Brownian motion and the rational price of a European call option given by Black-Scholes formula, respectively. Our results are consistent with distinct features exhibited by the movements of overleveraged equities, which is discussed below.

Now, assume that Lévy measure of $X(1)$ possesses the next density:

$$\pi(x) = \begin{cases} d_+ \cdot x^{-(\alpha_+ + 1)} \cdot e^{-\theta_+ \cdot x} & \text{if } x > 0, \\ d_- \cdot |x|^{-(\alpha_- + 1)} \cdot e^{\theta_- \cdot x} & \text{if } x < 0. \end{cases} \tag{2}$$

Here, $d_\pm$ and $\theta_\pm$ are arbitrary fixed non-negative real, $d_+ + d_- > 0$, and $\alpha_\pm \in [0,1) \cup (1,2)$.

The case when $\alpha_+ = \alpha_- = 0$ and $\theta_\pm > 0$ corresponds to the *variance-gamma* model (see Madan (2001)). The case when $d_+ = d_-$, $\alpha_+ = \alpha_-$, and $\theta_\pm > 0$ characterizes a *CGMY process* (see [CGMY]).

It is mathematically natural to generalize a CGMY process a step further. Namely, in contrast to [CGMY], we drop the condition that $d_+ = d_-$. Let us state the other assumptions of our model. Suppose that

$$\alpha_+ = \alpha_- =: \alpha \in (0,1) \cup (1,2), \tag{3}$$

and define

$$\theta := \theta_+ + \theta_-. \tag{4}$$

In the case when $d_+ > 0$, assume that

$$\theta_+ \geq 1. \tag{5}$$

The empirical data are typically consistent with (5) (compare to [CGMY, Table 2, Theorem 1 and (13)]). This set of constraints provides a reasonable level of generality at which our methods enable us to derive elegant mathematical properties, which are consistent with the three distinct features exhibited by the movements of overleveraged equities. In the case when $\alpha \in (1,2)$, set $\Gamma(1-\alpha) := \Gamma(2-\alpha)/(1-\alpha)$. Let

$$A := \Gamma(1-\alpha) \cdot (d_+ \cdot (\theta^\alpha - (\theta - 1)^\alpha) - d_-)/\alpha. \tag{6}$$

Note that price process $\{S(t), t \geq 0\}$ defined by (1)–(6) is a positive submartingale. Also, one can demonstrate that for each real $t \geq 0$,

$$\mathbf{E}S(t) = S(0) \cdot \exp\{g(\theta_\pm) \cdot t\}. \tag{7}$$

Here, the (exponential) growth rate $g(\theta_\pm)$ of an equity is as follows:

$$g(\theta_\pm) := \alpha^{-1} \cdot \Gamma(1-\alpha) \cdot \Big\{ d_+ \cdot \{(\theta_+^\alpha - (\theta_+ - 1)^\alpha) - (\theta^\alpha - (\theta - 1)^\alpha)\} + d_- \cdot (\theta_-^\alpha + 1 - (\theta_- + 1)^\alpha) \Big\} > 0. \tag{8}$$

Evidently, the growth rate also depends on α and $d_\pm$. However, we only emphasize its dependence on parameters $\theta_\pm$, since α and $d_\pm$ will remain unchanged throughout this work. Also, $S(t)$ generalizes the geometric Brownian motion $S_{0,\mu,\lambda}(t)$ to the discontinuous setting. Let us parametrize the geometric Brownian motion as follows:

$$S_{0,\mu,\lambda}(t) := S(0) \cdot \exp\left\{(\mu - 1/(2\cdot\lambda)) \cdot t + \lambda^{-1/2} \cdot B(t)\right\}. \tag{9}$$

Here, $B(t)$ stands for a standard univariate Brownian motion, $\mu \geq 0$ represents the growth rate of a stock in the sense that $\mathbf{E}S_{0,\mu,\lambda}(t) = S(0) \cdot \exp\{\mu \cdot t\}$, and $\lambda := 1/\sigma^2$, where $\sigma > 0$ denotes the volatility of an equity (compare to (7) and Hull (2002, (12.4))).

The next result concerns the tails of $X(t)$. In the case when $\alpha \in (1,2)$, we introduce two additional characteristics. Set $u_0^\pm := \pm\Gamma(2-\alpha)\cdot d_\mp \cdot \theta^{\alpha-1}/(\alpha-1)$. One can show that $u_0^- \leq \mathbf{E}X(1) \leq u_0^+$. Also, define

$$H(u_0^\pm) := (\alpha-1)^{-1}\cdot\Gamma(2-\alpha)\cdot\big\{\pm\theta_\pm \cdot d_\mp \cdot \theta^{\alpha-1} + (d_\pm \cdot \theta_\pm^\alpha - d_\mp \cdot (\theta^\alpha - \theta_\mp^\alpha))/\alpha\big\} \geq 0.$$

THEOREM 1. *Consider a driftless Lévy process $X(t)$ that has no Gaussian component and satisfies* (2)–(5). *Then* (i) *For each fixed real $t > 0$,*

$$\boldsymbol{P}\{X(t) > x\} = \frac{d_+ \cdot t}{\theta_+} \cdot \exp\left\{t \cdot \frac{\Gamma(1-\alpha)}{\alpha} \cdot (d_+ \cdot \theta_+^\alpha - d_- \cdot (\theta^\alpha - \theta_-^\alpha))\right\} \cdot e^{-\theta_+ \cdot x} \cdot x^{-(1+\alpha)} \cdot (1 + o(1)); \tag{10}$$

$$\boldsymbol{P}\{X(t) < x\} = \frac{d_- \cdot t}{\theta_-} \cdot \exp\left\{t \cdot \frac{\Gamma(1-\alpha)}{\alpha} \cdot (d_- \cdot \theta_-^\alpha - d_+ \cdot (\theta^\alpha - \theta_+^\alpha))\right\} \cdot e^{\theta_- \cdot x} \cdot |x|^{-(1+\alpha)} \cdot (1 + o(1)) \tag{11}$$

as $x \to +\infty$ and $x \to -\infty$, respectively.

(ii) *Suppose that $t \to \infty$ takes on integer values. Then*

(ii1) *In the case when $\alpha \in (0,1)$, the exact asymptotics of $\boldsymbol{P}\{X(t) > x\}$ and $\boldsymbol{P}\{X(t) < x\}$ as $x/t^{1/\alpha} \to \pm\infty$, respectively, are given by* (10)–(11).

(ii2) *In the case when $\alpha \in (1,2)$,*

$$\boldsymbol{P}\{X(t) > x\} = t \cdot \boldsymbol{P}\{X(1) > x - (t-1)\cdot u_0^+\} \cdot \exp\{-(t-1)\cdot H(u_0^+)\} \cdot (1+o(1));$$

$$\boldsymbol{P}\{X(t) < x\} = t \cdot \boldsymbol{P}\{X(1) < x - (t-1)\cdot u_0^-\} \cdot \exp\{-(t-1)\cdot H(u_0^-)\} \cdot (1+o(1))$$

as $(x - t\cdot u_0^+)/t^{1/\alpha} \to +\infty$ and $(x - t \cdot u_0^-)/t^{1/\alpha} \to -\infty$, respectively.

One can show that u_0^- constitutes a *critical point* in the formation of large deviations of $X(t)$ to $\mathbf{R}^1_-$. Similarly, u_0^+ constitutes a *critical point* in the formation of large deviations of $X(t)$ to $\mathbf{R}^1_+$.

