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SUR LES REPRÉSENTATIONS TEMPÉRÉES D'UN GROUPE RÉDUCTIF P-ADIQUE

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ABSTRACT. We study the tempered representations of $G(F)$, where G is a reductive group over local field F. There are two kinds of irreducible tempered representations which could be regarded as basic building blocks. On one hand are the elliptic representations, whose character does not vanish on the regular elliptic set of $G(F)$, while on the other hand are the essential representations, which cannot be properly induced from parabolic subgroups. We establish that if representation π is not elliptic, then π is a linear combination of representations which are properly induced from elliptic representations. These results are well known if $F = \mathbb{R}$ (see [K-Z]), but not in the case that F is p-adic.

1. **Paramétrage de $\Pi(G)$.** Soit $\mathbf{G}$ un groupe algébrique connexe réductif défini sur un corps local non archimédien F de charactéristique 0. Notons G l'ensemble des points F-rationnels de $\mathbf{G}$. Dans la suite on fixe une composante de Lévi F-rationnelle M_0 d'un certain sous-groupe parabolique minimal P_0 de $\mathbf{G}$ défini sur F. Un groupe de Lévi M de G est dit propre si $M \neq G$. On dit que (M, σ) est une *paire discrète* de G si M est un groupe de Lévi de G contenant M_0 et σ une classe d'isomorphisme de la série discrète de M. Jusqu'à la fin de ce chapitre fixons (M, σ) une telle paire. La composante déployée de M (resp. G) est notée A_M (resp. A_G et $\mathfrak{a}_M$ (resp. $\mathfrak{a}_G$) "l'algèbre de Lie" de A_M (resp. A_G). Notons $W(M) := W^G(M)$ le groupe de Weyl de (G, M) et $i_{G,M}(\sigma)$ la classe de la représentation induite $\mathcal{I}\mathrm{nd}_{P=MN}(\sigma)$ et $\mathfrak{R}_\sigma := \mathfrak{R}_\sigma^G$, le $\mathfrak{R}_\sigma$-groupe correspondant. Pour $r \in \mathfrak{R}_\sigma$, on définit:

$$\mathfrak{a}_M^r := \{H \in \mathfrak{a}_M; rH = H\} \quad \text{et} \quad \mathfrak{R}_{\sigma,\mathrm{reg}} := \mathfrak{R}_{\sigma,\mathrm{reg}}^G = \{r \in \mathfrak{R}_\sigma; \mathfrak{a}_M^r = \mathfrak{a}_G\}$$

Soit $\tilde{\mathfrak{R}}_\sigma := \tilde{\mathfrak{R}}_\sigma^G$ l'extention centrale du $\mathfrak{R}_\sigma$-groupe comme dans [A, section 2]:

$$1 \to Z_\sigma \to \tilde{\mathfrak{R}}_\sigma \to \mathfrak{R}_\sigma \to 1.$$

Il existe un caractère central χ_σ de Z_σ tel que l'ensemble $\Pi_\sigma(G)$ des constituants irréductibles de $i_{G,M}(\sigma)$ soit paramétré par l'ensemble $\Pi(\tilde{\mathfrak{R}}_\sigma, \chi_\sigma)$ des classes d'équivalence des représentations irréductibles de $\tilde{\mathfrak{R}}_\sigma$ ayant χ_σ comme Z_σ-caractère central [A, section 2]. La paramétrisation de $\Pi_\sigma(G)$ nous permet de classifier l'ensemble $\Pi(G)$ des classes d'isomorphismes des représentations irréductibles tempérées de G qui est réunion disjointe des $W^G(M_0)$-paires

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discrètes (M,σ) des ensembles $\Pi_\sigma(G)$, [H-C.1], [A, section 1]. On note Θ_π le caractère de $\pi \in \Pi(G)$ et $i_{G,M}(\Theta_\sigma)$ le caractère de $i_{G,M}(\sigma)$. Aussi on note $\mathcal{L}(\Re_\sigma)$ l'ensemble des sous-groupes de Lévi S de G contenant M tels que $\mathfrak{a}_M^r = \mathfrak{a}_S$ pour un certain $r \in \Re_\sigma$. Le groupe $\Re_\sigma$ est alors réunion disjointe des ensembles $\Re_{\sigma,\mathrm{reg}}^S$, $S \in \mathcal{L}(\Re_\sigma)$. J. Arthur [A, section 2] fait correspondre une distribution-caractère $\Theta^G(M,\sigma,r) := \Theta(M,\sigma,r)$ appelée *caractère virtuel* de G qui se décompose sous la forme:

$$\Theta(M,\sigma,r) = \sum_{\pi\in\Pi_\sigma(G)} \overline{\theta}_{\rho_\pi}(r)\Theta_\pi \tag{1}$$

oú θ_{ρ_π} est le caractère de $\rho_\pi \in \Pi(\tilde{\Re}_\sigma, \chi_\sigma)$ associé à $\pi \in \Pi_\sigma(G)$.

En inversant (1) on aura, pour tout $\pi \in \Pi_\sigma(G)$:

$$\Theta_\pi = |\tilde{\Re}_\sigma|^{-1} \sum_{r\in\tilde{\Re}_\sigma} \theta_{\rho_\pi}(r)\Theta^G(M,\sigma,r). \tag{2}$$

2. **Sur le groupe de Grothendieck de G.** Un élément $x \in G$ est dit *régulier* si $D_G(x) \neq 0$ où D_G est le facteur discriminant standard défini dans [H-C.2, section 15]. Un élément $x \in G$ est dit *elliptique* si son centralisateur est compact modulo la composante déployée A_G de G. Une représentation dans $\Pi(G)$ est dite elliptique si son caractère est non nul sur l'ensemble régulier elliptique de G.

THÉORÈME 1. *Toute représentation irréductible tempérée de G est combinaison linéaire d'induites d'elliptiques.*

PROOF. Soit $\pi \in \Pi(G)$. On sait que modulo conjugaison par $W^G(M_0)$, il existe une unique paire discrète (M,σ) de G tel que $\pi \in \Pi_\sigma(G)$. D'après l'expression (2), le caractère de $\pi = \pi_\rho \in \Pi_\sigma(G)$, $\rho_\pi \in \Pi(\tilde{\Re}_\sigma, \chi_\sigma)$ est égal à, [A, section 6]:

$$\begin{aligned}\Theta_\pi &= |\Re_\sigma|^{-1} \sum_{r\in\Re_\sigma} \theta_{\rho_\pi}(r)\Theta^G(M,\sigma,r) \\ &= |\Re_\sigma|^{-1} \sum_{S\in\mathcal{L}(\Re_\sigma)} \sum_{r\in\Re_{\sigma,\mathrm{reg}}^S} \theta_{\rho_\pi}(r)\Theta^G(M,\sigma,r).\end{aligned} \tag{3}$$

