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STRONGLY TWO-GENERATED IDEALS IN RINGS OF INTEGER-VALUED POLYNOMIALS DETERMINED BY FINITE SETS

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Presented by Vlastimil Dlab, FRSC

ABSTRACT. Let D be an integral domain, $E = \{e_1, \dots, e_k\}$ a finite nonempty subset of its quotient field K and $\mathrm{Int}(E, D)$ the ring of polynomials in $K[x]$ which map E into D. A 2-generated ideal I of $\mathrm{Int}(E, D)$ is called *strongly two-generated* if each of its nonzero elements can be chosen as one of two generators of I. We characterize the strongly two-generated ideals of $\mathrm{Int}(E, D)$. In the case where D is not a Bezout domain and $|E| > 1$, we show further that the strongly two-generated ideals of the ring $\mathrm{Int}(E, D)$ form a proper nontrivial subgroup of its ideal class group.

RÉSUMÉ. Soient D un anneau intègre de corps des fractions K, $E = \{e_1, \dots, e_k\}$ une partie non vide finie de K et $\mathrm{Int}(E, D)$ l'anneau des polynômes de $K[x]$ qui envoient E dans D. On dit qu'un idéal I de $\mathrm{Int}(E, D)$ engendré par 2 éléments est *fortement engendré par 2 éléments* si chacun de ses éléments non nuls peut être choisi pour l'un des deux générateurs. On caractérise les idéaux de $\mathrm{Int}(E, D)$ fortement engendrés par 2 générateurs. Dans le cas où D n'est pas un anneau de Bézout et où $|E| > 1$, on montre en outre que les idéaux fortement 2-engendrés de l'anneau $\mathrm{Int}(E, D)$ forment un sous-groupe non trivial de son groupe des classes.

Let D be an integral domain with field of fractions K. If a two-generated ideal I of a D has the property that the first of its two-generators can be chosen at random from the nonzero elements of I, then I is called *strongly two-generated.* A ring in which each two-generated ideal is strongly two-generated is said to have the *strong two-generator property.* If an element α of D can be chosen as one of two generators of every two-generated ideal I in which it is contained, then α is called a *strong two-generator of D.* This note is a sequel to the recent paper [2] by the current authors, in which we studied the strong two-generator property in certain rings of integer-valued polynomials determined by D. In particular, if E is a subset of K, then set

$$R = \mathrm{Int}(E, D) = \{f(x) \in K[x] \mid f(\alpha) \in D \text{ for every } \alpha \in E\}.$$

For ease of notation, if $E = D$, then set $\mathrm{Int}(D, D) = \mathrm{Int}(D)$. In [2] we showed for finite sets E that the ring $\mathrm{Int}(E, D)$ satisfies the strong two-generator property if and only if D is a Bezout domain. Moreover, if D is a Dedekind domain with nontrivial class group, we then characterized which elements of $\mathrm{Int}(E, D)$ are

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strong two-generators. In this current work, E is still a finite set but there is no hypothesis on the domain D. We characterize (in Proposition 4) the strongly two-generated ideals of $R = \text{Int}(E, D)$.

By a result of Lantz and Martin [6], the strongly two-generated ideals of a ring R form a subgroup of the invertible ideals of R. Hence, our discussion can be cast in the language of the class group. In the case where D is not a Bezout domain and $|E| > 1$, we show that the strongly two-generated ideals of R form a proper nontrivial subgroup of its ideal class group. We find this result of interest since it is not known if there exists a nonprincipal strongly two-generated ideal even in the much studied case $E = \mathbb{Z} = D$.

NOTATION 1. Given a domain R we let $\mathcal{F}(R)$ denote the monoid of finitely generated nonzero ideals, $\mathcal{I}(R)$ the group of invertible ideals and $\mathcal{P}(R)$ the group of nonzero principal ideals. Thus, the class group $\mathfrak{C}(R)$ is the quotient $\mathfrak{C}(R) = \mathcal{I}(R)/\mathcal{P}(R)$. As noted earlier, the strongly two-generated ideals form a subgroup of $\mathcal{I}(R)$ and we let $\mathfrak{C}_2(R)$ denote the subgroup of $\mathfrak{C}(R)$ represented by the strongly two-generated ideals.

In the case $R = \text{Int}(E, D)$, we make use of an additional reduction of the class of ideals needed to describe the class group $\mathfrak{C}(R)$. An ideal of $R = \text{Int}(E, D)$ is called *unitary* whenever $I \cap D \neq (0)$ (this is equivalent to the condition $IK[X] = K[X]$). One notes that if $\mathcal{I}_u(R)$ denotes the invertible unitary ideals, then $\mathcal{I}_u(R)$ is a subgroup of $\mathcal{I}(R)$. The following observation indicates that the class group of $\text{Int}(E, D)$ can be represented by the unitary ideals.

LEMMA 2. *Let $R = \text{Int}(E, D)$ where E is a non-empty subset of K. If I is in $\mathcal{I}(R)$, then there exists an ideal J in $\mathcal{I}_u(R)$ where $[I] = [J]$ in $\mathfrak{C}(R)$.*

PROOF. The argument for this result is a straight forward modification of the argument given in [1, Lemma VIII 1.2] in the case $E = D$. ■

For unitary ideals of R in the case where $E = \{e_1, \ldots, e_k\}$, we have the following simple description. As in [7], define the polynomial

$$F(x) = (x - e_1)(x - e_2) \cdots (x - e_k).$$

and for each $1 \leq r \leq k$

$$\varphi_r(x) = \frac{\prod_{j \neq r}(x - e_j)}{\prod_{j \neq r}(e_r - e_j)}.$$

We have that $\varphi_r(e_r) = 1$ and $\varphi_r(e_j) = 0$ when $j \neq r$. Hence each $\varphi_r(x) \in \text{Int}(E, D)$. The polynomials $\varphi_r(x)$ are called the Lagrange Interpolation Polynomials. We state here a basic result given in [7].

LEMMA 3. ([7, Proposition 5]) *Let D be an integral domain and $E = \{e_1, \ldots, e_k\}$ a finite nonempty subset of K. Each unitary ideal I of $R = \text{Int}(E, D)$ is uniquely represented as*

$$(*) \qquad I = I_1\varphi_1(x) + \cdots + I_k\varphi_k(x) + F(x)K[x]$$

where $I_1, \ldots, I_k$ are nonzero ideals of D. In addition, $I_i = I(e_i)$ for each i and $I \cap D = \cap_i I_i \neq (0)$. Conversely, any collection of nonzero ideals $I_1, \ldots, I_k$ in D give rise to such a unitary ideal I of R.

Lemma 3 also illustrates the presence of the "strong Hilbert property" for R; namely, if I and J are nonzero unitary ideals of R, then $I = J$ if and only if $I(e_i) = J(e_i)$ for all i. We remark that the Lemma remains valid in the context of fractional ideals (adjusting the equality to $I \cap K = \cap_1^k I_i$). We note I is invertible if and only if each $I_i = I(e_i)$ is invertible. The "if" part is immediate. but for the converse one uses the strong Hilbert property: if each $I(e_i)$ is invertible, letting J be the (unique) fractional unitary ideal such that $J(e_i) = I(e_i)^{-1}$ for each i, then $(IJ)(e_i) = D$ for each i and hence, by uniqueness, $IJ = \mathrm{Int}(E, D)$. Thus, the Lemma provides a natural mapping $\phi\colon \mathcal{I}_u(R) \to \mathcal{I}(D)^k$ and indeed gives us that this mapping is a group isomorphism. This idea will be considered after the following result is given regarding the strongly two-generated ideals.

PROPOSITION 4. *Let D be an integral domain and $E = \{e_1, \ldots, e_k\}$ a finite nonempty subset of K. Let I be a unitary ideal of $R = \mathrm{Int}(E, D)$ with unique representation $(*)$ where $I_1, \ldots, I_k$ are nonzero ideals of D. We have the following.*

(1) *I is principal if and only if each I_i is principal and $I_1 = I_2 = \cdots = I_k$.*
(2) *I is strongly two-generated if and only if each I_i is principal.*

PROOF. (1) ($\Rightarrow$) Suppose $I = d\,\mathrm{Int}(E, D)$ where $d \neq 0$ in D is a principal unitary ideal of $\mathrm{Int}(E, D)$. For each $1 \leq i \leq k$, we have that $I(e_i) = dD = I_i$. For ($\Leftarrow$), assume that $I_1 = I_2 = \cdots = I_k = dD$ for $d \neq 0$ in D. If $J = d\,\mathrm{Int}(E, D)$, then $I(e_i) = J(e_i)$ for $1 \leq i \leq k$ and $I = d\,\mathrm{Int}(E, D)$ by the strong Hilbert property.

(2) ($\Rightarrow$) Let I be a strongly two-generated unitary ideal of $\mathrm{Int}(E, D)$ with unique representation $(*)$. Since $F(x) \in I$, there exists a $g(x)$ in I with $I = (F(x), g(x))$. Thus $I_i = I(e_i) = (F(e_i), g(e_i)) = (0, g(e_i)) = (g(e_i))$ and hence each I_i is principal. For ($\Leftarrow$), suppose that each $I_i = (b_i)$ is principal and let $f(x)$ be a nonzero element of I. Using Lagrange interpolation, one easily gets a polynomial $g(x)$ in $\mathrm{Int}(E, D)$ with $g(e_i) = b_i$ for all i. In fact, one can construct such a polynomial where $f(x)$ and $g(x)$ are relatively prime in $K[X]$. This is done in the proof of [2, Theorem 4]. An alternate argument is to note that if $g(x)$ is a polynomial with $g(e_i) = b_i$, then $g_1(x) = g(x) + aF(x)$ has the same property for any constant a. Then, consider an extension L of K where $f(x)$ splits and choose an element a not in the set

$$\{ -g(z)/F(z) \mid z \text{ is a root of } f(x) \text{ with } z \neq e_i \text{ for all } i \}.$$

The polynomials $g_1(x)$ and $f(x)$ will have no common roots in L and, hence, $g_1(x)$ will be the desired polynomial. Finally, setting $J = (f(x), g_1(x))$ yields an unitary ideal with $J(e_i) = I(e_i)$ for all i. Hence $J = I$. ■

Combining Lemma 2 with Proposition 4 characterizes all strongly two-generated ideals in $\mathrm{Int}(E, D)$ when E is finite.