Now, we introduce some auxiliary notation. Set

$$p = p(\alpha) = (2-\alpha)/(1-\alpha) \in (-\infty, 0) \cup (2, +\infty). \tag{12}$$

Obviously, parameters p and α are in one-to-one correspondence. In the case when $\alpha \in (0,1)$, define

$$\lambda_\pm := (1-\alpha)^{-1} \cdot (\Gamma(1-\alpha) \cdot d_\pm)^{1/(1-\alpha)}; \tag{13}$$

$$\mu_\pm := \Gamma(1-\alpha) \cdot d_\pm \cdot \theta_\pm^{\alpha-1}. \tag{14}$$

In the case when $\alpha \in (1,2)$, let

$$\lambda_\pm := (\alpha-1)^{-1} \cdot (|\Gamma(1-\alpha)| \cdot d_\mp)^{1/(1-\alpha)}; \tag{15}$$

$$\mu_\pm := |\Gamma(1-\alpha)| \cdot d_\mp \cdot \theta_\mp^{\alpha-1}. \tag{16}$$

The meaning of $\mu_\pm$ and $\lambda_\pm$ is clarified by (25). Also, define

$$\lambda_0 := \lambda_+ \cdot \lambda_- / (\lambda_+ + \lambda_-); \tag{17}$$

$$\mu_0 := \mu_+ \cdot (1 + \lambda_+/\lambda_-). \tag{18}$$

Next, let us concentrate on option pricing. Let T be the time of the option to maturity, and $C_{T,p}$ denote the rational price of the European call option provided that the price of an equity at time t equals $S(t)$. This option may be exercised only at time T. Let $K > 0$ be the exercise price of the option, which gives its holder the right to buy a certain number of shares of the stock at price K at time T. Assume that the short-term risk-free rate

$$r \in [0, r_*(p)] \tag{19}$$

is constant over the time span $[0,T]$. Hereinafter, $r_*(p) := \alpha^{-1} \cdot \Gamma(1-\alpha) \cdot (d_+ + d_-) \cdot ((\theta-1)^\alpha + 1 - \theta^\alpha) > 0$.

Now, the fundamental theorem of option pricing yields that for each fixed $r \in [0, r_*(p)]$, the rational price $C_{T,p}$ of the European call option described below (18) can be evaluated as follows:

$$C_{T,p} = e^{-r \cdot T} \cdot \mathbf{E}_Q(S(T) - K)^+ \tag{20}$$

(cf., e.g., Bingham and Kiesel (1998, Theorem 6.1.4 and Section 7.1.2)). Here, $\mathbf{E}_Q$ stands for the expectation with respect to a certain *equivalent measure* hereinafter denoted by Q, and $a^+ := max(a, 0)$. We refer to process $S(\cdot)$ considered with respect to measure Q as a *risk-neutral* process, whereas measure Q is termed a *risk-neutral* measure. An important class of equivalent measures for process (1)–(6) can be derived from Proposition 1. Also, the *discounted price process* $e^{-r \cdot t} \cdot S(t)$ is a positive martingale with respect to Q.

Next, [CGMY] suggests that the distribution of the risk-neutral process considered with respect to the *'implied'* measure Q_* belongs to the same family of Lévy processes which satisfy (2)–(3). (The implied measure is a measure for a future price of an equity implied from option prices.)

PROPOSITION 1. *Let $X(t)$ satisfy (2)–(3). Consider a family of all absolutely continuous transformations of the original probability measure such that $X(t)$ considered with respect to an arbitrary measure from this family belongs to the same class of Lévy processes. Suppose that a transformed process possesses the spectral density denoted by $\widetilde{\pi}(\cdot)$. Thus, $\widetilde{\pi}(\cdot)$ satisfies (2)–(3) with a specific set of parameters $\widetilde{\alpha} \in (-\infty, 2)$, $\widetilde{\theta}_\pm \geq 0$, and $\widetilde{d}_\pm \geq 0$ such that $\widetilde{d}_+ + \widetilde{d}_- > 0$. Then $\widetilde{\alpha} = \alpha$ and $\widetilde{d}_\pm = d_\pm$.*

Model (1)–(6) is *not complete*, since it possesses more than one risk-neutral measure. Therefore, it is important to select and justify the choice of a *specific* risk-neutral measure to be used for option pricing. Here, we select the risk-neutral distribution by employing the techniques of Esscher transforms. We refer to Bingham and Kiesel (1998, Subsection 7.1.2) for the description of this transformation. This method differs from an approach utilized by [CGMY]. Note that Esscher transforms were frequently used for pricing stock options (cf., e.g., Gerber and Shiu (1994)). In our case, such choice of the risk-neutral measure yields that the values of $\widetilde{\theta}_\pm$ which emerge in the representation for $\widetilde{\pi}(\cdot)$ are as follows.

Suppose that r satisfies (19). Then there exists a unique value $v_p(r) \in [-\theta_-, \theta_+ - 1]$ such that for

$$\widetilde{\theta}_- := \theta_- + v_p(r), \tag{21}$$

$$\widetilde{\theta}_+ := \theta_+ - v_p(r), \tag{22}$$

one obtains that

$$g(\widetilde{\theta}_\pm) = r. \tag{23}$$

Here, $g(\widetilde{\theta}_\pm)$ is defined by (8). By (23), the discounted price process $e^{-r \cdot t} \cdot S(t)$ considered with respect to the corresponding risk-neutral measure constitutes a positive martingale.

Fix $r \in [0, r_*(p)]$, and denote the risk-neutral process considered with respect to the above-described (risk-neutral) Esscher transform by $\widetilde{S}(t)$. A subsequent combination of Proposition 1 with (21)–(23) implies that $\widetilde{S}(t)$ satisfies (1)–(6) with the same values of α and $d_\pm$, and with the *'risk-neutral'* values of parameters $\widetilde{\theta}_\pm$ given by (21)–(22).

Now, keeping in mind the conclusions of [CGMY], assume that the implied measure should encapsulate as much information about the original measure as possible and remain within the same family (2)–(3). By Proposition 1, α and $d_\pm$ remain invariant. On the other hand, the fundamental theorem of asset pricing implies that the preservation of the exponential tilting parameters is not possible, unless $\widetilde{\theta}_\pm = \theta_\pm$.

It appears that this argument supports the choice of a risk-neutral measure by employing Esscher transforms. In particular, the unique risk-neutral Esscher transform remains within the family (2)–(3). Moreover, it preserves the values of parameters α, $d_\pm$ and does not change the sum $\theta_+ + \theta_-$. Thus, the risk-neutral Esscher transform encapsulates a vast amount of information about the original measure. Also, it is known that the implied distribution of the logarithmic returns is typically skewed to the left (see Madan (2001, Subsection 2.5)). This observation is consistent with our method of option pricing.

In the sequel, we will employ decomposition (25) of $X(t)$ into the difference of two independent Lévy processes, which are generated by the power-variance family of distributions. This enables us to utilize various properties of the family which can be found in Vinogradov (2004). This family is comprised of r.v.'s W's which satisfy the next variance-to-mean relationship:

$$Var(W) = \mu^p/\lambda. \tag{24}$$

Here, location parameter $\mu = EW$, and $\lambda > 0$ is a certain scaling parameter (cf., e.g., Vinogradov (2004)). Following Vinogradov (2004), we denote r.v.'s from this class by $Tw_p(\mu, \lambda)$.

Next, we generate the three-parametric family of Lévy processes $\{X_{p,\mu,\lambda}(t), t \geq 0\}$ starting from the power-variance family. It suffices to define $X_{p,\mu,\lambda}(1) - X_{p,\mu,\lambda}(0) \stackrel{d}{=} Tw_p(\mu, \lambda)$. Following Vinogradov (2002), we refer to this class of Lévy processes as *Hougaard* processes. The sample paths of each such process belong to the càdlàg space $\mathbf{D}[0, \infty)$. Suppose that this space is equipped with the Skorokhod topology.