Pour $r \in \Re_{\sigma,\mathrm{reg}}^S$, la décomposition du caractère virtuel $\Theta^G(M,\sigma,r)$ est:

$$\Theta^G(M,\sigma,r) = \sum_{\delta\in\Pi_\sigma(S)} \overline{\theta_{\rho_\delta}}(r) i_{G,S}(\Theta_\delta)$$

(voir [A, section 5]). Ainsi (3) est équivalente à:

$$\Theta_\pi = \sum_{S\in\mathcal{L}(\Re_\sigma)} \sum_{\delta\in\Pi_\sigma(S)} \Big[|\Re_{\sigma,\mathrm{reg}}^S|^{-1} \sum_{r\in\Re_{\sigma,\mathrm{reg}}^S} \theta_{\rho_\pi}(r).\overline{\theta_{\rho_\delta}}(r)\Big] i_{G,S}(\Theta_\delta). \tag{4}$$

Si π est non elliptique alors la somme de droite de (4) est réduite aux Lévi propre de $S \in \mathcal{L}(\Re_\sigma)$ car θ_{ρ_π} est nul sur $\Re^G_{\sigma,\mathrm{reg}}$, [A, proposition 2.1]. De plus la somme des $\delta \in \Pi_\sigma(S)$ est réduite à son sous-ensemble formé d'éléments elliptiques car sinon θ_{ρ_δ} est nul sur $\Re^S_{\sigma,reg}$. ∎

REMARQUES. (1) Ce théorème est une généralisation de R. Herb [H] obtenu dans le cas où $G = \mathrm{Sp}(2n)$ et $G = \mathrm{SO}(2n)$.

(2) Kazhdan a montré seulement que si une représentation irréductible tempérée de G est non elliptique alors elle est une combinaison linéaire d'induites de tempérées [K].

RÉFÉRENCES

[A] J. Arthur, *On Elliptic Tempered Characters.* Acta Math. **171**(1993), 73–138.

[H] R. Herb, *Elliptic Representations for* Sp(2n) *and* SO(2n). Pacific J. Math. **161** (1993), 347–358.

[H-C.1] Harish-Chandra, *The Plancherel Formula for Reductive p-adic Groups.* In: Collected Papers, Vol IV, Springer Verlag, 353–367.

[H-C.2] Harish-Chandra, *Harmonic Analysis on Reductive p-adic Groups.* Proc. Sympos. Pure Math. **26**(1973), 167–192.

[K] D. Kazhdan, *Cuspidal Geometry of p-adic Groups.* J. Analyse Math **47**(1986), 1–36.

[K.Z] A. W. Knapp and G. Zuckerman, *Classification of irreducible tempered representations of semisimple groups.* Ann. of Math. **116**(1982), 389–501.

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SHAPE EQUIVALENCE OF AH INDUCTIVE LIMIT SYSTEMS: CUTTING DOWN BY PROJECTIONS

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Résumé. Quelques résultats concernant le concept d'équivalence de forme et sa compatibilité avec l'équivalence de Morita, qui entrent dans la classification des C*-algèbres simples limites inductives de type AH à dimension bornée, sont de'montrées.

The purpose of this note is to generalize several results of [EG] (e.g., 5.6 and 5.20) concerning shape equivalence of AH inductive limit systems

$$A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}) \quad \text{and}$$
$$B = \varinjlim(B_n = \bigoplus_{i=1}^{s_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m})$$

to the case that

$$A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}) \quad \text{and}$$
$$B = \varinjlim(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i}M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}, \psi_{n,m}),$$

where: $X_{n,i}, Y_{n,i}$ are connected finite simplicial complexes of dimension at most 3; $[n,i], \{n,i\}, t_n$ and s_n are positive integers; $M_{[n,i]}(C(X_{n,i}))$ (or $M_{\{n,i\}}(C(Y_{n,i}))$, respectively) are $[n,i] \times [n,i]$ (or $\{n,i\} \times \{n,i\}$, respectively) matrix algebras over the commutative C^*-algebras $C(X_{n,i})$ (or $C(Y_{n,i})$, respectively); and $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ (or $Q_{n,i} \in M_{\{n,i\}}(C(Y_{n,i}))$) are projections. That is, we will prove that the corresponding results of 5.6 and 5.20 of [EG] still hold if the direct sums of full matrix algebras over $C(X)$ are replaced by direct sums of corner subalgebras of full matrix algebras over $C(X)$ (obtained by cutting down the algebras by projections).

1. All the spaces appearing in this note (including $X_{n,i}$ and $Y_{n,i}$ in the inductive limit systems) are **connected finite simplicial complexes of dimension at most 3.** It is well known (see Remark 3.26 of [EG]) that if $p, q \in PM_n(C(X))P$ are two projections (where $\dim(X) \leq 3$) with $[p] = [q] \in K_0(PM_n(C(X))P)$, then there is a unitary $u \in PM_n(C(X))P$ such that $p = uqu^*$. More generally, if $\{p_1, p_2, \cdots, p_k\}$ and $\{q_1, q_2, \cdots, q_k\}$ are two sets of

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mutually orthogonal projections in $PM_n(C(X))P$ with

$$[p_i] = [q_i] \in K_0(PM_n(C(X))P), 1 \le i \le k,$$

then there is a unitary $u \in PM_n(C(X))P$ such that $p_i = uq_iu^*, 1 \le i \le k$.

2. All the inductive limit systems $(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ are supposed to have simple limit C^*-algebras A which are not isomorphic to either $M_n(\mathbb{C})$ or the algebra of compact operators $\mathcal{K}$. It is well known (see [BDR]) that, passing to subsequences, one may assume that the connecting maps $\phi_{n,m}$ are 12-full, i.e., $\mathrm{rank}\phi_{n,m}^{i,j}(P_{n,i})/\mathrm{rank}(P_{n,i}) \ge 12$ for all i, j.

Definition 3. Two inductive limit systems $A = \varinjlim(A_n, \phi_{n,m})$ and $B = \varinjlim(B_n, \psi_{n,m})$ are said to be *shape equivalent* if there are sequences $\{k_i\}, \{l_i\}$ and homomorphisms $\xi_i\colon A_{k_i} \to B_{l_i}$ and $\eta_i\colon B_{l_i} \to A_{k_{i+1}}$ such that the diagram

$$\text{(I)}\qquad \begin{array}{ccccccc} A_{k_1} & \xrightarrow{\phi_{k_1,k_2}} & A_{k_2} & \xrightarrow{\phi_{k_2,k_3}} & A_{k_3} & \xrightarrow{\phi_{k_3,k_4}} & \cdots \\ \downarrow \xi_1 & \nearrow \eta_1 & \downarrow \xi_2 & \nearrow \eta_2 & \downarrow \xi_3 & \nearrow \eta_3 & \cdots \\ B_{l_1} & \xrightarrow{\psi_{l_1,l_2}} & B_{l_2} & \xrightarrow{\psi_{l_2,l_3}} & B_{l_3} & \xrightarrow{\psi_{l_3,l_4}} & \cdots \end{array}$$

commutes at the level of homotopy, i.e.,

$$\phi_{k_i,k_{i+1}} \sim_h \eta_i \circ \xi_i \quad \text{and} \quad \psi_{l_i,l_{i+1}} \sim_h \xi_{i+1} \circ \eta_i.$$

The above two inductive limit systems are said to be KK-shape equivalent if

$$KK(\phi_{k_i,k_{i+1}}) = KK(\eta_i \circ \xi_i) \in KK(A_{k_i}, A_{k_{i+1}}) \quad \text{and}$$
$$KK(\psi_{l_i,l_{i+1}}) = KK(\xi_{i+1} \circ \eta_i) \in KK(B_{l_i}, B_{l_{i+1}}).$$

Obviously, shape equivalence implies KK-shape equivalence. For the simple AH inductive limit systems with $\dim(X_{n,i})$ and $\dim(Y_{n,i})$ at most 3, we have the following converse.