COROLLARY 5. *Let D be an integral domain with quotient field K, $E = \{e_1, \ldots, e_k\}$ a nonempty finite subset of K and I a finitely generated ideal of $R = \mathrm{Int}(E, D)$. I is strongly two-generated if and only if whenever J is a unitary ideal of $\mathrm{Int}(E, D)$ such that $[I] = [J]$ in $\mathfrak{C}(\mathrm{Int}(E, D))$, then $J(e_i)$ is a principal ideal of D for each i.*

We now use the previous results about the unitary ideals, the principal ideals, and the strongly two generated ideals to describe the groups $\mathfrak{C}(R)$ and $\mathfrak{C}_2(R)$. Lemma 3 provides a group isomorphism $\mathcal{I}(D)^k \to \mathcal{I}_u(R)$; on the other hand, since every ideal class is represented by a unitary ideal, we have a surjective homorphism $\mathcal{I}_u(R) \to \mathfrak{C}(R)$. Combining these produces a surjective homomorphism from $\mathcal{I}(D)^k$ onto $\mathfrak{C}(R)$. Proposition 4 yields the kernel of this homomorphism is Δ where

$$\Delta = \{(I, \ldots, I) \mid I \in \mathcal{P}(D)\}.$$

It is easy to see for any abelian group G and subgroup H that $G^k/\Delta \cong G/H \times G^{k-1}$ where $\Delta = \{(a, \ldots, a) \mid a \in H\}$. Thus, in our application we have

$$(\dagger) \qquad \mathfrak{C}(R) \cong \mathfrak{C}(D) \times \mathcal{I}(D)^{k-1}.$$

From this general statement we note two special cases.

THEOREM 6. *If D is a Dedekind domain, then*

$$\mathfrak{C}(R) \cong \mathfrak{C}(D) \times (\mathbb{Z}^{\mathbb{X}})^{k-1}$$

where $\mathbb{Z}^{\mathbb{X}}$ is a sum of copies of $\mathbb{Z}$ under the set $\mathbb{X}$ of prime ideals of D. In particular, if D is a discrete valuation domain, then $\mathfrak{C}(R) \cong \mathbb{Z}^{k-1}$.

PROOF. This is an immediate application of (†) for in case D is a Dedekind domain we have from the unique factorization of ideals that $\mathcal{I}(D) \cong \mathbb{Z}^{\mathbb{X}}$. The discrete valuation statement is just a special case. ∎

We also note here that in the special case $|E| = 1$ we have $\mathfrak{C}(R) \cong \mathfrak{C}(D)$. Moreover, when $|E| = 1$, Proposition 4 implies that $\mathfrak{C}_2(R)$ is trivial.

We now turn our attention to $\mathfrak{C}_2(R)$. Proposition 4 yields a natural homomorphism

$$\mathcal{I}_u(R) \to \mathfrak{C}(D)^k.$$

Since principal ideals map onto the trivial element of $\mathfrak{C}(D)^k$, we obtain a surjective homomorphism

$$\Psi\colon \mathfrak{C}(R) \to \mathfrak{C}(D)^k.$$

In addition, Proposition 4 yields the kernel of Ψ is $\mathfrak{C}_2(R)$. This result is summarized as follows.

PROPOSITION 7. $\mathfrak{C}(R)/\mathfrak{C}_2(R) \cong \mathfrak{C}(D)^k$.

Remark: Proposition 7 yields the result established in [2] that when D is a Bezout domain every invertible ideal is strongly two-generated (that is, one gets $\mathfrak{C}(R) = \mathfrak{C}_2(R)$). To be precise, $\mathfrak{C}_2(R)$ is a proper subgroup of $\mathfrak{C}(R)$ if and only if D is not a Bezout domain, and it is nontrivial if and only if $|E| > 1$. This last fact follows immediately from Proposition 4, or from (†) combined with Proposition 7.

We close with an example which illustrates the result of Proposition 7.

EXAMPLE 8. *Let D be an algebraic ring of integers with nontrivial class group (such as $D = \mathbb{Z}[\sqrt{-5}]$) and let E be any two element subset of K. Pick nonzero nonunit α and β in D such that α and β are not associates. Let I_3 be any nonprincipal finitely generated ideal of D. Consider the following three ideals in* Int(E, D).

$$I = \alpha D\varphi_1(x) + \alpha D\varphi_2(x) + F(x)K[x]$$
$$I^* = \alpha D\varphi_1(x) + \beta D\varphi_2(x) + F(x)K[x]$$
$$I^\# = \alpha D\varphi_1(x) + I_3\varphi_2(x) + F[x]K[x].$$

Then

1. *I is a principal ideal of* Int(E, D) *and hence $[I] = 0$ in $\mathfrak{C}$*(Int(E, D)).
2. *I^* is a strongly two-generated ideal of* Int(E, D) *which is not a principal ideal. Hence $[I^*]$ lies in $\mathfrak{C}_2$*(Int(E, D))$\backslash\{0\}$.
3. *$I^\#$ is finitely generated but neither principal nor strongly two-generated. Since under our hypothesis* Int(E, D) *is a Prüfer domain (see* [7, Theorem 8]*), $I^\#$ is invertible and thus $[I^\#]$ lies in $\mathfrak{C}$*(Int(E, D))$\backslash\mathfrak{C}_2$(Int(E, D)).

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PALINDROME-POLYNOMIALS WITH ROOTS ON THE UNIT CIRCLE

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ABSTRACT. Given a polynomial $f(x)$ of degree n, let $f^r(x)$ denote its *reciprocal*, i.e., $f^r(x) = x^n f(1/x)$. If a polynomial is equal to its reciprocal, we call it a *palindrome* since the coefficients are the same when read backwards or forwards. In this mathematical note we show that palindromes whose coefficients satisfy a certain magnitude-condition must have a root on the unit circle. More exactly our main result is the following. If a palindrome $f(x)$ of even degree n with real coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ satisfies the condition $|\epsilon_k| \geq |\epsilon_{n/2}| \cos(\pi/([\frac{n/2}{n/2-k}]+2))$, for some $k \in \{0, 1, \ldots n/2-1\}$, then $f(x)$ has unimodular roots. In particular, palindromes with coefficients 0 and 1 always have a root on the unit circle.

RÉSUMÉ. Soit $f(x)$ un polynôme de degré n. Soit $f^r(x) = x^n f(1/x)$. Le polynôme $f(x)$ s'appelle polynôme réciproque si $f^r(x) = f(x)$. Dans cet article nous prouvons que les polynômes réciproques dont les coefficients possèdent une certaine propriété, ont des zéros sur le cercle unité, c'est-à-dire ont des zeros de valeur absolue 1. Notre principal résultat est le théorème suivant. Soit $f(x)$ un polynôme réciproque dont le degré n est pair et dont les coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ sont des nombres réels tels que $|\epsilon_k| \geq |\epsilon_{n/2}| \cos(\pi/([\frac{n/2}{n/2-k}] + 2))$, pour au moins une valeur de $k \in \{0, 1, \ldots n/2-1\}$. Tel polynôme $f(x)$ possède des zéros de valeur absolue 1. Pour consequence, chaque polynôme réciproque avec des coefficients 0 et 1 a des zéros sur le cercle unité.

1. Introduction. We arrive at an interesting geometric property of palindrome-polynomials, i.e., of polynomials whose coefficients are the same when read backwards or forwards, by investigating polynomials with coefficients 0 and 1 or $(0, 1)$-polynomials. They are interesting because of their applications in various areas of pure and applied mathematics including algebra, number theory, combinatorics, and coding theory. Computer-assisted experiments lead us to conjecture that $(0, 1)$-palindromes necessarily have at least one unimodular root, that is, have a root of absolute value 1. In the sequel we will prove not only that this conjecture is true but that palindromes with real coefficients often have roots on the unit circle. This introductory section is dedicated to setting up notations and terminology. The second section contains the main results of the article. We begin by defining the notion of palindrome. Given a polynomial $f(x)$ of degree n, let $f^r(x)$ denote its *reciprocal*, i.e., $f^r(x) = x^n f(1/x)$. If a polynomial is equal to its reciprocal, we call it a *palindrome-polynomial* or simply

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a *palindrome.* In the next section we establish rather easily the following. *Let $f(x) \in \mathbb{R}[X]$ be a palindrome having even degree n and null middle coefficient. Then the polynomial $f(x)$ has unimodular roots,* (Lemma 2). Note that the existence of unimodular roots is interesting only for palindromes of even degree, because if the degree is odd, obviously -1 is a root. By $\mathbb{R}[X]$ we denote the ring of polynomials of one variable with coefficients in $\mathbb{R}$, the field of real numbers. If the middle coefficient is not null, a palindrome with real coefficients may lack unimodular roots, (consider e.g., $f(x) = x^2 - 3x + 1$). It turns out that the relative size of the middle coefficient with respect to the other coefficients is important for the existence of unimodular roots. Our proof relies on inequalities for the coefficients of non-negative trigonometric polynomials due to Szegő and Egerváry and Szász. For each real x let $[x]$ denote the largest integer which is less than or equal to x. The main result proved in Section 2 is the following.

Let $f(x) = \sum_{k=0}^{n} \epsilon_k x^k$, with $\epsilon_j = \epsilon_{n-j}$, $\forall j = 0, 1, \ldots, n/2$ be a palindrome having even degree n. If there exists $k \in \{0, 1, \ldots, n/2 - 1\}$ such that

$$|\epsilon_k| \geq |\epsilon_{n/2}| \cos\left(\frac{\pi}{[\frac{n/2}{n/2-k}] + 2} \right),$$

then $f(x)$ has unimodular roots (Theorem 1).

In particular, $(0,1)$-palindromes of any degree have roots on the unit circle, (Corollary 1).

Another consequence of Theorem 1 is the following.

Let $f(x)$ be a palindrome with real coefficients, having even degree n. If $2|\epsilon_n| \geq |\epsilon_{n/2}|$, then $f(x)$ has unimodular roots (Corollary 2).