Now, let all the conditions of Theorem 1 be fulfilled. Then there exist two independent Hougaard processes $X^+_{p,\mu_+,\lambda_+}(t)$ and $X^-_{p,\mu_-,\lambda_-}(t)$ such that

$$X(t) \stackrel{d}{=} X^+_{p,\mu_+,\lambda_+}(t) - X^-_{p,\mu_-,\lambda_-}(t). \tag{25}$$

Here, $p = p(\alpha)$, $\mu_\pm$ and $\lambda_\pm$ are expressed in terms of α, $d_\pm$ and $\theta_\pm$ by (12)–(16). Next, define

$$\Theta(p, \mu, \lambda) := \lambda \cdot \mu^{1-p}/|1 - p|; \tag{26}$$

$$\Theta^+_{r,\lambda_+,\lambda_-}(p) := \begin{cases} \Theta(p, \mu_+, \lambda_+) - v_p(r) & \text{if } p \in (2, \infty), \\ \Theta(p, \mu_+, \lambda_+) + v_p(r) & \text{if } p \in (-\infty, 0); \end{cases} \tag{27}$$

$$\Theta^-_{r,\lambda_+,\lambda_-}(p) := \theta - \Theta^+_{r,\lambda_+,\lambda_-}(p) \tag{28}$$

(compare to (21)–(22)). A combination of (8), (13)–(16) and (26)–(28) yields that $g(\Theta^{\pm}_{r,\lambda_+,\lambda_-}(p)) \equiv r$ (compare to (23)). Next, given $p \in (-\infty, 0) \cup (2, \infty)$, define $\mu^{\pm}_{r,\lambda_+,\lambda_-}(p) := (|1-p| \cdot \Theta^{\pm}_{r,\lambda_+,\lambda_-}(p)/\lambda_{\pm})^{1/(1-p)}$,

$$m_1^{\pm} := (|1-p| \cdot (\Theta^{\pm}_{r,\lambda_+,\lambda_-}(p) \mp \mathrm{sign}(p))/\lambda_{\pm})^{1/(1-p)} \cdot T^{1/(2-p)}, \tag{29}$$

$$m_2^{\pm} := \mu^{\pm}_{r,\lambda_+,\lambda_-}(p) \cdot T^{1/(2-p)}. \tag{30}$$

Now, given the exercise price K of an equity and its time to maturity T, set

$$L := \log\ (K/S(0)) \cdot T^{-(1-p)/(2-p)} + A \cdot T^{1/(2-p)}. \tag{31}$$

THEOREM 2. *Assume that $S(t)$ which is defined by* (1)–(6) *is the price process for a certain equity, and that* (19) *is fulfilled. Consider a European call option for this stock, which is described below* (18). *Then the rational price $C_{T,p}$ of this option defined by* (20) *is as follows:*

$$C_{T,p} = S(0) \cdot \boldsymbol{P}\{Tw_p^+(m_1^+, \lambda_+) - Tw_p^-(m_1^-, \lambda_-) > L\}$$
$$- K \cdot e^{-r \cdot T} \cdot \boldsymbol{P}\{Tw_p^+(m_2^+, \lambda_+) - Tw_p^-(m_2^-, \lambda_-) > L\}.$$

Here, $\{Tw_p^+(m_i^+, \lambda_+),\ Tw_p^-(m_i^-, \lambda_-)\}$, $i \in \{1, 2\}$, are two pairs of pairwise independent r.v.'s from the power-variance family with the specified parameters, and $\lambda_{\pm}$, $m_i^{\pm}$, L are defined by (13), (15), (29)–(31).

Recall that $S_{0,\cdot,\cdot}(t)$ is defined by (9). Let us denote convergence in $\mathbf{D}[0,T]$ equipped with the Skorokhod topology by " $\xrightarrow{\mathrm{D[0,T]}}$ ". Here, $\mathbf{D}[0,T]$ is the restriction of $\mathbf{D}[0,\infty]$ to $[0,T]$. Set $r_*(0) := (\lambda_+ \cdot \mu_+ + \lambda_- \cdot \mu_-)/\lambda_0$.

THEOREM 3. (i) *Assume that all the conditions of Theorem 2 are fulfilled. Then* (i1) $S(\cdot) \xrightarrow{\mathrm{D[0,T]}} S_{0,\mu_0,\lambda_0}(\cdot)$ *as $p \uparrow 0$, where μ_0 and λ_0 are given by* (17)–(18).

(i2) *For each fixed $r \in [0, r_*(0))$, the risk-neutral process $\widetilde{S}(\cdot) \xrightarrow{\mathrm{D[0,T]}} S_{0,r,\lambda_0}(\cdot)$ as $p \uparrow 0$.*

(ii) *For each fixed $r \in [0, r_*(0))$, $C_{T,p}$ converges as $p \uparrow 0$ to the price of the European call option given by Black-Scholes formula with the same r and the volatility $\lambda_0^{-1/2}$.*

Theorem 3 stipulates that in certain cases, the geometric Brownian motion and Black-Scholes formula can be used as reasonable approximations for stocks and options, respectively. Namely, they pertain to the market behavior near the equilibrium. In contrast, model (1)–(6) complies with the three distinct features exhibited by the stock price movements and their implied distributions. These properties are associated with the leverage effect, which was discovered by Black

(1976). One of these features is the fact that the the volatility and the growth rate are usually *negatively correlated* (see Hull (2002, pp. 334–335 and 341)). This is related to (24). Let us term the tails of the spectral densities which have the form (2) as *'semi-heavy'* tails. It is known that the tails of both the original and implied distributions are frequently semi-heavy. Proposition 1 and Theorem 2 yield that the risk-neutral Esscher transform is more skewed to the left than the original distribution. This is typical for chaotic movements of overleveraged equities and their option prices (compare to Madan (2001, Subsection 2.5)). The third feature is an increased likelihood of financial crashes. Voit (1999, Chapter 9) proposes their investigation by employing the critical state theory. This is consistent with the existence of critical points $u_0^{\pm}$ for our model (see Theorem 1(ii2)).

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FONCTIONS HYPERGÉOMÉTRIQUES DE LAURICELLA, PÉRIODES DE VARIÉTÉS ABÉLIENNES ET TRANSCENDANCE

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RÉSUMÉ. La fonction hypergéométrique de Lauricella généralise, en n variables, la fonction hypergéométrique classique dite de Gauss. On construit l'ensemble exceptionnel lié à la fonction de Lauricella. Cet ensemble généralise celui étudié par J. Wolfart dans le cas classique. On en déduit un nouveau contre-exemple à une conjecture de Coleman. On montre ensuite que des valeurs algébriques de fonctions de Lauricella, en des points algébriques, conduisent à la transcendance de valeurs d'autres de ces fonctions en des points associés. Enfin, on s'intéresse à une dernière fonction hypergéométrique et à son ensemble exceptionnel afin de donner une application.

ABSTRACT. The Lauricella hypergeometric function generalizes, in the case of n variables, the classical Gauss hypergeometric function. One constructs the exceptional set related to the Lauricella function. This set generalizes the one studied by J. Wolfart in the classical case. One deduces another counter-example of a Coleman conjecture. One then shows how algebraic values of Lauricella functions, at algebraic points, imply transcendental values of other such functions at related points. To finish with, one focuses on a new transcendental function in order to give an application.

1. Introduction. Sous certaines conditions sur les paramètres a, b et c, la fonction hypergéométrique classique, dite de Gauss — décrite ci-dessous — est transcendante :

$$F(a;b;c;x) = B(a;c-a)^{-1}\cdot\int_1^\infty u^{b-c}(u-1)^{c-a-1}(u-x)^{-b}\,du \quad \text{pour } x \in \mathbb{C}\backslash\{0;1\}$$

où B est la fonction Bêta.

Il est naturel de s'intéresser aux points algébriques en lesquels une telle fonction prend des valeurs algébriques. Dans [3] et [2], F. Beukers, J. Wolfart et N. Archinard donnent de tels exemples et dans [16], J. Wolfart étudie la finitude de l'ensemble suivant, appelé ensemble exceptionnel :

$$\mathcal{E} = \{x \in \overline{\mathbb{Q}} : F(a;b;c;x) \in \overline{\mathbb{Q}}\}.$$

L'utilisation d'outils géométriques permet de décrire cet ensemble en terme de variétés abéliennes de la même classe d'isogénie. C'est ce point de vue qui est

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utilisé dans [15] et dans [16] pour commencer de répondre à la question de la finitude. La réponse complète est donnée par P. Cohen et G. Wüstholz dans [6], elle utilise de façon essentielle la version afaiblie d'une conjecture d'André-Oort dans le cas d'une courbe, désormais démontrée par B. Edixhoven et A. Yafaev (voir [10]).