Lemma 4. *Let* $A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}\ M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ *and* $B = \varinjlim(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i}M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}, \psi_{n,m})$ *be two simple AH inductive limits as in* §1 *and* §2. *If the systems* $(A_n, \phi_{n,m})$ *and* $(B_n, \psi_{n,m})$ *are KK-shape equivalent, then they are shape equivalent.*

PROOF. Let $\xi_i\colon A_{k_i} \to B_{l_i}$ and $\eta_i\colon B_{l_i} \to A_{k_{i+1}}$ be homomorphisms inducing a KK-shape equivalence. Since $KK(\phi_{k_1,k_2}) = KK(\eta_1 \circ \xi_1) \in KK(A_{k_1}, A_{k_2})$, by §1, there is a unitary $u \in A_{k_2}$ such that $\phi_{k_1,k_2}(\mathbf{1}_{A_{k_1}^i}) = u(\eta_1 \circ \xi_1(\mathbf{1}_{A_{k_1}^i}))u^*$. Again we have $KK(\phi_{k_1,k_2}) = KK(\mathrm{Ad}\, u \circ \eta_1 \circ \xi_1)$. Applying Lemma 3.24 of [EG] to each pair of partial maps,

$$\phi_{k_1,k_2}^{i,j},\ (\mathrm{Ad}\, u \circ \eta_1 \circ \xi_1)^{i,j}\colon\ A_{k_1}^i \longrightarrow \phi_{k_1,k_2}^{i,j}(\mathbf{1}_{A_{k_1}^i})A_{k_2}^j\phi_{k_1,k_2}^{i,j}(\mathbf{1}_{A_{k_1}^i}),$$

we obtain a unitary $v \in A_{k_2}$ such that ϕ_{k_1,k_2} is homotopic to $\mathrm{Ad}\, v \circ \mathrm{Ad}\, u \circ \eta_1 \circ \xi_1$. Replacing η_1 by the new map $\mathrm{Ad}\, v \circ \mathrm{Ad}\, u \circ \eta_1$, we make the first triangle in the diagram (I) commute at the level of homotopy (instead of KK). One can continue this procedure to construct the shape equivalence. ■

5. A KK-shape equivalence between

$$A = \varinjlim(A_n, \phi_{n,m}) \text{ and } B = \varinjlim(B_n, \psi_{n,m})$$

induces an isomorphism of scaled ordered K-groups

$$\alpha\colon\ (K_0(A), K_0(A)^+, D(A), K_1(A)) \longrightarrow (K_0(B), K_0(B)^+, D(B), K_1(B)),$$

where $D(A)$ is the dimension range of A which is the subset $\{[p] \in K_0(A) : p \in A \text{ is a projection}\}$, and $K_0(A)^+$ is the semigroup generated by $D(A)$ (see §1.2 of [EG]). Note that for any stably finite C^*-algebra A, $(K_0(A), K_0(A)^+) \cong (K_0(A \otimes \mathcal{K}), K_0(A \otimes \mathcal{K})^+)$ and $D(A) \subseteq D(A \otimes \mathcal{K}) = K_0(A \otimes \mathcal{K})^+$.

6. Consider inductive limit systems

$$\tilde{A} = \varinjlim \left(\tilde{A}_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \tilde{\phi}_{n,m}\right) \quad \text{and}$$

$$A = \varinjlim \left(A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i}, \phi_{n,m}\right)$$

with $\phi_{n,m} = \tilde{\phi}_{n,m}|_{A_n}$. We will call the inductive limit system $(A_n, \phi_{n,m})$ a cut-down of the system $(\tilde{A}_n, \tilde{\phi}_{n,m})$. Obviously, $A \cong \overline{\bigcup_{n=1}^{+\infty} \phi_{n,\infty}(\mathbf{1}_{A_n}) \tilde{A} \phi_{n,\infty}(\mathbf{1}_{A_n})}$, if $\mathbf{1}_{A_n}$ is regarded as an element of $\tilde{A}_n$ (see §1.3 of [G] for details). In the rest of this paper, we will always assume $P_{n,i} \neq 0$ for any n, i. In these circumstances, the inclusion map $i\colon A \to \tilde{A}$ (or $i\colon A_n \to \tilde{A}_n$, respectively) induces a stable isomorphism $A \otimes \mathcal{K} \cong \tilde{A} \otimes \mathcal{K}$ (or $A_n \otimes \mathcal{K} \cong \tilde{A}_n \otimes \mathcal{K}$, respectively) and induces the identity element $[1] \in KK(A \otimes \mathcal{K}, \tilde{A} \otimes \mathcal{K})$ (or $[1] \in KK(A_n \otimes \mathcal{K}, \tilde{A}_n \otimes \mathcal{K})$, respectively). This inclusion also induces an order isomorphism of K_0-groups $(K_0(A), K_0(A)^+) \cong (K_0(\tilde{A}), K_0(\tilde{A})^+)$ (or $(K_0(A_n), K_0(A_n)^+) \cong (K_0(\tilde{A}_n), K_0(\tilde{A}_n)^+)$). But for the scale, we only have the inclusion $D(A) \subseteq D(\tilde{A})$ (or $D(A_n) \subseteq D(\tilde{A}_n)$).

Furthermore, if

$$C = \bigoplus_{i=1}^{s} Q_i M_{l_i}(C(Y_i)) Q_i \quad \text{is a cut-down of} \quad \tilde{C} = \bigoplus_{i=1}^{s} M_{l_i}(C(Y_i))$$

with $Q_i \neq 0$, then $KK(A_n, C) \cong KK(\tilde{A}_n, \tilde{C})$. If $\phi\colon \tilde{A}_n \to \tilde{C}$ is a homomorphism, then $KK(\phi) = KK(\phi|_{A_n})$, regarded in $KK(\tilde{A}_n, \tilde{C})$ $(= KK(A_n, C))$.