2. Palindromes and their zeros. Our approach is based on the following classical construction. Denoting $u = x + 1/x$ we associate to each palindrome $f(x) \in \mathbb{R}[X]$ having even degree n the polynomial $g(u)$ uniquely determined by the following identity.

$$f(x) = x^{n/2} g(u) \tag{1}$$

Several comments are in order here. For each $k = 1, 2, \ldots$ we denote $\sigma_k(x) = x^k + 1/x^k$. For $k = 0$ we set $\sigma_0(x) = 1$. Observe that $\sigma_1(x) = u$ and that the following identity holds.

$$u^k = \sigma_k(x) + \sum_{1 \leq j \leq k/2} \sigma_{k-2j}(x) \binom{k}{j} \qquad k = 2, 3, \ldots \tag{2}$$

Let $f(x) = \sum_{k=0}^{n} \epsilon_k x^k$, with $\epsilon_j = \epsilon_{n-j}$, $\forall j = 0, 1, \ldots, n/2$. Note that

$$f(x) = x^{n/2} h(x), \text{ where } h(x) = \sum_{k=0}^{n/2} \epsilon_k \sigma_{n/2-k}(x). \tag{3}$$

Equations (2) and (3) prove that $g(u)$ exists and is uniquely determined. After these preparations we can prove the following lemma, which appears also in [4], in essentially the same form. We include it with proof, to make the paper self-contained.

LEMMA 1. *Let $f(x) \in \mathbb{R}[X]$ and $g(u) \in \mathbb{R}[U]$ be as above. The polynomial $f(x)$ has a unimodular root if and only if the polynomial $g(u)$ has a real root in the interval $[-2, 2]$, respectively if and only if the following cosine polynomial has real zeros.*

$$\varphi(x) = \epsilon_{n/2} + \sum_{k=0}^{n/2-1} 2\epsilon_k \cos((n/2 - k)x) \tag{4}$$

PROOF. If $z = e^{i\theta}$ is a unimodular root of $f(x)$ then $u = 2\cos\theta$ is a root of $g(u)$ and clearly $-2 \le u \le 2$. Conversely if $g(u)$ has a root, $u \in [-2, 2]$, then there exists $\theta \in \mathbb{R}$ such that $2\cos\theta = u$. Let $z = e^{i\theta}$, then z is a unimodular root of $f(x)$. The fact that the existence of unimodular roots for the palindrome under consideration is equivalent to the existence of zeros for the cosine polynomial in (4) is an immediate consequence of the considerations above, equality (3), and the fact that if $x = e^{i\theta}$ then $\sigma_k(x) = 2\cos k\theta$ for $k \ge 1$. ∎

Based on the previous remarks we can prove the existence of unimodular roots for arbitrary palindromes with real coefficients, having even degree and null middle-term. More exactly the following is true.

LEMMA 2. *Let $f(x) \in \mathbb{R}[X]$ be a palindrome having even degree n and such that $\epsilon_{n/2} = 0$. Then the polynomial $f(x)$ has unimodular roots.*

PROOF. By Lemma 1 it will suffice to show that a cosine polynomial of the form

$$\varphi(x) = \sum_{k=0}^{n/2-1} \epsilon_k \cos((n/2 - k)x)$$

with real coefficients ϵ_k, necessarily has a zero in the interval $[0, 2\pi]$. The latter is an immediate consequence of the fact that the integral of $\varphi(x)$ on $[0, 2\pi]$ equals 0 and $\varphi(x)$ is a continuous function. ∎

As we observed in the introduction, if the middle coefficient is not null, a palindrome with real coefficients may fail to have unimodular roots. However, the relative size of the middle coefficient with respect to the other coefficients matters here. The crucial inequality needed in the proof of our next result states that the coefficients of a non-negative trigonometric polynomial

$$\frac{1}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \ge 0 \qquad x \in [0, 2\pi]$$

have the property

$$\sqrt{a_k^2+b_k^2} \le \cos\frac{\pi}{\left[\frac{n}{k}\right]+2} \qquad k=1,2,\dots,n;$$

see Egerváry and Szász [2], Szegő [6]. For further inequalities see Alzer [1], Losonczi [5]. In particular, for pure cosine polynomials, (where $b_1=b_2=\cdots=b_n=0$) we have

$$|a_k| \le \cos\frac{\pi}{\left[\frac{n}{k}\right]+2} \qquad k=1,2,\dots,n.$$

THEOREM 1. *Let $f(x)$ be a palindrome with real coefficients, having even degree n, such that for some $k\in\{0,1,\dots,n/2-1\}$ the following condition is satisfied*

$$|\epsilon_k| \ge \cos\left(\frac{\pi}{\left[\frac{n/2}{n/2-k}\right]+2}\right)|\epsilon_{n/2}|. \tag{5}$$

Such a palindrome has unimodular roots.

PROOF. The case $\epsilon_{n/2}=0$ is covered by Lemma 2. Let us consider the case $\epsilon_{n/2}\ne 0$. Since $f(x)$ and $-f(x)$ have the same roots we can assume without loss of generality that $\epsilon_{n/2}>0$. Denote by $\varphi(x)$ the cosine polynomial in equality (4). Since the integral of $\varphi(x)$ on $[0,2\pi]$ is positive, it follows that $\varphi(x)$ attains positive values on that interval. Thus, the only way it can have no zeros would be if it is a positive cosine polynomial. Assume by contradiction that this is the case, and observe that the following cosine polynomial, $\phi(x)$, is also positive.

$$\phi(x)=1/2+\sum_{k=0}^{n/2-1}\frac{\epsilon_k}{\epsilon_{n/2}}\cos((n/2-k)x)$$

Let $\delta=\min\{\phi(x):x\in[0,2\pi]\}$. Observe that $0<\delta<1/2$. Indeed, $\phi(x)-1/2$ is not the null function on $[0,2\pi]$ and has null integral on that interval, thus $\phi(x)-1/2$, being a continuous function, must assume both positive and negative values. Therefore the following cosine polynomial is non-negative on $\mathbb{R}$:

$$1/2+\sum_{k=0}^{n/2-1}\frac{\epsilon_k}{2\epsilon_{n/2}(1/2-\delta)}\cos((n/2-k)x).$$

By the Egerváry-Szász Inequality, in such a case one should have the following inequality satisfied for each $k\in\{0,1,\dots,n/2-1\}$

$$\left|\frac{\epsilon_k}{\epsilon_{n/2}(1-2\delta)}\right| \le \cos\left(\frac{\pi}{\left[\frac{n/2}{n/2-k}\right]+2}\right).$$

This leads to the fact that for each $k \in \{0, 1, \ldots, n/2 - 1\}$ one can write

$$\left|\frac{\epsilon_k}{\epsilon_{n/2}}\right| \leq \cos\left(\frac{\pi}{\left[\frac{n/2}{n/2-k}\right]+2}\right)(1-2\delta) < \cos\left(\frac{\pi}{\left[\frac{n/2}{n/2-k}\right]+2}\right),$$

which is contradictory under our assumptions. ∎

COROLLARY 1. *Let $f(x)$ be a palindrome with real coefficients, having even degree n, such that*

$$\max\{|\epsilon_k| : k \in \{0, 1, \ldots, n/2 - 1\}\} \geq |\epsilon_{n/2}|.$$

Such a palindrome has unimodular roots. In particular, $(0,1)$-palindromes of any degree have roots on the unit circle.

PROOF. The statement above is a direct consequence of Theorem 1 and the well-known fact that

$$\cos\left(\frac{\pi}{\left[\frac{n/2}{n/2-k}\right]+2}\right) \leq 1, \quad k \in \{0, 1, \ldots, n/2 - 1\}.$$

∎

COROLLARY 2. *Let $f(x)$ be a palindrome with real coefficients, having even degree n. If*

$$2|\epsilon_n| \geq |\epsilon_{n/2}| \tag{6}$$

then $f(x)$ has unimodular roots.

PROOF. Enter $k = 0$ in inequality (5) and recall that $\epsilon_n = \epsilon_0$. ∎

The author of [3] proves that palindromes of the form

$$l(z^n + z^{n-1} + \cdots + z + 1) + \sum_{k=1}^{[\frac{n}{2}]} a_k(z^{n-k} + z^k) \tag{7}$$

have all their roots on the unit circle if their coefficients satisfy the following condition

$$|l| \geq 2\sum_{k=1}^{[\frac{n}{2}]} |a_k|. \tag{8}$$

Palindromes of the form (7) are important because of their role in the investigation of spectral properties of the Coxeter transformation of certain oriented graphs. If one needs palindromes of form (7) to have some, (not necessarily all), roots on the unit circle, then the following less restrictive version of condition (8) can be proved.

COROLLARY 3. *Even-degree palindromes of the form* (7) *have unimodular roots if*

$$|l| \geq 2|a_{n/2}|. \tag{9}$$

PROOF. Note that the middle coefficient of such a palindrome is $2a_{n/2} + l$ and that if (9) holds then

$$|2a_{n/2} + l| \leq 2|a_{n/2}| + |l| \leq 2|l|,$$

that is, (6) holds, and hence the palindrome under consideration must have unimodular roots. ■

The statement in Theorem 1 does not extend to palindromes with complex coefficients. Indeed, consider $f(z) = z^2 + iz + 1$. The condition in Theorem 1 is sufficient, but not necessary, for the existence of unimodular roots. Indeed, consider $h(x) = -2x^4 - 2x^3 + 5x^2 - 2x - 2$ and use Lemma 1 to show that it has unimodular roots. Obviously this palindrome does not satisfy the condition in Theorem 1.

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CLASSIFICATION OF QUADRUPLE CANONICAL COVERS: GALOIS CASE

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Presented by M. Ram Murty, FRSC

Résumé. Le but de cet article est de décrire la classification obtenue dans [GP1] des revêtements galoisiens de degré 4 des surfaces de degré minimal qui sont définis par le morphisme canonique. Cette classification montre que ces revêtements sont ou bien bidoubles ou bien cycliques non simples. S'ils sont des revêtements bidoubles, alors ils sont tous, à une exception près, des produits fibrés de revêtements doubles. À partir de cette classification, on déduit des implications importantes, comme l'existence de familles d'un genre géométrique non borné et aussi, l'existence de familles avec irregularité non bornée. Cette situation est tres différente de celle des revêtements canoniques doubles et triples.