L'objectif est de décrire un tel ensemble exceptionnel pour le cas général des fonctions hypergéométriques de Lauricella — n variables. On utilise la description géométrique de cet ensemble, en terme de variétés abéliennes de la même classe d'isogénie — convenablement choisie pour généraliser le cas d'une variable — afin d'en donner une description en terme de valeurs algébriques de fonctions de Lauricella en des points algébriques. On utilise ensuite la construction de cet ensemble pour donner quelques applications. Enfin, dans une dernière partie, on s'intéresse aux cas d'une et de deux variables afin d'introduire une autre fonction hypergéométrique, dite de Picard. La contruction de son ensemble exceptionnel permet de donner une application du résultat de B. Edixhoven et A. Yafaev.

2. La fonction hypergéométrique de Lauricella. Afin de simplifier les notations, on considère l'ensemble

$$\mathcal{Q} := \mathbb{P}_1(\mathbb{C})^n \backslash \Big\{(x_2;\dots;x_{n+1}) \in \mathbb{P}_1(\mathbb{C})^n \ : \\ x_i = 0;1;\infty;x_j \text{ pour } i;j \in \{2;\dots;n+1\} \text{ et } j \neq i\Big\}$$

et on note :

$$\omega(x_2;\dots;x_{n+1}) := u^{\sum_{i=2}^{n+1} b_i - c}(u-1)^{c-a-1}\prod_{i=2}^{n+1}(u-x_i)^{-b_i}du.$$

La fonction hypergéométrique de Lauricella est définie sur l'ensemble $\mathcal{Q}$ (à l'aide d'intégrales euleriennes) par l'expression :

$$F(a;b_2;\dots;b_{n+1};c;x_2;\dots;x_{n+1}) = B(a;c-a)^{-1}\cdot\int_1^{\infty}\omega(x_2;\dots;x_{n+1}).$$

REMARQUE. Les conditions $x_i = 0;1;\infty;x_j$ pour $i;j \in \{2;\dots;n+1\}$ et $j \neq i$, qui permettent de définir $\mathcal{Q}$, correspondent aux singularités régulières d'un système d'équations différentielles linéaires partielles dont l'espace vectoriel des solutions est de dimension $n+1$. La fonction hypergéométrique de Lauricella est la solution holomorphe de ce système qui est égale à 1 au point $(0;\dots;0)$. (Voir [1, chapitre VII].)

Afin d'utiliser des outils géométriques et de s'assurer que la fonction est transcendante, on impose les conditions suivantes sur les paramètres :

$$(*) \qquad 0<a<1, \quad 0<c<1, \quad 0<\sum_{i=2}^{n+1} b_i < c.$$

On note N le plus petit dénominateur commun des a, $b_2, \dots, b_{n+1}$ et c.

On peut alors voir la fonction de Lauricella comme le quotient de périodes de variétés abéliennes :

- l'une, $P_0 = B(a; c-a)$ est une période non nulle de première espèce sur une sous-variété abélienne, notée A, à C.M. par $\mathbb{Q}(\zeta_N)$ ou par un sous-corps de $\mathbb{Q}(\zeta_N)$ (ζ_N étant une racine N-ième primitive de l'unité), extraite de la jacobienne de la courbe de Fermat :
$$\mathcal{F}_N : u^N + w^N = 1,$$
voir [16] pour plus de détails;
- l'autre, $P(x_2; \dots; x_{n+1}) = \int_1^\infty \omega(x_2; \dots; x_{n+1})$, est une période non nulle de première espèce sur une variété abélienne, notée $T(x_2; \dots; x_{n+1})$ obtenue de la façon suivante : pour $f \in \{1; \dots; N\}$ un diviseur de N, on note
$$\mathcal{X}_f(x_2; \dots; x_{n+1}) : w^f = u^{(c-\sum_{i=2}^{n+1} b_i)N}(u-1)^{(a+1-c)N} \prod_{i=2}^{n+1} (u - x_i)^{b_i N}$$
l'application de $\mathcal{X}_N$ vers $\mathcal{X}_f$ qui envoie $(u; w)$ sur $(u; w^{\frac{N}{f}})$ induit un morphisme m_f de $\mathrm{Jac}(\mathcal{X}_N)$ vers $\mathrm{Jac}(\mathcal{X}_f)$, on note alors $T(x_2; \dots; x_{n+1})$ la composante connexe à l'origine de l'intersection des noyaux des morphismes m_f pour f diviseur propre de N, voir [16] pour plus de détails.

Cette variété abélienne $T(x_2; \dots; x_{n+1})$, liée à la fonction hypergéométrique de Lauricella est de dimension $\frac{(n+1)\varphi(N)}{2}$ (où φ est la fonction d'Euler), de type IV par le corps cyclotomique $\mathbb{Q}(\zeta_N)$ et son type est connu et exprimé en fonction des paramètres a, $b_2, \dots, b_{n+1}$ et c. Ces informations permettent, grâce aux travaux de G. Shimura (voir par exemple [12]), de déterminer l'espace qui paramétrise de telles variétés abéliennes, on le note H.

REMARQUE. Dans le cas des fonctions hypergéométriques de Gauss et d'Appell, l'espace qui paramétrise les classes d'isomorphisme de ces variétés abéliennes a une structure sous-jacente de variété quasi-projective définie sur $\overline{\mathbb{Q}}$, dont les points complexes peuvent être décrits comme quotient par un groupe arithmétique Γ d'une copie de plusieurs boules unité (de la dimension correspondante), une certaine condition étant automatiquement vérifiée (voir la condition 2 du paragraphe 5 de [13]). L'utilisation des boules unité est utile notamment pour se servir des groupes de monodromie qui sont d'action discontinue sur de tels espaces (voir [4], [5] et [6]). Dans le cadre général, l'espace de paramètres est plus compliqué mais on sait le décrire explicitement.

3. Suite aux travaux de H. Shiga et J. Wolfart, voir [13, paragraphe 5], on dispose d'un morphisme régulier entre variétés quasi-projectives définies sur $\overline{\mathbb{Q}}$:
$$\Phi : \begin{array}{ccc} \mathcal{Q} & \longrightarrow & H \\ (x_2; \dots; x_{n+1}) & \longmapsto & Z(T(x_2; \dots; x_{n+1})) \end{array}$$

où $Z(T(x_2;\ldots;x_{n+1}))$ est un point de l'espace H correspondant à la variété abélienne $T(x_2;\ldots;x_{n+1})$ dont l'orbite par Γ correspond à la classe d'isomorphisme de $T(x_2;\ldots;x_{n+1})$.

REMARQUE. On peut étendre ce morphisme à certaines surfaces caractéristiques, celles dites stables, (voir [7, p. 669] pour la définition de ces surfaces).

Dans l'objectif de la construction de l'ensemble exceptionnel, on s'intéresse à l'ensemble des variétés abéliennes de la même classe d'isogénie. Pour ce faire, on fixe la variété abélienne :

$$T_0 := A \times \prod_{k=2}^{n+1} B_k$$

où A (resp. B_k) est la variété abélienne de dimension $\varphi(N)/2$ à multiplication complexe par $\mathbb{Q}(\zeta_N)$ ou l'un de ses sous-corps, extrait de la jacobienne de la courbe de Fermat $\mathcal{F}_N : u^N + w^N = 1$ et caractérisée par la période de première espèce $B(a; c-a)$ (resp. $B(b_k; c-\sum_{i=2}^{n+1} b_i)$). Le théorème qui suit fait le lien entre l'aspect arithmétique et l'aspect géométrique de l'étude des valeurs spéciales des fonctions de Lauricella.