Theorem 7. *Let* $A = \varinjlim \left(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}\ M_{[n,i]}(C(X_{n,i})) P_{n,i}, \phi_{n,m}\right)$ *and* $B = \varinjlim \left(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i} M_{\{n,i\}}(C(Y_{n,i})) Q_{n,i}, \psi_{n,m}\right)$ *be cut-downs of the simple AH inductive limit systems* $\tilde{A} = \varinjlim(\tilde{A}_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \tilde{\phi}_{n,m})$ *and* $\tilde{B} = \varinjlim(\tilde{B}_n = \bigoplus_{i=1}^{s_n} M_{\{n,i\}}(C(Y_{n,i})), \tilde{\psi}_{n,m})$*, respectively. If there is a shape* *equivalence between* $(\tilde{A}_n, \tilde{\phi}_{n,m})$ *and* $(\tilde{B}_n, \tilde{\psi}_{n,m})$ *which induces an isomorphism*

$$\alpha\colon (K_0(\tilde{A}), K_0(\tilde{A})^+, D(\tilde{A})) \longrightarrow (K_0(\tilde{B}), K_0(\tilde{B})^+, D(\tilde{B}))$$

satisfying $\alpha(D(A)) = D(B)$*, then there is a shape equivalence between the cut-down systems* $(A_n, \phi_{n,m})$ *and* $(B_n, \psi_{n,m})$*, which may be chosen to induce the same isomorphism as* α *(identifying* $K_0(A) = K_0(\tilde{A}), K_0(B) = K_0(\tilde{B})$*).*

PROOF. By Lemma 4, it is enough to prove that the systems $(A_n, \phi_{n,m})$ and $(B_n, \psi_{n,m})$ are KK-shape equivalent. Passing to subsequences, we may assume that there is a homotopy intertwining

$$\begin{array}{ccccccccc}
\tilde{A}_1 & \xrightarrow{\tilde{\phi}_{1,2}} & \tilde{A}_2 & \xrightarrow{\tilde{\phi}_{2,3}} & \tilde{A}_3 & \xrightarrow{\tilde{\phi}_{3,4}} & \cdots & \longrightarrow & \tilde{A} \\
\downarrow{\scriptstyle \tau_1} & {\scriptstyle \mu_1}\nearrow & \downarrow{\scriptstyle \tau_2} & {\scriptstyle \mu_2}\nearrow & \downarrow{\scriptstyle \tau_3} & {\scriptstyle \mu_3}\nearrow & \cdots & & \\
\tilde{B}_1 & \xrightarrow{\tilde{\psi}_{1,2}} & \tilde{B}_2 & \xrightarrow{\tilde{\psi}_{2,3}} & \tilde{B}_3 & \xrightarrow{\tilde{\psi}_{3,4}} & \cdots & \longrightarrow & \tilde{B}
\end{array}$$

which induces $\alpha\colon K_0(\tilde{A}) \to K_0(\tilde{B})$. We will construct a diagram

$$\begin{array}{ccccccccc}
A_{k_1} & \xrightarrow{\phi_{k_1,k_2}} & A_{k_2} & \xrightarrow{\phi_{k_2,k_3}} & A_{k_3} & \xrightarrow{\phi_{k_3,k_4}} & \cdots & \longrightarrow & A \\
\downarrow{\scriptstyle \xi_1} & {\scriptstyle \eta_1}\nearrow & \downarrow{\scriptstyle \xi_2} & {\scriptstyle \eta_2}\nearrow & \downarrow{\scriptstyle \xi_3} & {\scriptstyle \eta_3}\nearrow & \cdots & & \\
B_{l_1} & \xrightarrow{\psi_{l_1,l_2}} & B_{l_2} & \xrightarrow{\psi_{l_2,l_3}} & B_{l_3} & \xrightarrow{\psi_{l_3,l_4}} & \cdots & \longrightarrow & B
\end{array}$$

which commutes at the level of KK-theory.

Let $k_1 = 1$. Consider $[\mathbf{1}_{A_1}] \in K_0(\tilde{A}_1)$. Since $\alpha(D(A)) = D(B)$, one has

$$((\tilde{\psi}_{1,\infty})_* \circ (\tau_1)_*)([\mathbf{1}_{A_1}]) = (\alpha \circ (\phi_{1,\infty})_*)([\mathbf{1}_{A_1}]) \in \alpha(D(A)) = D(B).$$

This implies that there are an l_1 and a projection $P \in B_{l_1} \subset \tilde{B}_{l_1}$ such that $[\tilde{\psi}_{1,l_1} \circ \tau_1(\mathbf{1}_{A_1})] = [P] \in K_0(\tilde{B}_{l_1})$. Hence by §1, there is a unitary $u \in \tilde{B}_{l_1}$ such that $u\tilde{\psi}_{1,l_1}(\tau_1(\mathbf{1}_{A_1}))u^* = P \in B_{l_1}$. Consequently, $(\mathrm{Ad}u \circ \tilde{\psi}_{1,l_1} \circ \tau_1)(A_{k_1}) \subseteq B_{l_1}$. Let $\xi_1\colon A_{k_1} \to B_{l_1}$ be defined by $\xi_1 = (\mathrm{Ad}u \circ \tilde{\psi}_{1,l_1} \circ \tau_1)|_{A_{k_1}}$. Then $KK(\xi_1) = KK(\tilde{\psi}_{1,l_1} \circ \tau_1)$, where we identify $KK(A_{k_1}, B_{l_1})$ with $KK(\tilde{A}_{k_1}, \tilde{B}_{l_1})$. Similarly, for B_{l_1}, there are A_{k_2} and a unitary $v \in A_{k_2}$ such that $(\mathrm{Ad}v \circ \tilde{\phi}_{l_1+1,k_2} \circ \mu_{l_1})(B_{l_1}) \subseteq A_{k_2}$. Set $\eta_1 = (\mathrm{Ad}v \circ \tilde{\phi}_{l_1+1,k_2} \circ \mu_{l_1})|_{B_{l_1}} : B_{l_1} \to A_{k_2}$. Obviously,

$$KK(\eta_1 \circ \xi_1) = KK(\tilde{\phi}_{l_1+1,k_2} \circ \mu_{l_1} \circ \tilde{\psi}_{1,l_1} \circ \tau_1) = KK(\tilde{\phi}_{k_1,k_2}) = KK(\phi_{k_1,k_2}).$$

Hence we obtain the first commuting triangle of the desired diagram. The other parts of the diagram can be constructed in the same way. ■

Proposition 8. *Suppose that the inductive system*

$$(A_n = \textstyle\bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$$

has simple limit A *with* $A \ncong M_n(\mathbb{C})$ *or* $\mathcal{K}$*. Then there are a subsequence* (A_{k_n})

of (A_n) *and maps* $\psi_{k_n,k_{n+1}}: A_{k_n} \to A_{k_{n+1}}$ *which are homotopic to the maps* $\phi_{k_n,k_{n+1}}: A_{k_n} \to A_{k_{n+1}}$ *such that* $\varinjlim(A_{k_n}, \psi_{k_n,k_{n+1}})$ *is simple and of real rank zero.*

PROOF. By Proposition 5.33 of [EG], there exist such A_{k_n} and $\phi_{k_n,k_{n+1}}$ making $A' = \varinjlim(A_{k_n}, \psi_{k_n,k_{n+1}})$ of real rank zero. Since shape equivalence preserves the ordered K_0-group, $(K_0(A'), K_0(A')^+) \cong (K_0(A), K_0(A)^+)$ is a simple ordered group by 6.3.6 of [Bl] applied to A. Consequently, A' is also simple, by 2.2 of [GL]. ∎

As a consequence of Theorem 7, Lemma 5.6 (see also 5.1–5.5) of [EG] can be generalized to the following result.