The purpose of this research announcement is to describe the resultsin [GP1] on the classification of Galois quadruple canonical covers of *surfaces of minimal degree* W (*i.e.*, an embedded projective algebraic surface whose degree is equal to its codimension in projective space plus 1.) Let X be a surface of general type with a base point free canonical bundle K_X. We say $X \xrightarrow{\varphi} W$ is a *canonical cover* of W if φ is induced by the complete linear series $|K_X|$. Note that, since W maybe singular, the Galois canonical covers of this article need not be flat, though they are always finite.

The earlier work on the classification of double covers is due to Horikawa in [Ho1], [Ho2] and for triple covers to Konno [K]. The surfaces of general type with a base point free canonical bundle which admit a morphism to a surface of minimal degree play a fundamental role in numerous contexts including the classification of surfaces of general type with low K_S^2 ([Ho1], [Ho2]) and their moduli, the construction of new examples of surfaces of general type by various authors, the mapping of the so-called geography of surfaces of general type and the study of the generators of the canonical ring of a variety of general type, among other things.

The classification done in [GP1] yields important implications and we will mention only two of them (due to space constraints) at the end of this article. Our work together with that of Horikawa and Konno seems to reveal a striking numerology depending on the degree n of the cover. There exist many more canonical covers if n is even: p_g is unbounded for $n = 2, 4$ while $p_g \leq 4$ if $n = 3$. On the other hand p_g is bounded if W is singular, φ is Galois and $n \leq 4$. This seems to point out to the non-existence of higher degree canonical covers when W is singular. This together with other facts, like the nonexistence of regular Galois

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canonical covers of prime degree $n \geq 5$ (proved in [GP1]), suggests obstacles to the existence in general of canonical covers of higher degree.

We fix now some notation and conventions. Throughout this article W will be an embedded projective algebraic surface of minimal degree and X will be a projective algebraic normal surface of general type with at worst canonical singularities. By $\mathbf{F}_e$ we denote the Hirzebruch surface with e, C_0 and f as in Hartshorne [H]. By $S(a,b)$ we denote a rational normal scroll (which can be a cone) as in Eisenbud-Harris [EH]. This scroll is the image of $\mathbf{F}_e$ by the complete linear series $|C_0+mf|$, with $e=|b-a|$, $m \geq e+1$ if $e=0,1$ and $m \geq e$ if $e \geq 2$. If $a=b$, the linear series $|mC_0+f|$ also gives a minimal degree embedding of $\mathbf{F}_0$, equivalent to the previous one by the automorphism of $\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{F}_0$ swapping the factors. In this case the convention will always be to choose C_0 and f so that the surface is embedded by $|C_0+mf|$.

We first summarize the results dealing with the classification of canonical Galois covers of smooth surfaces of minimal degree, starting with the case in which the Galois group is $\mathbf{Z}_2 \times \mathbf{Z}_2$ (these are also called bi-double covers).

THEOREM 1. *Let W be a smooth surface of minimal degree. If $X \xrightarrow{\varphi} W$ is a canonical Galois cover with Galois group $\mathbf{Z}_2 \times \mathbf{Z}_2$, then W is either linear $\mathbf{P}^2$ or a smooth rational normal scroll and X is the product fibered over W of two double covers of $X_1 \xrightarrow{p_1} W$ and $X_2 \xrightarrow{p_2} W$ and φ is the morphism $X_1 \times_W X_2 \to W$. Let D_2 and D_1 be the branch divisors of p_1, p_2 and if W is a rational normal scroll, let $D_2 \sim 2a_2C_0+2b_2f$ and $D_1 \sim 2a_1C_0+2b_1f$. Then we have in addition:*

I. *If $W=\mathbf{P}^2$, then D_1 and D_2 are quartics. In this case X is regular.*
II. *If W is a rational normal scroll and X is regular, then $W=S(m-e,m)$, $0 \leq e \leq 2$, $m \geq e+1$ and $a_1=1, a_2=2, b_1=m+1$ and $b_2=e+1$.*
III. *If X is an irregular surface, then $W=S(m,m)$ and one of the following happens:*

(1) $a_1=0$, $a_2=3$, $b_1=m+1$, $b_2=1$. *In this case,* $q(X)=m$;

(2) $a_1=0$, $a_2=3$, $b_1=m+2$, $b_2=0$. *In this case,* $q(X)=m+3$;

(3) $a_1=1$, $a_2=2$, $b_1=m+2$, $b_2=0$. *In this case,* $q(X)=1$.

Conversely, if $X \xrightarrow{\varphi} W$ is the fiber product over W of two double covers $X_1 \xrightarrow{p_1} W$ and $X_2 \xrightarrow{p_2} W$, branched respectively along divisors D_2 and D_1 as in one of the above cases, then $X \xrightarrow{\varphi} W$ is a Galois canonical cover with Galois group $\mathbf{Z}_2 \times \mathbf{Z}_2$.

SKETCH OF PROOF. Since W is smooth, if $X \xrightarrow{\varphi} W$ is Galois of degree 4, then φ is flat and $\varphi_*\mathcal{O}_X$ splits as a vector bundle as $\mathcal{O}_W \oplus L_1 \oplus L_2 \oplus L_3$, with L_1, L_2 and L_3 line bundles. This implies in particular, that if W is isomorphic to $\mathbf{P}^2$, X is regular. The subalgebras $\mathcal{O}_W \oplus L_i$ correspond to three intermediate degree 2 covers p_i induced by the three index 2 subgroups of $\mathbf{Z}_2 \times \mathbf{Z}_2$. The general description of the branch divisors of φ and of the intermediate covers, and of the multiplicative structure of $\varphi_*\mathcal{O}_X$ is well-known (see [HM].) Using this general description we first show that a bidouble cover is a fibered product if and only if $L_1 \otimes L_2 = L_3$. Using relative duality we show that, if φ is a canonical cover, then

$$L_3 = L_1 \otimes L_2 = \omega_W(-1) \tag{1}$$

and in particular, a fiber product. If W is isomorphic to $\mathbf{P}^2$, then it has to be linear $\mathbf{P}^2$ and not the Veronese surface. This follows as the splitting type of $\varphi_* \mathcal{O}_X$ is incompatible with the one proved in Lemma 2.3, [GP2]. This lemma also gives the branch divisors when W is linear $\mathbf{P}^2$. If W is a scroll, one determines the branch divisors of the double covers through a detailed study of the different possibilities which arise, the key ingredients in the argument being (1), the fact that φ is induced by a complete linear series and the computation of $H^1(\mathcal{O}_X)$ from the cohomology of the L_i's. This analysis gives for instance, that, if X is regular, then

$$\begin{aligned} p_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y\big(-C_0 - (m+1)f\big) \oplus \mathcal{O}_Y\big(-2C_0 - (e+1)f\big) \\ \oplus \mathcal{O}_Y\big(-3C_0 - (m+e+2)f\big). \end{aligned}$$

From such a description it is easy to find out the linear equivalence classes of the branch divisors. Finally, the bound $e \leq 2$ on $W \simeq \mathbf{F}_e$ comes from the restrictions imposed on the fixed part of the branch divisors by the fact that X is normal. Knowing the branch divisors allows us to determine exactly the L_i's and from them the irregularity of X. ∎

We study now canonical cyclic covers (*i.e.*, those with Galois group $\mathbf{Z}_4$) of smooth surfaces of minimal degree. Among other things, next theorem together with Theorem 3 shows the nonexistence of *simple cyclic canonical* quadruple covers of surfaces of minimal degree.

THEOREM 2. *Let W be a smooth surface of minimal degree r and let $X \xrightarrow{\varphi} W$ be a canonical Galois cover with Galois group $\mathbf{Z}_4$. Then φ is the composition of two double covers $X_1 \xrightarrow{p_1} Y$ branched along a divisor D_2 and $X \xrightarrow{p_2} X_1$, branched along the ramification of p_1 and $p_1^* D_1$ and with trace zero module $p_1^* \mathcal{O}_W(-\frac{1}{2}D_1 - \frac{1}{4}D_2)$ and W, D_1 and D_2 satisfy*

I. $W = \mathbf{P}^2$, *D_1 is a conic and D_2 is a quartic.*
II. $W = S(m-e, m)$, $0 \leq e \leq 2$, $m \geq e+1$, $D_1 \sim (2m-e+1)f$ *and* $D_2 \sim 4C_0 + (2e+2)f$.
III. $W = S(1,1)$ *(i.e., W is quadric hypersurface in $\mathbf{P}^3$), $D_1 \sim 3C_0$ and $D_2 \sim$* $2C_0 + 4f$.

In cases I, II *and* III, *X is regular and X is singular having at best $4(r+1)$ singular points of type A_1.*

If X is irregular, then $q(X) = 1$, X is singular having at best $4(r+4)$ singular points of type A_1 and

IV. $W = S(m,m)$, $m \geq 1$, $D_1 \sim (2m+4)f$, $D_2 \sim 4C_0$.

Conversely, if $X \xrightarrow{\varphi} W$ *is the composition of two double covers* $X_1 \xrightarrow{p_1} Y$ *branched along a divisor* D_2 *and* $X \xrightarrow{p_2} X_1$*, branched along the ramification of* p_1 *and* $p_1^* D_1$ *and with trace zero module* $p_1^* \mathcal{O}_W(-\frac{1}{2}D_1 - \frac{1}{4}D_2)$*, with* D_1 *and* D_2 *as in one of the above cases, then* $X \xrightarrow{\varphi} W$ *is a Galois canonical cover with Galois group* $\mathbf{Z}_4$.