Pour A et B deux variétés abéliennes, on note $A \hat{=} B$ lorsque ces variétés sont isogènes.

THÉORÈME 1. *Pour $(x_2;\ldots;x_{n+1}) \in \mathcal{Q} \cap \overline{\mathbb{Q}}^n$, on a $T(x_2;\ldots;x_{n+1}) \hat{=} T_0$ si et seulement si*

$$\begin{cases} (h) : F(a; b_2;\ldots;b_n; c; x_2;\ldots;x_{n+1}) \in \overline{\mathbb{Q}} \\ \text{et si } n \geq 2, \text{ pour tout } l \in \{0;\ldots;n-2\}, \text{ il existe un } k \in \{2;\ldots;n+1\} \\ \text{tel que } (h_l^{(k)}) : F(b_k; 1-b_2;\ldots;c-a;\ldots;1-b_{n+1}; \\ \qquad c+l-\sum_{i=2,i\neq k}^{n+1} b_i; \frac{x_k}{x_k-x_2};\ldots;\frac{x_k}{x_k-1};\ldots;\frac{x_k}{x_k-x_{n+1}}) \in \overline{\mathbb{Q}} \end{cases}$$

où le k-ième paramètre est donné par $c-a$ et la $(k-1)$-ième variable est donnée par $\frac{x_k}{x_k-1}$.

On peut maintenant donner l'écriture de *l'ensemble exceptionnel relatif à la fonction hypergéométrique de Lauricella* :

$$\mathcal{E}_n = \Big\{(x_2;\ldots;x_{n+1}) \in \mathcal{Q} \cap \overline{\mathbb{Q}}^n : \begin{cases} (h) : F(a; b_2;\ldots;b_n; c; x_2;\ldots;x_{n+1}) \in \overline{\mathbb{Q}} \\ \forall l \in \{0;\ldots;n-2\}, \exists k \in \{2;\ldots;n+1\}, (h_l^{(k)}) \text{ est vraie} \end{cases} \Big\}.$$

Les résultats obtenus dans ce paragraphe généralisent la construction des ensembles exceptionnels des fonctions hypergéométriques de Gauss (introduit par J. Wolfart dans [15]) et d'Appell (voir [8]).

4. Un contre-exemple. Il existe une infinité de valeurs de $(x; y; z) \in \overline{\mathbb{Q}}^3$ pour lesquelles la jacobienne de la courbe algébrique de genre 4,

$$\mathcal{X}_3(x) : v^3 = u(u-1)(u-x)(u-y)(u-z),$$

est à multiplication complexe par $\mathbb{Q}(\zeta_3)$ ou par un de ses sous-corps. Ceci contredit une conjecture de Coleman qui affirmait qu'il n'existe qu'un nombre fini de classes d'isomorphisme de courbes algébriques de genre $g \geq 4$, dont la jacobienne est à multiplication complexe.

REMARQUE. Trois contre-exemples sont déjà connus, deux dans le cas d'une variable trouvés par J. de Jong et R. Noot voir [9] ou [6], et un dans le cas de deux variables voir [8].

5. Dans l'énoncé ci-après, on exprime toutes les fonctions présentes comme quotient de périodes non nulles de première et de deuxième espèce sur des variétés abéliennes. Des résultats sur l'indépendance linéaire sur $\overline{\mathbb{Q}}$ de telles périodes, dûs à J. Wolfart et G. Wüstholz [16], permettent de conclure.

THÉORÈME 2. *Pour tous les n-uplets* $(x_2; \ldots; x_{n+1})$ *situés dans l'ensemble exceptionnel* $\mathcal{E}_n$*, les* $2n$ *nombres suivants sont transcendants :*

$$\begin{cases} \textit{pour } l = 0; \ldots; n-1, \\ \quad F(n-l-a; 1-b_2; \ldots; 1-b_{n+1}; n+1-l-c; x_2; \ldots; x_{n+1}) \\ \textit{pour } k = 2; \ldots; n+1, \\ \quad F\big(1-b_k; b_2; \ldots; a+1-c; \ldots; b_{n+1}; 2+\sum_{i=2, i\neq k}^{n+1} b_i - c; \\ \qquad \frac{x_k}{x_k-x_2}; \ldots; \frac{x_k}{x_k-1}; \ldots; \frac{x_k}{x_k-x_{n+1}}\big) \end{cases}$$

où le k-ième paramètre est donné par $a+1-c$ *et la* $(k-1)$*-ième variable est donnée par* $\frac{x_k}{x_k-1}$.

COROLLAIRE. *Soient* $a, b, c \in \mathbb{Q} \cap]0; 1[$*,* $a, b < c$*. Pour tout* $x \in \overline{\mathbb{Q}} \setminus \{0; 1\}$*, si* $F(a; b; c; x)$ *algébrique alors* $F(b+1-c; a+1-c; 2-c; x)$ *est transcendant.*

REMARQUE. Les travaux de F. Beukers et J. Wolfart [3] et N. Archinard [2] donnent des valeurs algébriques explicites de la fonction de Gauss lorsqu'elle est transcendante (conditions $(*)$), ce résultat permet donc d'en déduire des valeurs transcendantes.

6. La fonction hypergéométrique de Picard. À partir de cas particuliers des fonctions hypergéométriques précédentes, celle de Gauss (une variable) et celle d'Appell (deux variables), on construit une nouvelle fonction transcendante d'une variable complexe : la fonction de Picard Φ_λ — cette appelation est relative à une remarque p. 73 de [1]. Pour $\lambda \in \mathbb{C} \setminus \{0;1\}$ fixé et $a, b, b', c \in \mathbb{Q} \cap]0;1[$, $a < c$, $b + b' < c$,

$$\text{pour } y \in \mathbb{C} \setminus \{0;1;\lambda\}, \quad \Phi_\lambda(a;b;b';c;y) = \frac{F(a;b;b';c;\lambda;y)}{F(a;b;c;\lambda)}.$$

Lorsque les conditions ci-dessus sur les paramètres sont réalisées, la fonction de Picard est transcendante. Afin d'utiliser les techniques précédentes, on peut la regarde comme le quotient de périodes non nulles de première espèce de variétés abéliennes.

Pour deux nombres complexes α et β, on note $\alpha \sim \beta$ lorsqu'il existe un nombre algébrique non nul δ tel que $\alpha = \delta\beta$.

Ainsi, *l'ensemble exceptionnel* décrit en terme de variétés abéliennes de la même classe d'isogénie, puis à l'aide de valeurs, modulo $\overline{\mathbb{Q}}$, de la fonction de Picard est donné par :

$$\begin{aligned} \mathcal{E}_\lambda &= \{y \in \overline{\mathbb{Q}} \setminus \{0;1;\lambda\} : T(\lambda;y) \hat{=} A_0\} \\ &= \left\{ y \in \overline{\mathbb{Q}} \setminus \{0;1;\lambda\} : \begin{cases} (h_1) : \Phi_\lambda(a;b;b';c;y) \in \overline{\mathbb{Q}} \\ (h_2) \end{cases} \right\} \end{aligned}$$

où $A_0 := T(\lambda) \times A$ et

$$\begin{aligned} (h_2) : \Phi_{\frac{\lambda}{\lambda-1}}(b; c-a; 1-b'; c-b'; \frac{\lambda}{\lambda-y}) \\ \sim \frac{B(b;c-b).F(c-b;c-a;c;\lambda)}{B(b;c-b-b').F(c-b-b';c-a;c-b';\lambda)}. \end{aligned}$$

Dans un dernier temps, on utilise cet ensemble $\mathcal{E}_\lambda$ afin de donner une application du théorème qui suit. Ce théorème est la version faible d'une conjecture d'André-Oort dans le cas d'une courbe désormais démontrée par B. Edixhoven et A. Yafaev (et dont la réciproque est connue) :

THÉORÈME (B. Edixhoven et A. Yafaev [10]). *On suppose que sur une courbe algébrique C irréductible d'une variété de Shimura, il existe un point P — dit de base — qui est C.M. (c'est à dire qui correspond à une variété abélienne à multiplication complexe). S'il existe une infinité de points de C dont la variété abélienne correspondante est isogène à la variété abélienne correspondant à P, alors C est de type Hodge.*

Ce théorème traite de la distribution de variétés abéliennes de la même classe d'isogénie. Or, on a construit l'ensemble exceptionnel de cette façon : $\mathcal{E}_\lambda = \{y \in \overline{\mathbb{Q}} \setminus \{0;1;\lambda\} : T(\lambda;y) \hat{=} A_0\}$. Pour pouvoir appliquer le théorème précédent, il reste à s'assurer que le point de base A_0 est à multiplication complexe. La deuxième ligne du théorème 3 correspond à cette condition. Ce dernier résultat permet de faire le lien entre une propriété géométrique de la courbe C_λ et la finitude du nombre d'apparitions de certaines valeurs de la fonction de Picard.