Corollary 9. *Let* $A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}\, M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ *and* $B = \varinjlim(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i} M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}, \psi_{n,m})$ *be simple inductive limit* C^**-algebras, where* $X_{n,i}, Y_{n,i}$ *are connected finite simplicial complexes of dimensions at most 3 with finite cohomology groups* $H^2(X_{n,i}), H^3(X_{n,i}), H^2(Y_{n,i})$*, and* $H^3(Y_{n,i})$*. Suppose that*

$$\alpha\colon (K_0(A), K_0(A)^+, D(A), K_1(A)) \longrightarrow (K_0(B), K_0(B)^+, D(B), K_1(B))$$

is an isomorphism of the scaled ordered K-groups. Then there is a shape equivalence between $(A_n, \phi_{n,m})$ *and* $(B_n, \psi_{n,m})$ *which induces the isomorphism* α.

PROOF. By Lemma 1.3.3 of [G], $(A_n, \phi_{n,m})$ (or $(B_n, \psi_{n,m})$ respectively) can be regarded as a cut down system of $(\tilde{A}_n, \tilde{\phi}_{n,m})$ (or $(\tilde{B}_n, \tilde{\psi}_{n,m})$ respectively) as described in Theorem 7. Obviously, one can further assume that $\tilde{A} = \varinjlim(\tilde{A}_n, \tilde{\phi}_{n,m})$ and $\tilde{B} = \varinjlim(\tilde{B}_n, \tilde{\psi}_{n,m})$ are stable. Hence $D(\tilde{A}) = K_0(\tilde{A})^+ = K_0(\tilde{A})^+$ and $D(\tilde{B}) = K_0(\tilde{B})^+ = K_0(\tilde{B})^+$. It follows that α induces an isomorphism

$$\tilde{\alpha}\colon (K_0(\tilde{A}), K_0(\tilde{A})^+, D(\tilde{A}), K_1(\tilde{A})) \longrightarrow (K_0(\tilde{B}), K_0(\tilde{B})^+, D(\tilde{B}), K_1(\tilde{B})).$$

By Theorem 7, the proof of the result is reduced to the case of full matrix algebras which is the case of Lemma 5.6 of [EG] according to Proposition 8. ∎

Similarly, the shape equivalence part of Lemma 5.20 (see also 5.18 and 5.19) of [EG] can be generalized:

Corollary 10. *Let* $A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ *and* $A' = \varinjlim(A'_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}, \psi_{n,m})$ *be simple inductive limits with* $A_n = A'_n$ *and let the connecting maps* $\phi_{n,m}$ *and* $\psi_{n,m}$ *satisfy*

$$(\phi_{n,m})_* = (\psi_{n,m})_*\colon K_*(A_n) \longrightarrow K_*(A_m).$$

In particular,

$$(K_0(A), K_0(A)^+, D(A), K_1(A)) \cong (K_0(A'), K_0(A')^+, D(A'), K_1(A')).$$

It follows that the systems $(A_n, \phi_{n,m})$ and $(A'_n = A_n, \psi_{n,m})$ are shape equivalent. Furthermore the shape equivalence may be chosen to induce the above isomorphism of K-theory.

(In the application in [EGL], we only need the case that $X_{n,i}$ are of the forms $\{pt\}$, $[0,1]$, $S^1, T_{II,k}, T_{III,k}$, and S^2, and $\psi_{n,m}$ satisfy the condition (*) in 4.1 of [EGL], which corresponds to the conditions in 5.18 of [EG]. But the proof of 5.20 of [EG] (referred to below) holds for the above more general form for the case of full matrix algebras.)

PROOF. By §1, one can modify the homomorphism $\psi_{n,n+1}$ by conjugating by a unitary to make the following true: $\psi_{n,n+1}(1_{A_n^i}) = \phi_{n,n+1}(1_{A_n^i})$ for any block A_n^i of A_n. Then, as in the construction in 2.1.2 of [EGL], the two homomorphisms

$\phi^{i,j}_{n,n+1}, \psi^{i,j}_{n,n+1}$:

$$P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}) \to \phi^{i,j}_{n,n+1}(P_{n,i})M_{[n+1,j]}(C(X_{n+1,j}))\phi^{i,j}_{n,n+1}(P_{n,i})$$

can be dilated to unital homomorphisms

$$\phi \text{ and } \psi \colon M_{[n,i]}(C(X_{n,i})) \longrightarrow QM_{\bullet}(C(X_{n+1,j}))Q$$

respectively, with the same projection Q. As a consequence (see §1.3 of [G] and 4.24 of [EG]), there are inductive limit systems

$$\varinjlim(\tilde{A}_n = \textstyle\bigoplus_{i=1}^{t_n} M_{[n,i]^{\sim}}(C(X_{n,i})), \tilde{\phi}_{n,m}) \text{ and}$$

$$\varinjlim(\tilde{A}'_n = \tilde{A}_n = \textstyle\bigoplus_{i=1}^{t_n} M_{[n,i]^{\sim}}(C(X_{n,i})), \tilde{\psi}_{n,m})$$

(with the same $\tilde{A}_n$) such that for each n, A_n is a corner of $\tilde{A}_n$, and $\phi_{n,m} = \tilde{\phi}_{n,m}|_{A_n}$ and $\psi_{n,m} = \tilde{\psi}_{n,m}|_{A_n}$. Obviously

$$(\tilde{\phi}_{n,m})_* = (\tilde{\psi}_{n,m})_* \colon K_*(\tilde{A}_n) \to K_*(\tilde{A}_m).$$

By Proposition 8, $(\tilde{A}_n, \tilde{\phi}_{n,m})$ and $(\tilde{A}_n, \tilde{\psi}_{n,m})$ are shape equivalent to inductive limit systems of the kind (with real rank zero limit algebras) described in 5.18–5.20 of [EG]. (Note that the only property of the inductive limit systems (besides that the limit algebras are simple) actually used in the proof of Lemma 5.20 of [EG] is that $(\tilde{\phi}_{n,m})_* = (\tilde{\psi}_{n,m})_* \colon K_*(\tilde{A}_n) \longrightarrow K_*(\tilde{A}_m)$; see lines 7-8 on page 597 of [EG]. With this property, the maps $\mathrm{id}_n \colon K_*(\tilde{A}_n) \to K_*(\tilde{A}'_n)$ give an intertwining at the level of K-theory (not KK-theory). Such identity maps id_n can be used to construct a shape equivalence because they take $K_*(\tilde{A}_n^0)$ to $K_*(\tilde{A}'^0_n)$.) Hence by Lemma 5.20 of [EG], $(\tilde{A}_n, \tilde{\phi}_{n,m})$ and $(\tilde{A}_n, \tilde{\psi}_{n,m})$ are shape equivalent. The corollary follows by Theorem 7. ∎

(If one checks the proofs of Lemmas 5.6 and 5.20 carefully, one will notice that the real rank zero property was not used in the proof. In fact, in the proofs, one needs to pass to C^*-algebras which are not simple and not of real rank zero. But the simplicity of the ordered K-groups is used in the proof. Hence, in principle, one does not need Proposition 8 in the above proofs of Corollaries 9 and 10.)