SKETCH OF PROOF. We first prove that φ cannot be simple cyclic. If it were, $\varphi_* \mathcal{O}_X = \mathcal{O}_W \oplus L \oplus L^{\otimes 2} \oplus L^{\otimes 3}$ for some line bundle L and $\omega_X = \omega_W \otimes L^{-3}$. An analysis of the branch divisor together with the fact that φ is induced by the complete canonical series of X yields a contradiction. A cover with Galois group $\mathbf{Z}_4$ is a composition of two double covers which corresponds to the index 2 subgroup of $\mathbf{Z}_4$. The bundle $\varphi_* \mathcal{O}_X$ splits as $\mathcal{O}_W \oplus L_1 \oplus L_2 \oplus L_3$ by the action of $\mathbf{Z}_4$, as direct sum of eigenspaces, where L_1, L_2, L_3 corresponds to eigenspaces associated to i, -1 and $-i$ respectively. The general description of the multiplicative structure of $\varphi_* \mathcal{O}_X$ and its relation with the branch locus of the double covers is summarized in [GP1] (see also [PA] and [HM]). Using relative duality, we show that if φ is a canonical cover, then (1) holds also in this case. This simplifies the multiplicative structure of $\varphi_* \mathcal{O}_X$ for an arbitrary cyclic cover of degree 4. For instance, if W is a scroll and X is regular, we obtain a splitting for $\varphi_* \mathcal{O}_X$ as in Theorem 1 and, from it, the description of the branch divisors. One shows that the bound $e \leq 2$ when $W \simeq \mathbf{F}_e$ is imposed by the normality of X. A more involved analysis when X is irregular gives us the description of the branch divisors in this case. The number and type of singular points that X possesses comes from the intersection between the branch divisors. ∎

Finally we classify Galois quadruple canonical covers of singular surfaces W of minimal degree. The proof is much more subtle than the smooth case and it uses the classification of the smooth case together with some ideas on how to "control the singularities" to be made precise later. First we construct a commutative diagram.

Let W be a singular rational normal scroll and let $X \xrightarrow{\varphi} W$ be a canonical cover. Let w the singular point of W and let $Y \xrightarrow{q} W$ be the minimal desingularization of W. Then there exists the following commutative square:

$$(*) \qquad \begin{array}{ccc} \bar{X} & \xrightarrow{\bar{q}} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{q} & W \end{array}$$

where $\bar{X}$ is the normalization of the reduced part of $X \times_W Y$, which is irreducible, and p and $\bar{q}$ are induced by the projections from the fiber product onto each factor. Note that if φ is a Galois cover with Galois group G so is p.

THEOREM 3. *Let* W *be a singular rational normal scroll and let* $X \xrightarrow{\varphi} W$ *be a Galois, quadruple canonical cover. Let* $\bar{X}$, Y, q, $\bar{q}$ *and* p *be as in commutative* *diagram* $(*)$. *Then* $W = S(0,2)$ *(and hence,* $Y = \mathbf{F}_2$*), the (normal) surface* $\bar{X}$ *has at worst canonical singularities and* X *is regular. Moreover*

I. *If* $\bar{q}$ *is crepant, then* X *is the canonical model of* $\bar{X}$ *and one of the following happens:*

(1) *If* $G = \mathbf{Z}_2 \times \mathbf{Z}_2$*, then* $\bar{X}$ *is the product over* Y *of two double covers branched along divisors* D_2 *and* D_1 *which are linearly equivalent to* $2C_0 + 6f$ *and* $4C_0 + 6f$ *respectively.*

(2) *If* $G = \mathbf{Z}_4$*, then* p *is the composition of two double covers:* $\bar{X}_1 \xrightarrow{p_1} Y$ *branched along a divisor* D_2 *linearly equivalent to* $4C_0 + 6f$ *and* $\bar{X} \xrightarrow{p_2} \bar{X}_1$*, branched along the ramification of* p_1 *and* $p_1^*D_1$*, with* D_1 *linearly equivalent to* $3f$*, and with trace zero module* $p_1^*\mathcal{O}_Y(-C_0-3f)$*. Then* $\bar{X}$ *has at best* 12 *singular points of type* A_1*,* 3 *of them lying on the line* F*, inverse image of* C_0*, and* 9 *outside* F*, and* X *has only one point, which is singular, lying over* w *and has at best another* 9 *singular points of type* A_1*.*

II. *If* $\bar{q}$ *is noncrepant and* $G = \mathbf{Z}_2 \times \mathbf{Z}_2$*, then* $\bar{X}$ *is the normalization of the fiber product over* Y *of two double covers of* Y *each branched along a divisor linearly equivalent to* $4C_0 + 6f$*. In this case* $\varphi^{-1}\{w\}$ *consists of two smooth points and* $\bar{X} \xrightarrow{\bar{q}} X$ *is the blowing up of* X *at these two points.*

III. *If* $\bar{q}$ *is noncrepant and* $G = \mathbf{Z}_4$*, then the inverse image of* C_0 *consists of a single line* F *with* $F^2 = -\frac{1}{2}$*,* $\bar{q}$ *contracts only* F*,* $K_{\bar{X}} = \bar{q}^*K_X + 2F$ *and* F *is singular only at a one point, which has a singularity of type* A_1*. In this case* $\bar{1}X$ *has at best* 9 *singular points outside* F *of type* A_1 *and* X *has at best another* 9 *singular points of type* A_1*.*

Moreover, p *is the composition of two double covers:* $\bar{X}_1 \xrightarrow{p_1} Y$ *branched along a divisor* D_2 *and* $\bar{X} \xrightarrow{p_2} \bar{X}_1$*, branched along the ramification of* p_1 *and* $p_1^*D_1$ *and with trace zero module* $p_1^*\mathcal{O}_Y(-\frac{1}{2}(D_1+C_0)-\frac{1}{4}D_2)\otimes\mathcal{O}_{\bar{X}_1}(\bar{C}_0)$*, where* $\bar{C}_0 = p^{-1}C_0$*, and one of the following happens:*

(1) $D_1 \sim C_0 + 3f$ *and* $D_2 \sim 4C_0 + 6f$*.*
(2) $D_1 \sim 4C_0 + 9f$ *and* $D_2 \sim 2C_0 + 2f$*.*

Conversely, let $\bar{X}$ *be a normal surface with at worst canonical singularities and let* $Y = \mathbf{F}_2$*. If* $\bar{X} \xrightarrow{p} Y$ *is as described in one of* I.1, I.2, II, III.1, *or* III.2, *then* p *is Galois with Galois group* $\mathbf{Z}_2 \times \mathbf{Z}_2$ *or* $\mathbf{Z}_4$ *as indicated in* I.1, I.2, II, III.1, *or* III.2, *and there exists a commutative diagram like* $(*)$ *where* φ *is the canonical morphism of* X *and* $\bar{q}$ *is as described in* I.1, I.2, II, III.1, *or* III.2.

Sketch of Proof. First we study the 0-cycle which is the inverse image of the singular point of W under φ. Since φ is Galois, this study splits into three cases according to the cardinality of the support of the 0-cycle, which is 4, 2 and 1. In each of these situations, except for one possible exception which corresponds to case III and is settled once we obtain a detailed description of the branch divisors, we prove that $\bar{X}$ has canonical singularities. This is done by first showing that it has rational singularities and then bounding the discrepancies of the intermediate double covers induced by the action of the index 2-subgroups. Next we show that the morphism $\bar{q}$, except in two cases (corresponding to II and III) is crepant. This step is closely intertwined with the above mentioned steps. In the process of proving the above, we show that $W = S(0,2)$. Then one proceeds to study separately the cases when $\bar{q}$ is crepant and noncrepant. If $\bar{q}$ is crepant we apply to p arguments similar to those used to prove Theorems 1 and 2. If $\bar{q}$ is noncrepant a separate, more subtle analysis is required. ∎

TWO IMPORTANT IMPLICATIONS. (a) We have constructed examples in [GP1] to show that all the cases that appear in the above classification do indeed occur. This shows that there exist families of quadruple covers with *unbounded* p_g, in sharp contrast to triple covers classified by [K] where $p_g \leq 5$, and *unbounded irregularity* q in sharp contrast with the picture for double covers classified in [Ho1], where the surfaces are all regular. In fact all Horikawa surfaces are simply connected.

(b) The existence of *non-simple cyclic* canonical covers are rare in the case of surfaces and not easy to construct. The classification shows there indeed exist bounded families of non-simple cyclic covers.

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WHEN THE CENTRAL NORM EQUALS 2 IN THE SIMPLE CONTINUED FRACTION EXPANSION OF A QUADRATIC SURD

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Presented by M. Ram Murty, FRSC

ABSTRACT. We complete the task, begun in [19], of determining when the central norm (determined by the infrastructure of the underlying real quadratic field) is equal to 2 in the simple continued fraction expansion of the associated quadratic surd.

RÉSUMÉ. Nous poursuivons le travail amorcé dans l'article [19], et étudions le problème de déterminer quand la norme centrale (déterminée par l'infracture du corps quadratique réel en question) est égale à 2 dans le développement en fraction continue de l'irrationnalité quadratique associée au corps quadratique.

1. Introduction In [19], we showed that when the integer $D > 1$ is not a perfect square and $D = 2^a c$ where $a > 1$ and c is odd, then the central norm (defined in the next section) being 2 for $\sqrt{D}$ is directly related to the central norm of $\sqrt{D/2^{a-1}}$ being 2. Then we settled the case for $D = 2c$ for all except the case where c is divisible only by primes congruent to 1 modulo 8. In this note, we solve that case as well and give a new general criterion for all cases, thereby completing the project, which was motivated by correspondence with Irving Kaplansky as outlined in [19].

2. Notation and Preliminaries We write the simple continued fraction expansions of $\sqrt{D}$, $D \in \mathbb{N}$ (the natural numbers), D not a perfect square, by:

$$\sqrt{D} = \langle q_0; \overline{q_1, q_2, \ldots, q_{\ell-1}, 2q_0} \rangle,$$

where $\ell = \ell(\sqrt{D})$ is the period length of $\sqrt{D}$.