THÉORÈME 3. *Soient* $a, b, b', c \in \mathbb{Q} \cap]0;1[$, $a < c$, $b + b' < c$.

Pour $\lambda \in \overline{\mathbb{Q}} \setminus \{0;1\}$ *tel que* $F(a;b;c;\lambda) \sim \frac{B(a+1-c;1-a)}{B(c-b;b)} \cdot F(b+1-c;$ $a+1-c; 2-c; \lambda)$,

$$C_\lambda \text{ est de type Hodge si et seulement si } \mathrm{Card}(\mathcal{E}_\lambda) = \infty$$

où $C_\lambda = \{[T(\lambda;y)] : y \in \mathbb{P}_1(\mathbb{C}) \setminus \{0;1;\lambda;\infty\}\}$ *est une famille de dimension* 1, *de classes d'isomorphisme de variétés abéliennes, tracée sur une certaine variété de Shimura. (*$[A]$ *désigne la classe d'isomorphisme de la variété abélienne* A.*)*

REMARQUES. Pour certaines valeurs des paramètres a, b, b' et c, on connait des λ pour lesquels la condition sur les fonctions de Gauss est vérifiée (voir [3] ou [2]). On sait même qu'elle peut se réaliser une infinité de fois pour certains paramètres (voir [16, remarque 2, p. 98]).

Par contre, on ne sait pas préciser quand la courbe C_λ est de type Hodge ; sauf les cas limites où $\lambda \in \{0;1;\infty\}$ où l'on est ramené à la fonction hypergéométrique de Gauss. Par exemple pour $\lambda = 0$, l'arithméticité du groupe triangulaire $\Delta(|b'-a|^{-1}; |1-c|^{-1}; |a+b'-c|^{-1})$ (que l'on sait donner : voir [14]) fournit le caractère de type Hodge de la courbe C_0 grâce au théorème de l'ensemble exceptionnel de [6].

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ON THE SYMMETRY OF ARITHMETICAL FUNCTIONS IN ALMOST ALL SHORT INTERVALS

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ABSTRACT. We study the mean-square (over $x \in [N, 2N]$) of the "symmetry sum" of arithmetical functions f, namely $\sum_{|n-x|\le h} \mathrm{sgn}(n-x)f(n)$; in this way we may check the symmetry of f in "almost all" short intervals by elementary methods; using the Large Sieve alone, together with elementary considerations.

1. Introduction and statement of the results. In this paper we study the symmetry, in almost all short intervals, of a particular class of arithmetical functions; we generalize, in fact, the results of our previous papers [2], [3] and [4], using (mainly) the Large Sieve applied to the periodic function $\chi_q(x)$.

We introduce a new argument (our Lemma 3), in order to generate "sporadic" terms, which can be estimated by a "Fragmented Large-Sieve" (our Lemma 2); this doesn't apply to "extremely large" moduli $d > N/(2h)$ (see the following). In order to avoid these divisors, we have two choices: we can exclude them, providing a result only for a particular class of functions (see our Theorem 2); or, also, if the Dirichlet convolution $f * \mathbf{1}$ is "symmetric enough" (see also [3]), we can get a result on the "average-symmetry" of distribution of f (see our Corollary 1). In both cases, we use the Large Sieve for the "small" divisors, see Lemma 1, and the fragmented Large-Sieve, see Lemma 2, for the "large" divisors. Finally, in Section 4, we provide some examples for our three theorems and we also give individual results, *i.e.*, we get enough symmetry for *all* short intervals, rather than for almost all. Actually, this is possible for the square-free numbers (and, in the same way, for k-free numbers).

First of all, we consider n to be in a short interval of the kind $[x-h, x+h]$ (we define it short, as usual, if $h = o(x)$ and $h(x) \to \infty$ as $x \to \infty$) and we form the "symmetry sum" (here $\mathrm{sgn}(t) := t/|t|$, $\forall t \in \mathbf{R} - \{0\}$, $\mathrm{sgn}(0) = 0$; in the sequel $x \in \mathbf{N}$ and $x \to \infty$):

$$S_f^{\pm}(x) := \sum_{x-h\le n\le x+h} f(n)\,\mathrm{sgn}(n-x);$$

then we consider its "mean square" (with $h = h(N) \in \mathbf{N}$ independent of x, $N \in \mathbf{N}$, $N \to \infty$):

$$I_f(N,h) := \sum_{N<x\le 2N} |S_f^{\pm}(x)|^2.$$

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In the sequel we'll write $d \sim D$ for $D < d \le 2D$; and $L := \log N$. We now give our results. We start from the generalization of our Theorem 1 of [2]:

THEOREM 1. *Let N, h be natural numbers, with $h = h(N) \to \infty$ and $h = o(N)$ as $N \to \infty$; assume that $A = A(N,h) \ge 1$ is a real parameter. Then, defining $M_f := \max_{n \le A\sqrt{N}} |f(n)|$,*

$$\sum_{x \sim N} \Big| \sum_{|n-x| \le h} \operatorname{sgn}(n-x) \sum_{d|n, d \le A\sqrt{N}} f(d) \Big|^2 \ll M_f^2 N A^2 h L^3.$$

REMARK. We might also consider sums on d with $f(d)g(\frac{n}{d})$ instead of $f(d)$, for a suitable $g \in \mathcal{S}$ (see before Theorem 3), but we will not, for clarity.

Actually, the divisors range can be even wider, thanks to the "Fragmented-Large-Sieve".

THEOREM 2. *Let $N, h \in \mathbf{N}$, with $h = h(N) \to \infty$ and $h = o(N)$, as $N \to \infty$. Assume that $M_f := \max_{n \le N/(2h)} |f(n)|$. Then*

$$\sum_{x \sim N} \Big| \sum_{|n-x| \le h} \operatorname{sgn}(n-x) \sum_{d|n, d \le N/(2h)} f(d) \Big|^2 \ll M_f^2 N^{4/3} h L^{13/3}.$$

(We emphasize that the same remark of Theorem 1 is still valid.)

Of course, both the theorem and Corollary 1 (see Section 4) are non-trivial when $h \ge N^{1/3} L^B$, $B > 0$ suitable (see also [3]). We now come to the generalization of the results of [4]. For this reason, we define a set of arithmetical functions, all of which are *quasi-constant* (see the following, in particular requirement 3), because a convolution of two such functions is *quasi-symmetric* (*i.e.*, the mean-square of its symmetry sum is "small enough"). The functions f and g appearing in Theorem 3 will satisfy the requirements:

(1) f is a completely additive or completely multiplicative arithmetical function with complex values, whose domain is extended to positive rationals in the obvious way;
(2) $\max_{\mathbf{Q}^+} |f| \ll N^c$, for a certain (absolute) constant $c > 0$;
(3) $|f(m) - f(\frac{x}{d})| \ll |m - \frac{x}{d}| h^{-C}$, for a fixed $C > 0$, uniformly in all the variables involved, *i.e.*, $\forall x \in [N, 2N]$, $\forall d \le \sqrt{x}$, $\forall m \in [\frac{x-h}{d}, \frac{x+h}{d}]$.