To conclude this note, we announce the following classification theorem which is proved in [EGL]. Suppose that A and B are unital simple AH algebras as described in §1 and §2. Suppose that there is an order isomorphism

$$\alpha\colon (K_0(A), K_0(A)^+, [1_A], K_1(A)) \longrightarrow (K_0(B), K_0(B)^+, [1_B], K_1(B))$$

and there is an isomorphism between compact convex sets $\xi\colon TB \to TA$ such that α and ξ are compatible in the sense $\tau(\alpha(g)) = \xi(\tau)(g)$ for any $g \in K_0(A)$ and $\tau \in TB$. Then there is an isomorphism $\phi\colon A \to B$ which induces α and ξ.

Corollary 9 and Corollary 10 are used in the proof of the above theorem. These two corollaries are not used in the real rank zero case in [EG], because in [EG], we are not assuming the C^*-algebras are unital, so we can replace the cut-downs of full matrix algebras by full matrix algebras themselves (see 4.24 of [EG] and §1.3 of [G]). Notice that when a cut-down system is replaced by the corresponding inductive limit system of full matrix algebras, the system (and the limit algebra) may not be unital any more (see Remark 1.3.4 of [G]). The techniques of [EGL] apply specifically to the unital case. On the other hand, the classification theorem for the non-unital case follows as a corollary from the unital case.

References

[Bl] B. Blackadar, *K-Theory for Operator Algebras*, Springer-Verlag, New York, Berlin, Heidelberg, 1986.

[BDR] B. Blackadar, M. Dadarlat, and M. Rørdam, *The real rank of inductive limit C^*-algebras*, Math. Scand. **69** (1991), 211–216.

[EG] G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero, II*, Ann. of Math. **144** (1996), 497–610.

[EGL] G. A. Elliott, G. Gong, and L. Li, *On the classification of simple inductive limit C^*-algebras, II: The isomorphism theorem*, preprint.

[G] G. Gong, *On the classification of simple inductive limit C^*-algebras, I: The reduction theorem*, Documenta Math. **7** (2002), 255–461.

[GL] G. Gong and H. Lin, *The exponential rank of inductive limit C^*-algebras*, Math. Scand. **71** (1992), 301–319.

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A CONGRUENCE THEOREM OF GEODESICS ON SOME NATURALLY REDUCTIVE RIEMANNIAN HOMOGENEOUS MANIFOLDS

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Presented by Vlastimil Dlab, FRSC

ABSTRACT. We establish a congruence theorem of geodesics on real hypersurfaces of type A, which are naturally reductive homogeneous Riemannian manifolds, in a nonflat complex space form by using submanifold theory.

RÉSUMÉ. Nous establissons un théorème sur la congruence des géodésiques sur les hypersurfaces du type A, qui est des variétés homogène riemanniennes naturellement reductible, dans des espaces projectives complexes et des espaces hyperboliques complexes s'aidant de la théorie des subvariétés.

1. Introduction. Let $\tilde{M}_n(c)$ denote a nonflat complex space form of constant holomorphic sectional curvature c ($\neq 0$), which is either a complex n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c or a complex n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. It is well-known that there is no isometrically immersed real hypersurface which is locally symmetric in a nonflat complex space form $\tilde{M}_n(c)$ with $n \geq 3$. However it admits the following real hypersurfaces which are typical examples of *naturally reductive Riemannian homogeneous manifolds* (for details, see [N]): In $\mathbb{C}P^n(c)$ we have

(A_1) geodesic spheres of radius r ($0 < r < \pi/\sqrt{c}$),
(A_2) tubes around totally geodesic $\mathbb{C}P^k(c)$ of radius r ($0 < r < \pi/\sqrt{c}$), where $1 \leq k \leq n-2$,

and in $\mathbb{C}H^n(c)$ we have

(A_0) horospheres,
(A_1) geodesic spheres of radius r ($r > 0$) and tubes around totally geodesic $\mathbb{C}H^{n-1}(c)$ of radius r ($r > 0$),
(A_2) tubes over totally geodesic $\mathbb{C}H^k(c)$ of radius r ($r > 0$), where $1 \leqq k \leqq n-2$.

These real hypersurfaces are called *hypersurfaces of type A* in nonflat complex space forms. In *intrinsic* Riemannian geometry, naturally reductive homogeneous Riemannian manifolds are interesting objects. On a naturally reductive

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	U	V	$\mathbb{R}\xi$
geodesic sphere of radius r in $\mathbb{C}P^n(4)$	$\cot r$	—	$2\cot 2r$
tube of radius r around totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n(4)$	$\cot r$	$-\tan r$	$2\cot 2r$
horosphere in $\mathbb{C}H^n(-4)$	1	—	2
geodesic sphere of radius r in $\mathbb{C}H^n(-4)$	$\coth r$	—	$2\coth 2r$
tube of radius r around totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}P^n(4)$	—	$\tanh r$	$2\coth 2r$
tube of radius r around totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n(4)$	$\coth r$	$\tanh r$	$2\coth 2r$

Table 1: Principal curvatures.

homogeneous Riemannian manifold M, every geodesic is a one-parameter subgroup of the isometry group $\mathrm{Iso}(M)$ of M. These hypersurfaces are also interesting from the viewpoint of *extrinsic* geometry, that is, submanifold theory (see Section 2). The purpose of this paper is to give a necessary and sufficient condition on congruence of geodesics on hypersurfaces of type A by using submanifold theory (Theorem 2). For this purpose we make use of two geometric invariants of geodesics. We also characterize hypersurfaces of type A in the class of all real hypersurfaces in nonflat complex space forms in terms of these invariants for geodesics.