The jth *convergent* of α for $j \geq 0$ is given by

$$\text{(1)} \qquad \frac{A_j}{B_j} = \langle q_0; q_1, q_2, \ldots, q_j \rangle = \frac{q_j A_{j-1} + A_{j-2}}{q_j B_{j-1} + B_{j-2}}.$$

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If we set $P_0 = 0$, $Q_0 = 1$, then for $j \geq 1$,

$$P_{j+1} = q_j Q_j - P_j, \tag{2}$$

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \tag{3}$$

$$D = P_{j+1}^2 + Q_j Q_{j+1}. \tag{4}$$

We will also need the following facts (which can be found in most introductory texts in number theory, such as [14], or see [13] for a more advanced exposition).

$$A_j B_{j-1} - A_{j-1} B_j = (-1)^{j-1}, \tag{5}$$

$$A_{j-1} = P_j B_{j-1} + Q_j B_{j-2}, \tag{6}$$

$$A_{j-1}^2 - B_{j-1}^2 D = (-1)^j Q_j. \tag{7}$$

When ℓ is even, $P_{\ell/2} = P_{\ell/2+1}$, so by Equation (2),

$$Q_{\ell/2} \mid 2P_{\ell/2}, \tag{8}$$

where $Q_{\ell/2}$ is called the *central norm*, (via Equation (7)), and

$$q_{\ell/2} = 2P_{\ell/2}/Q_{\ell/2}. \tag{9}$$

We will need the following in the next section. Note that this result corrects the oversights in [15, Theorem 1.3, p. 334], [16, Theorem 1.3, p. 101], and [17, Theorem 1.2, p. 221]. (Fortunately, the correct version below is the one actually used in those papers, rather than the incorrectly stated ones. The problem only arises when the norm is not squarefree.)

LEMMA 1. *If $D > 1$ is not a perfect square, then $Q_j \mid 2D$ and $Q_j \mid 2A_{j-1}$ for some $j < \ell$ (where the Q_j, A_j, and ℓ are as defined in the previous section for $\sqrt{D}$) if and only $j = \ell/2$.*

PROOF. If $Q_j \mid 2D$ and $Q_j \mid A_{j-1}$, then by Equation (6) $Q_j \mid 2P_j$ (since $\gcd(A_{j-1}, B_{j-1}) = 1$ by Equation (5)). Now the proof follows exactly as in [18, Theorem 2.3, p. 64] (where Q_j was assumed therein to be squarefree in order to achieve the latter divisibility condition).

The converse is proved exactly as in [18, Theorem 2.3, p. 63] since no squarefreeness was needed or assumed therein. ■

For work related to the work herein, which helped to inspire this author's work along with the aforementioned correspondence with Kaplansky see [1]–[12], and [21]–[23].

3. The Criterion The following is a general criterion for $Q_{\ell/2} = 2$ (where ℓ is defined in he previous section as are the symbols used below) when $D \equiv 2 \pmod 4$ which completes the general case for reasons cited in the introduction.

THEOREM 1. *Let $D = 2c$ where $c > 1$ is odd (possibly a perfect square) . If ℓ is even, then the following are equivalent.*

1. $Q_{\ell/2} = 2$.
2. $q_{\ell/2} = P_{\ell/2}$.
3. *There exists a solution to the Diophantine equation $x^2 - Dy^2 = \pm 2$.*
4. *There does not exist a factorization $c = ab$ with $2 < a < b$ for which there is a solution to the Diophantine equation $ax^2 - by^2 = \pm 1$.*
5. *There does not exist a divisor $a > 2$ of D such that $a \mid A_{\ell/2-1}$.*

PROOF. The equivalence of 1 and 2 is a consequence of Equation (9). The equivalence of 1 and 3 was proved in [20], as was the equivalence of 1 and 4. It remains to show the equivalence of 1 and 5.

Suppose that 5 holds and $Q_{\ell/2} = a$. Then by Lemma 1, $a \mid 2D$ and $a \mid 2A_{\ell/2-1}$. If a is odd, then 5 forces $a = 1$, a contradiction. If a is divisible by 4, then by Equation (8), $2 \mid P_{\ell/2}$, so by Equations (5) and (7), $4 \mid D$, a contradiction. Hence, $a \equiv 2 \pmod 4$. If $a > 2$, then $(a/2) \mid D$ and $(a/2) \mid A_{\ell/2-1}$, so by 5 $a/2 = 1$. Conversely, if 1 holds, then by Lemma 1, 5 must hold. ∎

The new condition in terms of criteria for $Q_{\ell/2} = 2$ is 5 in Theorem 1. This completes the work done in [19]. The following completes the proof of a conjecture of Kaplansky begun in that paper.

COROLLARY 1. *If $D = 2pq$ where p and q are distinct odd primes, then*

1. *If $p \equiv q \equiv 7 \pmod 8$, then ℓ is even. Also, $Q_{\ell/2} = 2$ if and only if $\ell/2$ is even and $A_{\ell/2-1} \equiv 2 \pmod 4$.*
2. *If $p \equiv q \equiv 3 \pmod 8$, then ℓ is even and $Q_{\ell/2} = 2$.*
3. *If $p \equiv 1 \pmod 8$ and $q \equiv 3 \pmod 8$, then ℓ is even. Also, $Q_{\ell/2} = 2$ if and only if $\ell/2$ is odd and $A_{\ell/2-1} \equiv 2 \pmod 4$.*
4. *If $p \equiv 1 \pmod 8$ and $q \equiv 7 \pmod 8$, with $p > 2q$, then ℓ is even. Also, $Q_{\ell/2} = 2$ if and only if $\ell/2$ is even and $A_{\ell/2-1} \equiv 0 \pmod 4$.*
5. *If ℓ is even and $p \equiv q \equiv 1 \pmod 8$, then $Q_{\ell/2} = 2$ if and only if $\gcd(A_{\ell/2-1}, pq) = 1$.*

PROOF. Parts 1–4 were proved in [19]. Part 5 follows from Theorem 1. ∎

We conclude with an example that illustrates case 5.

EXAMPLE 1. *Let $D = 2 \cdot 17 \cdot 41$. Then $\ell = 6$ and $Q_{\ell/2} = 2$. Here $A_{\ell/2-1} = 112$, which is relatively prime to $c = 17 \cdot 41$.*

If $D = 2 \cdot 41 \cdot 113$, then $\ell = 8$, $Q_{\ell/2} = 82$ and

$$\gcd(A_{\ell/2-1}, c) = \gcd(2214, 4633) = 41.$$

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PERIODIC INTEGRAL TRANSFORMS AND C^*-ALGEBRAS

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Presented by G. A. Elliott, FRSC

ABSTRACT. We construct canonical integral transforms, analogous to the Fourier transform, that have periods six and three. The existence of this transform is shown to arise naturally from the expectation that the Schwartz space on the real line, viewed as the Heisenberg module of Rieffel and Connes over the rotation C^*-algebra, should extend to a module action over the crossed product of the latter by the canonical automorphisms of orders three and six (which does in fact happen and is shown here).

RÉSUMÉ. On construit deux transformations intégrales, analogues à la transformation de Fourier, d'ordre six et trois, respectivement. Ces transformations sont reliées à la théorie des automorphismes canoniques d'ordre six et trois de l'algèbre associées à une rotation irrationelle du cercle.

1. Introduction. It is a well known classical fact that the Fourier transform of a Schwartz function f

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e(-tx)\, dx, \tag{1}$$

has period four and extends to a unitary operator on $L^2(\mathbb{R})$. (Throughout the paper we write $e(t) := e^{2\pi i t}$.) This stems from the fact that $\hat{\hat{f}}(t) = f(-t)$. In this paper we show that if the product tx in (1) is replaced by a suitable quadratic, then one obtains transforms of period three and six. More specifically, one has a one-parameter family of *hexic* transforms

$$(Hf)(t) = i^{1/6}\sqrt{2\mu} \int_{-\infty}^{\infty} f(x) e(2\mu t x - \mu x^2)\, dx, \tag{2}$$

for $\mu > 0$ and f in the Schwartz space $\mathcal{S}(\mathbb{R})$. Thus, H extends to a unitary operator on $L^2(\mathbb{R})$ of period six (*i.e.*, $H^6 = I$). (The "ideal" transform is when $\mu = \frac{1}{2}$ as is explained in Remark 2 below.) Note that H is a composition of the multiplication operator by the complex Gaussian $e(-\mu x^2)$ and an inverse Fourier transform (up to scaling), and hence is itself a unitary operator on $L^2(\mathbb{R})$ that leaves invariant $\mathcal{S}(\mathbb{R})$.

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THEOREM 1. *One has $(H^3f)(t) = f(-t)$ for all $f \in \mathcal{S}(\mathbb{R})$, so that the transform H has period six and extends to a unitary operator on $L^2(\mathbb{R})$. Further, its square H^2 (the cubic transform) is given by*

$$(H^2f)(t) = \frac{\sqrt{2\mu}}{i^{1/6}} e(\mu t^2) \int_{-\infty}^{\infty} f(x)e(2\mu tx)\,dx. \tag{3}$$

Therefore, $H^{-1} = H^5 = H^3H^2$ and by (3) one gets the formula for the inverse hexic transform

$$(H^{-1}f)(t) = \frac{\sqrt{2\mu}}{i^{1/6}} e(\mu t^2) \int_{-\infty}^{\infty} f(x)e(-2\mu tx)\,dx.$$

One similarly gets a formula for the inverse cubic transform $(H^{-2}f) = (H^3Hf)(t) = (Hf)(-t)$.

REMARK 1. It is interesting to note that although the Fourier transform has period four, if it is composed with multiplication by a complex Gaussian, as in (3), it can be made to have period three. Though this may seem a little surprising, it can be shown that from the C^*-algebra point of view it is not (see Section 3).

REMARK 2. By analogy with the fact that $e^{-\pi x^2}$ is invariant under the Fourier transform, one can easily check that $e^{-\pi(\sqrt{3}-i)\mu x^2}$ is invariant under H (and hence also under the cubic transform). (It can be checked that this is the only function among the Gaussian exponentials, up to scalars, that is invariant under the cubic or hexic transform.) The reason we referred to $\mu = 1/2$ as the "ideal" case is that in this case one has $\frac{1}{2}(\sqrt{3} - i) = i^{-1/3}$ is of modulus 1 so that one has the invariant Gaussian $e^{-\pi i^{-1/3}x^2}$.