(The last one gives enough smooth functions, with no regularity condition on the derivatives.)

We will indicate the set of these functions with $\mathcal{S}$; also, we write $\mathcal{S}_{\mathcal{A}}$ for those $f \in \mathcal{S}$ completely additive and $\mathcal{S}_{\mathcal{M}}$ for those $f \in \mathcal{S}$ completely multiplicative. In order to give Theorem 3, we need to abbreviate the remainders, so we set

$$R_{f,g}(N,h) := (M_f^2 + M_g^2) N h^{2-2C} L^2 + M_f^2 M_g^2 (h^2/N + N),$$

for each arithmetical function f, where $m_f := \min_{1<n\le\sqrt{2N}} |f(n)|$, $M_f := \max_{n\le\sqrt{2N}} |f(n)|$.

Then we have the following:

THEOREM 3. *Let $N, h \in \mathbf{N}$ with $h = h(N) \to \infty$ and $h = o(N)$ as $N \to \infty$. Then*

(i) $f, g \in \mathcal{S_A} \Rightarrow I_{f*g}(N,h) \ll NhL^3M_f^2M_g^2 + R_{f,g}(N,h)$.

(ii) $f \in \mathcal{S_A}$, $g \in \mathcal{S_M} \Rightarrow I_{f*g}(N,h) \ll NhL^3(\frac{M_f^2M_g^2}{m_g^4} + M_f^2M_g^4) + R_{f,g}(N,h)$.

(iii) $f, g \in \mathcal{S_M} \Rightarrow I_{f*g}(N,h) \ll NhL^3(\frac{M_f^4M_g^2}{m_g^4} + \frac{M_g^4M_f^2}{m_f^4}) + R_{f,g}(N,h)$.

The paper is organized as follows:

- in Section 2 we give the lemmas to prove our theorems;
- in Section 3 we prove our theorems;
- in Section 4 we give some examples.

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2. Lemmas.

LEMMA 1. *Let Q and N be natural numbers, $M \in \mathbf{Z}$ and $\lambda_{a,q}$ be complex numbers ($\forall a, q \in \mathbf{N}$); then*

$$\sum_{n=M+1}^{M+N} \Big|\sum_{q\le 2Q} \sideset{}{^*}\sum_{a\le q} \lambda_{a,q} e_q(an)\Big|^2 = (N + \mathcal{O}(Q^2)) \sum_{q\le 2Q} \sideset{}{^*}\sum_{a\le q} |\lambda_{a,q}|^2.$$

(Here, as usual, the $*$ in the sum means that $(a, q) = 1$.)

This is a version obtained by duality from the well-known Large Sieve inequality (see [1], or [5]). We can also prove it using Hilbert's inequality (see the proof of Lemma 3 of [4] and its remarks). Here we will apply the upper bound $\mathcal{O}(N+Q^2)$, for the proof of our Theorems 1 and 2 (but we give this "asymptotic" version for completeness, see also [4]).

LEMMA 2. *Let J, N be natural numbers, with $J = o(N)$ as $N \to \infty$, and let f be an arithmetical function with $M_f := \max_{n\in\mathbf{N}} |f(n)|$. Then, defining*

$$\chi_q(x) := \sum_{\substack{|n-x|\le h \\ n\equiv 0(\bmod q)}} \operatorname{sgn}(n-x),$$

we have

$$\sum_{x\sim N} \Big|\sum_{q\sim J} \chi_q(x) f\Big(\Big[\frac{x+h}{q}\Big]\Big)\Big|^2 \ll M_f^2 NJhL^3.$$

(Here the implied constant is absolute.)

PROOF. We set $S_J(x) := \sum_{q\sim J} \chi_q(x) f([(x+h)/q])$, whence (as in the proof of Theorem 1): $S_J(x) = \sum_{d\le 2J} \sum_{n\sim \frac{J}{d}} \frac{1}{n} f\left(\left[\frac{x+h}{nd}\right]\right) \sum_{j\le d}^{*} c_{j,d} e_d(jx)$, and then expand its square (see also [3]) to get $\sum_{x\sim N} |S_J(x)|^2$ as

$$\sum_{n_1,n_2\le 2J} \frac{1}{n_1}\frac{1}{n_2} \sum_{\substack{d_1\sim J/n_1\\ d_2\sim J/n_2}} \sum_{j_1\le d_1}^{*} \sum_{j_2\le d_2}^{*} c_{j_1,d_1}\overline{c_{j_2,d_2}}$$

$$\times \sum_{x\sim N} f\left(\left[\frac{x+h}{n_1 d_1}\right]\right) \overline{f\left(\left[\frac{x+h}{n_2 d_2}\right]\right)} e\left(\left(\frac{j_1}{d_1} - \frac{j_2}{d_2}\right)x\right).$$

This last sum over x is estimated trivially, when $j_1/d_1 = j_2/d_2$, as $\mathcal{O}(NM_f^2)$, while it can be bounded by means of Lemma 2 of [4] when j_1/d_1 and j_2/d_2 are distinct Farey fractions. In fact, we break the range $[N, 2N]$ into $\mathcal{O}(N/J)$ "fragmented" ranges, each of length $\mathcal{O}(J)$, in which the f are constant (being constant their arguments); then, we argue as in [4, Lemma 2], with $k := j/d$ and $C_k := c_{j,d}$ (using the classical estimate for exponential sums [9]):

$$\sum_{x\sim N} |S_J(x)|^2$$

$$\ll M_f^2 \sum_{n_1,n_2\le 2N} \frac{1}{n_1}\frac{1}{n_2} \sum_{k_r,k_s} |C_{k_r}|\,|C_{k_s}| \min\left(N, \frac{N}{J\|k_r - k_s\|}\right)$$

$$\ll M_f^2 \sum_{n_1,n_2\le 2N} \frac{1}{n_1 n_2}\left(N\sum_k |C_k|^2 + \frac{N}{J}\sum_{k_r\ne k_s} |C_{k_r}|\,|C_{k_s}| \frac{1}{\|k_r - k_s\|}\right);$$

by Lemma 2 of [4], choosing $\delta = 1/J^2$ (the k are δ-well-spaced), this is at most

$$\ll M_f^2 \sum_{n_1,n_2\le 2N} \frac{1}{n_1 n_2} L(N + NJ) \sum_k |C_k|^2 \ll M_f^2 NJhL^3,$$

since $\sum_k |C_k|^2 \ll h$, [4]; this concludes the proof of the lemma.

REMARK. We explicitly point out that the remainders of this lemma are non-optimal, due to the non-optimality of this argument; also, for particular functions f, we may try to estimate the exponential sums arising from the fragmented x-range, to get more cancellation.