2. **Characterization of hypersurfaces of type A.** Let M^{2n-1} be an orientable real hypersurface in a nonflat complex space form $\tilde{M}_n(c)$ and $\mathcal{N}_M$ a unit normal vector field on M in $\tilde{M}_n(c)$. The Riemannian connections $\tilde{\nabla}$ of $\tilde{M}_n(c)$ and ∇ of M are related by $\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle\mathcal{N}_M$ and $\tilde{\nabla}_X\mathcal{N}_M = -AX$ for vector fields X and Y tangent to M, where $\langle\ ,\ \rangle$ denotes the Riemannian metric of M induced from the standard metric of $\tilde{M}_n(c)$, and A is the shape operator of M in $\tilde{M}_n(c)$. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the complex structure of the ambient space $\tilde{M}_n(c)$, which satisfies $\phi^2 = -I + \eta\otimes\xi$, $\eta(\xi) = 1$ and $\langle\phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y)$, where I denotes the identity map of the tangent bundle TM of M. They satisfy $(\nabla_X\phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi$ and $\nabla_X\xi = \phi AX$. The Codazzi equation for the real hypersurface M in $\tilde{M}_n(c)$ is expressed as

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle\phi X, Y\rangle\xi\}.$$

We consider two geometric invariants for a geodesic γ of unit speed on a real hypersurface M^{2n-1} in $\tilde{M}_n(c)$ with $c \neq 0$. One is the *normal curvature*

$\kappa_\gamma(s)$ of γ which is defined by $\kappa_\gamma(s) = \langle A\dot\gamma(s), \dot\gamma(s)\rangle$, where $\dot\gamma(s)$ is the tangent vector along γ with respect to arclength s. We see by use of the first equality in (2.1) that $|\kappa_\gamma| = \|\tilde\nabla_{\dot\gamma}\dot\gamma\|$ is the first curvature of γ in the sense of Frenet formula by regarding it as a curve in the ambient space $\tilde M_n(c)$ if it does not vanish anywhere. The other is the *structure torsion* $\rho_\gamma(s)$ of γ which is defined by $\rho_\gamma(s) = \langle\dot\gamma(s), \xi_{\gamma(s)}\rangle$, where ξ is the characteristic vector field of the real hypersurface M (see [AMY]).

We now restrict ourselves to hypersurfaces of type A which are the simplest examples of real hypersurfaces in a complex space form. They have the principal curvatures as in Table 1. In the table, $TM = U \oplus V \oplus \mathbb{R}\xi$ is the splitting of the tangent bundle into principal curvature vector subbundles, and $1 \le k \le n-2$. The dimensions of subbundles U, V are $2(n-k-1)$ and $2k$, respectively.

It is well-known that these real hypersurfaces are characterized in the following way in terms of the structure tensor and the shape operator (see for example [NR]).

PROPOSITION. *Let M be a real hypersurface in $\tilde M_n(c)$ with $c \neq 0$. The following conditions are mutually equivalent:*

(1) *M is a hypersurface of type A.*
(2) *The structure tensor ϕ and the shape operator A of M in $\tilde M_n(c)$ are commutative: $\phi A = A\phi$.*
(3) *The covariant derivative of the shape operator A satisfies*

$$(\nabla_X A)Y = -(c/4)\{\langle\phi X, Y\rangle\xi + \eta(Y)\phi X\}. \tag{2.1}$$

In this section we provide the following characterization of hypersurfaces of type A in a complex space form in terms of geometric invariants for their geodesics.

THEOREM 1. *Let M be a real hypersurface in $\tilde M_n(c)$ with $c \neq 0$. The following conditions are mutually equivalent:*

(1) *M is a hypersurface of type A.*
(2) *For every geodesic γ on M the normal curvature κ_γ is constant along γ.*
(3) *For every geodesic γ on M the structure torsion ρ_γ is constant along γ.*

PROOF. (1) $\Leftrightarrow$ (2). If M is a hypersurface of type A, then (2.1) yields $\nabla_{\dot\gamma}\kappa_\gamma = \nabla_{\dot\gamma}\langle A\dot\gamma, \dot\gamma\rangle = \langle(\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma\rangle = 0$ for every geodesic γ on M, which leads us to the second condition. Conversely, under the second condition, our real hypersurface M satisfies $\langle(\nabla_X A)X, X\rangle) = 0$ for every tangent vector X of M. This is equivalent to the condition $\langle(\nabla_X A)Y, Z\rangle + \langle(\nabla_Y A)Z, X\rangle + \langle(\nabla_Z A)X, Y\rangle = 0$ for arbitrary X, Y and Z tangent to M. On the other hand, by virtue of Codazzi equation, we have

$$\langle(\nabla_Z A)X, Y\rangle - \langle(\nabla_X A)Z, Y\rangle = \frac{c}{4}\{\eta(Z)\langle\phi X, Y\rangle - \eta(X)\langle\phi Z, Y\rangle - 2\eta(Y)\langle\phi Z, X\rangle\}.$$

Exchanging the roles of X and Y, we get

$$\langle(\nabla_Z A)Y, X\rangle - \langle(\nabla_Y A)Z, X\rangle = \frac{c}{4}\{\eta(Z)\langle\phi Y, X\rangle - \eta(Y)\langle\phi Z, X\rangle - 2\eta(X)\langle\phi Z, Y\rangle\}.$$

Summing these three equalities, we obtain

$$\langle(\nabla_Z A)X, Y\rangle = -\frac{c}{4}\{\eta(X)\langle\phi Z, Y\rangle + \eta(Y)\langle\phi Z, X\rangle\}$$

for every X, Y and Z tangent to M. This is equivalent to the equality (2.1), so that M is a hypersurface of type A.

(1) $\Leftrightarrow$ (3). By Proposition, in order to see that M is a hypersurface of type A, it suffices to show that the condition $\phi A = A\phi$ is equivalent to the third condition. As we find that $\rho'_\gamma = \nabla_{\dot\gamma}\langle\dot\gamma, \xi\rangle = \langle\dot\gamma, \nabla_{\dot\gamma}\xi\rangle = \langle\dot\gamma, \phi A\dot\gamma\rangle$ for every geodesic γ on M, we obtain under the third condition that $\langle X, \phi AX\rangle = 0$ for each tangent vector $X \in TM$. In particular, we see $\langle X+Y, \phi A(X+Y)\rangle = 0$ for every tangent vectors $X, Y \in TM$. This guarantees $\langle(\phi A - A\phi)X, Y\rangle = 0$ for arbitrary tangent vectors $X, Y \in TM$, hence we see $\phi A = A\phi$. On the other hand, if M is a hypersurface of type A, as $\phi A = A\phi$, we find by the equality $\rho'_\gamma = \langle\dot\gamma, \phi A\dot\gamma\rangle$ that the structure torsion ρ_γ is constant for every geodesic on M. ■

3. Congruence theorem on geodesics of type A hypersurfaces.

In this section we establish the following congruence theorem for geodesics on hypersurfaces of type A by use of two geometric invariants, normal curvature and structure torsion. This is an extension of our preceding result in [AMY] of congruence theorem for geodesics on geodesic spheres in a nonflat complex space form.

THEOREM 2. *Let γ_1, γ_2 be two geodesics on a hypersurface M of type A in a nonflat complex space form $\tilde M_n(c)$. The following conditions are equivalent:*

(1) *These geodesics are congruent, that is, there exists an isometry φ of $\tilde M_n(c)$ satisfying that $\gamma_1(s) = (\varphi\circ\gamma_2)(s+s_0)$ for every s with some s_0.*
(2) *They have the same absolute values of normal curvatures and structure torsions.*

By the result of [AMY] we only need to show this theorem when M is a real hypersurface of type A_2. To do this we need the following lemmas.