REMARK 3. The Fourier transform is a "canonical" transform in the sense that it intertwines the translation and phase multiplication operators in the well known way. Similarly, the hexic transform, with $\mu = \frac{1}{2}$, is also canonical. In fact, letting $(T_xf)(t) = f(t-x)$ and $(E_xf)(t) = e(-xt)f(t)$, one checks the following relations (see Section 3 below):

$$T_xH = HE_x, \quad E_xHT_x = e(-\tfrac{1}{2}x^2)T_xH.$$

Since the group $\mathrm{SL}(2,\mathbb{Z})$ is known to contain finite order elements only of orders 2, 3, 4, and 6, it follows that a periodic canonical transform can only have these orders. This therefore gives us canonical transforms for each allowed order.

REMARK 4. It may be worthwhile investigating properties that the above transforms have that are analogous to those that are well known to hold of the

Fourier transform (as for example in Rudin [8]). For example, is there a multiplication # on the space of Schwartz functions (or L^1 functions) such that $H(f\#g) = H(f)H(g)$? (For the Fourier transform this multiplication is convolution.) It may also be of interest to explore the extension of the transforms H and H^2 to $\mathbb{R}^n$, or even to locally compact Abelian groups.

An application of Theorem 1 is the existence of finitely generated projective modules over crossed products $6_\theta := A_\theta \rtimes_\rho \mathbb{Z}_6$ and $3_\theta := A_\theta \rtimes_{\rho^2} \mathbb{Z}_3$, where A_θ is the rotation C^*-algebra and ρ is the canonical order six automorphism on A_θ (see Section 3). These modules will give rise to primary classes in the corresponding K_0-groups $K_0(6_\theta)$ and $K_0(3_\theta)$. We write 6_θ^∞ and 3_θ^∞ for the respective canonical smooth dense $*$-subalgebras. It is well known [5] that there are natural isomorphisms $K_*(6_\theta) = K_*(6_\theta^\infty)$ and $K_*(3_\theta) = K_*(3_\theta^\infty)$. (See Section 3.)

THEOREM 2. *Under the action* (2), *the Schwartz space* $\mathcal{S}(\mathbb{R})$ *is a finitely generated projective right module* $\mathcal{M}_6$ *over* 6_θ^∞ *(thus giving rise to a class in* $K_0(6_\theta^\infty)$*). Similarly, under the action of the order three unitary given by* (3), *the Schwartz space* $\mathcal{S}(\mathbb{R})$ *is a finitely generated projective right module* $\mathcal{M}_3$ *over* 3_θ^∞ *(thus giving rise to a class in* $K_0(3_\theta^\infty)$*). Further, one has*

$$\tau_*[\mathcal{M}_6] = \frac{\theta}{6}, \quad \tau_*[\mathcal{M}_3] = \frac{\theta}{3},$$

for $j = 0, 1, \ldots, 5$, *where* τ_* *is the induced map by the canonical trace* τ *on* K_0.

In [1], Buck and the author compute the Connes-Chern characters of the hexic and cubic modules $\mathcal{M}_6$, $\mathcal{M}_3$ and show that there are explicit injections $\mathbb{Z}^{10} \to K_0(6_\theta)$ and $\mathbb{Z}^8 \to K_0(3_\theta)$ for each $\theta > 0$. The author believes that, just as in the Fourier case [10], these injections will turn out to be isomorphisms (at least for a dense G_δ set of θ).

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2. Proof of Theorem 1.

We will make free use of the following identity

$$\int_{-\infty}^{\infty} e(Ax)\, e^{-\pi b x^2}\, dx = \frac{1}{\sqrt{b}} e^{-\pi A^2/b}$$

which holds for $b, A \in \mathbb{C}$, $\mathrm{Re}(b) > 0$, and $\sqrt{b}$ is the principal square root.

The theorem follows once we show that:

(A) the set of Gaussians $f_\alpha(x) := e(-\alpha x)e^{-2\pi\mu x^2}$, where $\alpha \in \mathbb{R}$, is a total set in $L^2(\mathbb{R})$,

(B) $(H^3 f_\alpha)(t) = f_\alpha(-t)$ for all t, α,

(C) equality (3) holds for each f_α.

PROOF OF (A). It is enough to show that if $g \in L^2(\mathbb{R})$ is such that

$$\int_{-\infty}^{\infty} g(x)e(-\alpha x)e^{-2\pi\mu x^2}\, dx = 0$$

for each α, then $g = 0$. Setting $g(x)e^{-2\pi\mu x^2} = h(x)$ we note that h is in $L^1(\mathbb{R})$ since it is a product of two L^2 functions. Hence one has

$$0 = \int_{-\infty}^{\infty} h(x)e(-\alpha x)\,dx = \hat{h}(\alpha)$$

for each α. Therefore, $\hat{h} = 0$ and hence $h = 0$, *i.e.*, $g = 0$. This proves (A) and shows that the set of linear combinations of functions of the form f_α is a dense subspace of $L^2(\mathbb{R})$.

PROOF OF (B). One has

$$\begin{aligned}(Hf_\alpha)(t) &= i^{1/6}\sqrt{2\mu}\int_{-\infty}^{\infty} e(-\alpha x)e^{-2\pi\mu x^2}e(2\mu tx - \mu x^2)\,dx \\ &= i^{1/6}\sqrt{2\mu}\int_{-\infty}^{\infty} e\big((2\mu t - \alpha)x\big)\,e^{-2\pi\mu(1+i)x^2}\,dx \\ &= \frac{i^{1/6}}{\sqrt{1+i}}e^{-\pi(2\mu t-\alpha)^2/(2\mu(1+i))}.\end{aligned}$$

Applying H again gives

$$\begin{aligned}(4)\qquad (H^2f_\alpha)(t) &= \frac{i^{1/3}\sqrt{2\mu}}{\sqrt{1+i}}\int_{-\infty}^{\infty} e^{-\pi(2\mu x-\alpha)^2/(2\mu(1+i))}e(2\mu tx - \mu x^2)\,dx \\ &= \frac{i^{1/3}\sqrt{2\mu}}{\sqrt{1+i}}e(C)\int_{-\infty}^{\infty} e\big((2\mu t + D)x\big)e^{-\pi\beta x^2}\,dx \\ &= \frac{i^{1/3}\sqrt{2\mu}}{\sqrt{(1+i)\beta}}e(C)e^{-\pi(2\mu t+D)^2/\beta} = \frac{i^{1/3}}{\sqrt{i}}e(C)e^{-\pi(2\mu t+D)^2/\beta}\end{aligned}$$

where

$$\beta = \mu(1+i), \quad C = \frac{1}{8\mu}(i+1)\alpha^2, \quad D = \frac{-(i+1)}{2}\alpha.$$

A third iteration gives

$$\begin{aligned}(H^3f_\alpha)(t) &= \sqrt{2\mu}e(C)\int_{-\infty}^{\infty} e^{-\pi(2\mu x+D)^2/\beta}e(2\mu tx - \mu x^2)\,dx \\ &= \sqrt{2\mu}e^{-\frac{\pi\alpha^2}{2\mu}}\int_{-\infty}^{\infty} e\big((2\mu t - i\alpha)x\big)e^{-2\pi\mu x^2}\,dx \\ &= e^{-\frac{\pi\alpha^2}{2\mu}}\,e^{-\pi(2\mu t-i\alpha)^2/(2\mu)} = e(\alpha t)e^{-2\pi\mu t^2}.\end{aligned}$$

Therefore, $(H^3f_\alpha)(t) = f_\alpha(-t)$ is the usual flip map. Since this holds for all α, and $\{f_\alpha\}$ is a total set of functions in $L^2(\mathbb{R})$, this relation holds for all L^2 functions on $\mathbb{R}$. Hence H^6 is the identity.

PROOF OF (C). The right hand side of (3) evaluated at f_α is

$$\frac{\sqrt{2\mu}}{i^{1/6}} e(\mu t^2) \int_{-\infty}^{\infty} e(-\alpha x) e^{-2\pi\mu x^2} e(2\mu t x)\, dx = \frac{1}{i^{1/6}} e(\mu t^2) e^{-\pi(2\mu t-\alpha)^2/2\mu}$$

and it is easy to check that this is exactly (4), namely $(H^2 f_\alpha)(t)$. This completes the proof of Theorem 1.

3. Application to C^*-algebras. The following shows how by means of C^*-algebras one can discover the above transforms.

Let $\theta > 0$, $\lambda = e(\theta)$, and consider the rotation C^*-algebra A_θ generated by unitaries U, V satisfying $VU = \lambda UV$. The (noncommutative) *hexic* transform of A_θ is the canonical order six automorphism ρ defined by

$$\rho(U) = V, \quad \rho(V) = \lambda^{-1/2} U^{-1} V.$$

Its square $\kappa := \rho^2$ is the canonical order three automorphism, which we call the *cubic* transform, and ρ^3 is the usual flip automorphism studied in great detail in [2], [3], and [4]. The corresponding crossed product $6_\theta := A_\theta \rtimes_\rho \mathbb{Z}_6$ is the universal C^*-algebra generated by unitaries U, V, W enjoying the commutation relations

(5) $\quad VU = \lambda UV, \quad WUW^{-1} = V, \quad WVW^{-1} = \lambda^{-1/2} U^{-1} V, \quad W^6 = I.$

One may view the crossed product $3_\theta = A_\theta \rtimes_\kappa \mathbb{Z}_3$ as the C^*-subalgebra of 6_θ generated by U, V, and W^2. We write 6_θ^∞ and 3_θ^∞ for their respective canonical smooth dense $*$-subalgebras. (For example, the elements of 6_θ^∞ consist of sums of terms of the form aW^j where $a \in A_\theta^\infty$.) Using Rieffel's Theorem 2.15 [7] (with an appropriate lattice group in $\mathbb{R} \times \hat{\mathbb{R}}$) one obtains a smooth Heisenberg module structure on the Schwartz space $\mathcal{S}(\mathbb{R})$, with A_θ^∞ acting on the right, given by

$$(fU)(t) = f(t-\alpha), \quad (fV)(t) = e(-\alpha t) f(t),$$

where $\alpha = \sqrt{\theta}$. To extend this action so as to obtain a right 6_θ^∞-module action on $\mathcal{S}(\mathbb{R})$, we need W to act as an integral transform

$$(fW)(t) = \int_{-\infty}^{\infty} f(x) K(x,t)\, dx,$$

for suitable kernel function K, so that the relations (5) are satisfied. Rewrite the commutation relations in (5) involving W in the form