LEMMA 3. *Let N, h be natural numbers, with $h = h(N) \to \infty$ and $h = o(N)$ as $N \to \infty$, and assume that $A = A(N) \ge 1$ is a real parameter. Then, defining* $M_f := \max_{A\sqrt{N} < n \le \frac{2N+h}{2h}} |f(n)|$,

$$\sum_{x\sim N} \Big| \sum_{A\sqrt{N} < d \le \frac{x+h}{2h}} f(d)\chi_d(x) \Big|^2 \ll \left(\frac{N^{3/2}}{A} hL^5 + \frac{h^2}{A^2} + NL^3\right) M_f^2.$$

PROOF. First of all, write $\Sigma_2(x) := \sum_{A\sqrt{N}<d\le(x+h)/(2h)} f(d)\chi_d(x)$ and use $\chi_d(x)$ definition:

$$\Sigma_2(x) = \sum_{j\in\mathbb{N}}\Big(\sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h} \\ \frac{x}{j}<d<\frac{x+h}{j}}} f(d) - \sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h} \\ \frac{x-h}{j}<d<\frac{x}{j}}} f(d)\Big)$$
$$+ \mathcal{O}\big((d(x-h)+d(x)+d(x+h))M_f\big)$$

($d = \frac{x-h}{j}$, $d = \frac{x}{j}$ and $d = \frac{x+h}{j}$ give the divisor-functions-remainders; their mean-square is $\mathcal{O}(NM_f^2L^3)$, see [7], which will be omitted for the moment). This series is genuinely finite:

$$\Sigma_2(x) = \sum_{\frac{2(x-h)h}{x+h}<j<\frac{x+h}{A\sqrt{N}}} \Big(\sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h} \\ \frac{x}{j}<d<\frac{x+h}{j}}} f(d) - \sum_{\substack{A\sqrt{N}<d\le\frac{x+h}{2h} \\ \frac{x-h}{j}<d<\frac{x}{j}}} f(d)\Big)$$
$$= \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}} \sum_{|d-\frac{x}{j}|\le\frac{h}{j}} \mathrm{sgn}(d-x/j)f(d)$$
$$+ \mathcal{O}\Big(M_f\Big(\sum_{\frac{2(x-h)h}{x+h}<j\le 2h} + \sum_{\frac{x-h}{A\sqrt{N}}<j\le\frac{x+h}{A\sqrt{N}}} \Big)\Big(\frac{h}{j}+1\Big)\Big)$$
$$= \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}} \sum_{|d-\frac{x}{j}|\le\frac{h}{j}} \mathrm{sgn}(d-x/j)f(d) + \mathcal{O}\Big(\Big(\frac{h^2}{N} + \frac{h}{A\sqrt{N}} + 1\Big)M_f\Big),$$

since $h = o(N)$ implies $2h(x-h)/(x+h) = 2h + o(h)$; then the d-sum is *sporadic*, *i.e.*, has at most *one* term (and the bound $\mathcal{O}(1)$ is non-optimal). Henceforth, we have this sporadic term if and only if $[(x-h)/j, (x+h)/j]$ contains an integer, which is exactly $[\frac{x+h}{j}]$; the function $\chi_j(x)$ detects it, also giving (by the sign) its side w.r.t. $\frac{x}{j}$ (right or left of).

Then (apart from the remainders, which we'll join later)

$$S(x) := \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}} \sum_{\frac{x-h}{j}<d\le\frac{x+h}{j}} \mathrm{sgn}(d-x/j)f(d) = \sum_{2h<j\le\frac{x-h}{A\sqrt{N}}} \chi_j(x)f\Big(\Big[\frac{x+h}{j}\Big]\Big);$$

we argue as in [3], by a dyadic dissection of the j-range, to get

$$S(x) \ll L \max_{2h<J\le\frac{N-h}{A\sqrt{N}}} \sum_{j\sim J} \chi_j(x)f([(x+h)/j]) + \sum_{\frac{N-h}{A\sqrt{N}}<j\le\frac{x-h}{A\sqrt{N}}} \chi_j(x)f([(x+h)/j]).$$

Now the proof follows by applying Lemma 2.

3. Proof of the theorems.

PROOF OF THEOREM 1. Using $\chi_d(x)$ definition (see Lemma 2) and letting $\Sigma_1(x) := \sum_{d\le A\sqrt{N}} \chi_d(x) f(d)$, it will suffice to show that $\sum_{x\sim N} |\Sigma_1(x)|^2 \ll NA^2 M_f^2 h L^3$. We rearrange the finite Fourier expansion [4] of $\chi_d(x)$, defined in Lemma 2, as follows. Since ($c_{j,d}$ being the Fourier coefficients) $dc_{j,d}$ depends only on j/d, we get $c_{at,bt} = \frac{1}{t} c_{a,b}$ and

$$\chi_d(x) = \sum_{j<d} c_{j,d} e_d(jx)$$

$$= \sum_{t|d} \frac{1}{t} \sum_{\substack{j'<(d/t) \\ (j',(d/t))=1}} c_{j',d/t} e_{d/t}(j'x) = \sum_{t|d} \frac{t}{d} \sum_{\substack{r\le t \\ (r,t)=1}} c_{r,t} e_t(rx),$$

whence $\Sigma_1(x) = \sum_{t\le A\sqrt{N}} (\sum_{n\le (A\sqrt{N})/t} f(tn)/n) \sum_{j\le t}^{*} c_{j,t} e_t(jx)$. We can apply Lemma 1 to get

$$\sum_{x\sim N} \Big| \sum_{q\le A\sqrt{N}} \alpha_f(q) \sum_{j\le q}^{*} c_{j,q} e_q(jx) \Big|^2 \ll NA^2 \sum_{q\le A\sqrt{N}} |\alpha_f(q)|^2 \sum_{j<q} |c_{j,q}|^2,$$

where, say, $\alpha_f(q) := \sum_{n\le (A\sqrt{N})/q} f(qn)/n$ and (by definition of $c_{j,q}$) $\sum_{j<q} |c_{j,q}|^2 \ll \frac{h}{q}$; then

$$\sum_{x\sim N} \Big| \sum_{d\le A\sqrt{N}} f(d) \chi_d(x) \Big|^2 \ll NA^2 M_f^2 h L^3.$$

REMARK. Lemma 3 of [4] gives the same result, with an additional logarithm, for $d \le \sqrt{x}$.

PROOF OF THEOREM 2. We have

$$\sum_{|n-x|\le h} \operatorname{sgn}(n-x) \sum_{d|n, d\le \frac{N}{2h}} f(d) = \sum_{d\le \frac{N}{2h}} f(d) \chi_d(x)$$

(since Lemma 3 holds also when $A\sqrt{N} < d \le \frac{N}{2h}$: the terms $\frac{N}{2h} < d \le \frac{x+h}{2h}$ are contained in the remainders, see its proof); applying Theorem 1 and Lemma 3, together with the optimal choice $A = N^{1/6} L^{2/3}$, we get the theorem.

PROOF OF THEOREM 3. We start writing

$$S_{f*g}^{\pm}(x) = \sum_{|md-x|\le h} \Big(\sum_{d\le m} f(d) g(m) + \sum_{m\le d} g(m) f(d) \Big) \operatorname{sgn}(md-x)$$

$$+ O\Big(M_f M_g \Big(\frac{h}{\sqrt{N}} + 1 \Big) \Big)$$

As an example for Theorem 1, take $a(n) := \sum_{d|n, d\le\sqrt{n}} \log(d)$ ($f(n) := \log n$ and $A = 1$); it has symmetry sum: $\sum_{|n-x|\le h} \mathrm{sgn}(n-x)a(n) \ll \sqrt{h}\log^{5/2+\varepsilon} x$, a.a. $x \in [N, 2N]$ ("a.a." = "almost all", *i.e.*, all $x \in [N, 2N]$ with at most $o(N)$ exceptions).

As an example for Theorem 2, we consider the function [6] $M_z(n) := \sum_{d|n, d<z} \mu(d)$, where we can choose any $z \le N/(2h)$. The same function is an example of Corollary 1 (in fact, the second mean-square in the remainders is actually 0). Of course, the best instance of Theorem 3 is $f = g = \mathbf{1}$ [4]. More generally, consider $f(d) = d^{-s}$, with $s = \sigma + it$, and $g = \mathbf{1}$; in this case we apply Theorem 3, case (iii), provided $\sigma > -1$ (for requirement (3), whatever $t \in \mathbf{R}$ is (but, of course, $s \in \mathbf{C}$ has to be fixed). We will give asymptotics for this in a future paper.

Finally, we present an "individual" result: square-free numbers in *all* short intervals:

THEOREM 4. *Let $x, h \in \mathbf{N}$, with $h = h(x) \to \infty$ and $h = o(x)$, as $x \to \infty$. Then we have $S^{\pm}_{\mu^2}(x) \ll h \log^{-A} x$, provided $h \ge \sqrt{N} \log^A x$. (Write $\mu^2(n) = \sum_{d^2|n} \mu(d)$, see [8].)*

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Index—Volume 26, 2004