(6) $\quad WU = VW, \quad VW = \lambda^{1/2} UWV, \quad W^6 = I.$

For the second of these relations one has

$$(fUWV)(t) = e(-\alpha t)(fUW)(t) = e(-\alpha t) \int_{-\infty}^{\infty} f(x) K(x+\alpha, t)\, dx$$

and

$$(fVW)(t) = \int_{-\infty}^{\infty} e(-\alpha x) f(x) K(x,t)\, dx.$$

Hence, doing the same thing for the first relation in (6), one gets the relations

$$K(x, t-\alpha) = e(-\alpha x)\, K(x,t), \quad \lambda^{1/2} K(x+\alpha, t) = e(\alpha t - \alpha x)\, K(x,t).$$

Now it is easy to check that the kernel function $K(x,t) = i^{1/6} e(tx - \frac{1}{2}x^2)$ (of the transform H above with $\mu = \frac{1}{2}$) satisfies these relations. Therefore, by Theorem 1 we can define the right action of W on $\mathcal{S}(\mathbb{R})$ by:

$$(fW)(t) = i^{1/6} \int_{-\infty}^{\infty} f(x) e\Big(tx - \frac{1}{2}x^2\Big)\, dx,$$

so that the three relations in (6) hold. This, together with the above actions of U, V gives rise to a right 6_θ^∞ module structure on $\mathcal{S}(\mathbb{R})$. We shall denote this module by $\mathcal{M}_6$ and call it the *hexic* module. Also by Theorem 1, one has the order three action

$$(fW^2)(t) = i^{-1/6} e\Big(\frac{1}{2}t^2\Big) \int_{-\infty}^{\infty} f(x) e(-tx)\, dx = i^{-1/6} e\Big(\frac{1}{2}t^2\Big) \hat{f}(t),$$

which makes $\mathcal{S}(\mathbb{R})$ into a right 3_θ^∞-module—we call it the *cubic* module and denote it by $\mathcal{M}_3$. In view of Rieffel's inner product formulas [7], the A_θ^∞-valued inner on the Heisenberg module $\mathcal{S}(\mathbb{R})$ can be defined by

$$\langle f, g\rangle_{A_\theta^\infty} = \sum_{m,n} \langle f, g\rangle_{A_\theta^\infty}(m,n) \cdot V^n U^m,$$

where $f, g \in \mathcal{S}(\mathbb{R})$ and

$$\langle f, g\rangle_{A_\theta^\infty}(m,n) = \int_{-\infty}^{\infty} \overline{f(t+\alpha m)} g(t) e(-\alpha n t)\, dt.$$

The associated 6_θ^∞ and 3_θ^∞-valued inner product $\langle\ ,\ \rangle_{6_\theta^\infty}$, $\langle\ ,\ \rangle_{3_\theta^\infty}$ are defined by symmetrization

$$\langle f, g\rangle_{6_\theta^\infty} = \sum_{j=0}^{5} \langle f, gW^{-j}\rangle_{A_\theta^\infty} W^j, \quad \langle f, g\rangle_{3_\theta^\infty} = \sum_{j=0}^{2} \langle f, gW^{-2j}\rangle_{A_\theta^\infty} W^{2j}.$$

With these inner products, and exactly as was done in the proof for the Fourier module in [9], one sees that the hexic and cubic modules are finitely generated projective, giving classes in the corresponding K_0-group, and therefore one obtains Theorem 2.

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ON THE INFINITE-DIMENSIONAL HOLOMORPHIC STRUCTURE ON TOPOLOGICAL BUNDLES

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ABSTRACT. Let (X, d) be a complete metrizable space, $f: X \to \mathbf{C}$ a continuous map and E a topological vector bundle on X. Here we prove the existence of a complex Banach space V and a continuous closed injective map $j: X \to V$ such that $j(X)$ is the zero-locus of a family of holomorphic functions on V, f is holomorphic and E is a holomorphic vector bundle with respect to the complex structure on X induced by j.

RÉSUMÉ. Soient (X, d) une espace metrizable complete, $f: X \to \mathbf{C}$ une application continue et E une fibré vectoriel topologique sur X. Nous prouvons ici l'existence d'un espace de Banach complexe V et d'une application injectif, continue et fermé $j: X \to V$ tel que $j(X)$ soit l'ensemble des zeros d'une ensemble des fonctions analytiques et E soit une fibré analytique pour la structure complexe de X donnée par j.

1. Introduction In [2] A. Douady introduced Banach analytic spaces and Banach analytic sets (i.e., zero-sets of holomorphic functions on a complex Banach manifold) to solve an important moduli problem concerning finite-dimensional complex analytic geometry. In [3] (or see [9], Prop. II.1.3) he remarked that Banach analytic sets are not a reasonable field of research (just by itself) because every compact metric space K is homeomorphic to the zero-set of a family of holomorphic function on a complex Banach space. He also conjectured that the same is true for any complete metric space. This conjecture was proved by V. Pestov in a stroger form (i.e., in a "metric form") in [8]. Here we will use the methods of [7] and [8] to prove the following results concerning the existence of analytic structures on topological vector bundle (see Theorem 1) and the realization of any continuous complex valued function on a metric space X as a holomorphic function on a certain analytic structure on X (see Lemma 2). In section 2 we will prove the following result.

THEOREM 1. *Let X be a complete metrizable space and E_i, $1 \le i \le n$, finitely many topological vector bundles on X. Then then there exist a complex Banach space V and a continuos injective map $u: X \to V$ such that $j(X)$ is the zero-locus of a family of holomorphic functions on V and the topological vector bundles $u^{-1*}(E_i)$, $1 \le i \le n$, on $u(X)$ have a holomorphic structure.*

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To prove Theorem 1 we will use the following result.

LEMMA 2. *Let (X, d) be a complete metric space and* $\mathrm{Lip}(X)$ *the algebra of all complex valued Lipschitz functions on X. Fix a finite set S of continuous complex valued functions on X and call A the* algebra of all complex valued functions on X generated by $\mathrm{Lip}(X) \cup S$. *Then then there exist a complex Banach space V and a continuous injective map $u\colon X \to V$ such that $u(X)$ is the zero-locus of a family of holomorphic functions on V and for every $a \in A$ the continuous function $u^{-1*}(a)\colon u(X) \to \mathbf{C}$ is holomorphic.*

2. The proofs *Proof of Lemma* 2. The notion of a free Banach space was introduced in [1] and througly investigated in [4]. For all $x, y \in X$, set $d'(x,y) := d(x,y) + \sum_{f \in S} |f(x) - f(y)|$. It is easy to check that d' is a metric on X equivalent to the metric d. Since $d' \geq d$, X is complete for the metric d'. Since $d' \geq d$, every Lipschitz function on (X, d) is a Lipschitz function for (X, d'). Every function f_i is a Lipschitz function with Lipschitz constant one with respect to the metric d'. Fix any $Q \in X$ and let $B_{X,d',Q}$ be the free Banach space of (X, d') with respect to Q and $j\colon X \to B_{X,d',Q}$ the embedding constructed in [8]. Let $g\colon X \to \mathbf{C}$ be any Lipschitz function on (X, d') and call $L > 0$ any Lipschitz constant for g. Hence the function g/L is a nonexpanding map on (X, d') in the sense of [8]. By the defining property of the free Banach space $B_{X,d',Q}$ ("property 2" before the Lemma at page 70 of [8]), there is a continuous linear map $h\colon B_{X,d',Q} \to \mathbf{C}$ such that $g/L = h - g(Q)/L$. There is an embedding $j\colon X \to B_{X,d',Q}$ with $j(Q) = 0$. Since (X, d') is complete, $j(X)$ is closed in $B_{X,d',Q}$ ([7], Theorem 2). First assume that X has diameter at most two with respect to d'. In this case $j(X)$ is defined in $B_{X,d',Q}$ by a quadratic map from $B_{X,d',Q}$ to another Banach space ([7], Theorem 3) and hence it is a closed analytic subset of $B_{X,d',Q}$. We just saw that every Lipschitz function on (X, d') is induced by a holomorphic function on $j(X)$ with respect to this complex structure. The general case in which (X, d') may have infinite diameter is reduced to the previous case using a contraction map on the Banach space $B_{X,d',Q}$ ([7], Lemma 2 and proof of the Main Theorem at p. 1047).

Proof of Theorem 1. Let C_X be the sheaf of all complex valued continuous functions on X. Set $r_i := \mathrm{rank}(E_i)$. By a theorem of Nagata and Smirnov any metric space is paracompact. Hence every topological vector bundle on X is isomorphic to the pull-back of the tautological quotient bundle on a suitable complex Grassmannian G by a continuous map $X \to G$ ([5], Corollary 5.3 and Theorem 5.5). Thus there are integers $a_i \geq 0$, $b_i \geq r_i$, $1 \leq i \leq n$, and an $a_i \times b_i$-matrix of continuous complex valued functions $M_i = (f_{i;j,h})$, $1 \leq i \leq n$, $1 \leq j \leq a_i$, $1 \leq h \leq b_i$, such that the associated map $f_i\colon C_X^{\oplus m_i} \to C_X^{\oplus n_i}$ has constant rank $n_i - r_i$ and $\mathrm{Coker}(f_i) \cong E_i$ as topological vector bundles. The proof of Lemma 2 shows how to define an equivalent metric d' on X such that all continuous functions $f_{i;j,h}$, $1 \leq i \leq n$, $1 \leq j \leq n_i$, $1 \leq h \leq n_i$, are Lipschitz with respect to the metric d'. By Lemma 2 there is an embedding u of X in a Banach space B with $u(X)$ closed analytic subset of B and such that all continuous functions $f_{i;j,h}$ are holomorphic with respect to the holomorphic structure on X induced by u. The vector bundles Coker(f_i), $1 \leq i \leq n$, are holomorphic with respect to the holomorphic structure on X induced by u.

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