LEMMA 1. *Let M be a tube around totally geodesic $\mathbb{C}P^k$ $(1 \le k \le n-2)$ in $\mathbb{C}P^n$. For arbitrary principal vectors $u \in U_x$, $v \in V_x$ and $u' \in U_y$, $v' \in V_y$ with $\|u\| = \|v\| = \|u'\| = \|v'\| = 1$ at arbitrary points x, y, there exist isometries φ^+, φ^- of $\mathbb{C}P^n$ with*

(i) $\tilde\varphi^+(M) = \tilde\varphi^-(M) = M$ *and* $\tilde\varphi^+(x) = \tilde\varphi^-(x) = y$,
(ii) $d\tilde\varphi^\pm_x(u) = u'$, $d\tilde\varphi^\pm_x(v) = v'$ *and* $d\tilde\varphi^+_x(\xi_x) = \xi_y$, $d\tilde\varphi^-_x(\xi_x) = -\xi_y$.

PROOF. It is enough to study on $\mathbb{C}P^n(4)$. Let $\varpi\colon S^{2n+1} \to \mathbb{C}P^n$ denote the Hopf fibration of a unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ and $\hat{M}_k$ a hypersurface given by

$$\left\{w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1} \;\middle|\; \begin{array}{l} |w_0|^2 + \cdots + |w_k|^2 = \cos^2 r, \\ |w_{k+1}|^2 + \cdots + |w_n|^2 = \sin^2 r \end{array}\right\}.$$

We see that $\varpi^{-1}(M)$ is isometric to $\hat{M}_k$.

For the sake of simplicity we only treat the case $n = 3, k = 1$ and $M = \varpi(\hat{M}_1)$. At a point $w = (w_0, w_1, w_2, w_3) \in \hat{M}_1$ the tangent space of $\hat{M}_1$ is represented as

$$\begin{aligned} T_w\hat{M}_1 &= \{(w, v) \in \{w\} \times \mathbb{C}^3 \mid \mathrm{Re}(w_0\bar{v}_0 + w_1\bar{v}_1) = \mathrm{Re}(w_2\bar{v}_2 + w_3\bar{v}_3) = 0\} \\ &= \big\{\big(w, (\sqrt{-1}aw_0 - \alpha\bar{w}_1, \sqrt{-1}aw_1 + \alpha\bar{w}_0, \\ &\qquad \sqrt{-1}bw_2 - \beta\bar{w}_3, \sqrt{-1}bw_3 + \beta\bar{w}_2)\big) \;\big|\; a, b \in \mathbb{R}, \alpha, \beta \in \mathbb{C}\big\}, \end{aligned}$$

where $\mathrm{Re}(\alpha)$ denotes the real part of a complex number α. Let $T_wS^7 = \mathcal{H}_w \oplus \mathcal{V}_w$ denote the horizontal and vertical decomposition of the tangent space of S^7 at w with respect to the Hopf fibration. They are represented as $\mathcal{H}_w = \{(w, v) \mid w_0\bar{v}_0 + w_1\bar{v}_1 + w_2\bar{v}_2 + w_3\bar{v}_3 = 0\}$ and $\mathcal{V}_w = \{(w, \sqrt{-1}aw) \mid a \in \mathbb{C}\}$. We denote by $(T_w\hat{M}_1)^\perp$ the orthogonal complement of $T_w\hat{M}_1$ in T_wS^7. We find that the horizontal lift $\hat{\mathcal{N}}_w \in (T_w\hat{M}_1)^\perp \cap \mathcal{H}_w$ of the unit normal $\mathcal{N}_M$ at $\varpi(w)$ is of the form $\hat{\mathcal{N}}_w = \big(w, (-\tan rw_0, -\tan rw_1, \cot rw_2, \cot rw_3)\big)$. We set $\hat{\xi}_w = -J\hat{\mathcal{N}}_w$ and denote by $\langle\hat{\xi}_w\rangle$ the real linear subspace spanned by $\hat{\xi}_w$. Here we denote the complex structure of $\mathbb{C}^4$ also by J. The horizontal part $\langle\hat{\xi}_w\rangle^\perp \cap \mathcal{H}_w$ of the orthogonal complement subspace $\langle\hat{\xi}_w\rangle^\perp$ in $T_w\hat{M}_1$ is represented as $\langle\hat{\xi}_w\rangle^\perp \cap \mathcal{H}_w = \big\{\big(w, (-\alpha\bar{w}_1, \alpha\bar{w}_0, -\beta\bar{w}_3, \beta\bar{w}_2)\big) \;\big|\; \alpha, \beta \in \mathbb{C}\big\}$. If we set $\hat{U}_w = d\varpi^{-1}(U_{\varpi(w)}) \cap \mathcal{H}_w$ and $\hat{V}_w = d\varpi^{-1}(V_{\varpi(w)}) \cap \mathcal{H}_w$, we see $T_w\hat{M} \cap \mathcal{H}_w = \langle\hat{\xi}_w\rangle \oplus \hat{U}_w \oplus \hat{V}_w$, and $\hat{U}_w$ and $\hat{V}_w$ are complex linear subspaces spanned by $\eta_w = \big(w, (-\sec r\bar{w}_1, \sec r\bar{w}_0, 0, 0)\big) \in \hat{U}_w$ and $\zeta_w = \big(w, (0, 0, -\csc r\bar{w}_3, \csc r\bar{w}_2)\big) \in \hat{V}_w$, respectively.

Consider a point $z = (\cos r, 0, \sin r, 0) (\in \hat{M}_1)$. For arbitrary $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$, the unitary matrix

$$\begin{pmatrix} \sec rw_0 & -\alpha\sec r\bar{w}_1 & 0 & 0 \\ \sec rw_1 & \alpha\sec r\bar{w}_0 & 0 & 0 \\ 0 & 0 & \csc rw_2 & -\beta\csc\bar{w}_3 \\ 0 & 0 & \csc rw_3 & \beta\csc\bar{w}_2 \end{pmatrix}$$

induces an isometry $\hat{\varphi}^+_{\alpha,\beta}$ of S^7 such that

(i) $\hat{\varphi}^+_{\alpha,\beta}(z) = w$ and $\hat{\varphi}^+_{\alpha\beta}(\hat{M}) = \hat{M}$,

(ii) $(d\hat{\varphi}^+_{\alpha\beta})_z(\hat{\mathcal{N}}_z) = \hat{\mathcal{N}}_w$ and $(d\hat{\varphi}^+_{\alpha,\beta})_z(\langle\hat{\xi}_z\rangle^\perp \cap \mathcal{H}_z) = \langle\hat{\xi}_w\rangle^\perp \cap \mathcal{H}_w$,

(iii) $(d\hat{\varphi}^+_{\alpha,\beta})_z(\eta_z) = \alpha\eta_w$ and $(d\hat{\varphi}^+_{\alpha,\beta})_z(\zeta_z) = \beta\zeta_w$.

This guarantees the existence of an isometry $\tilde{\varphi}^+$ of $\mathbb{C}P^n$ with desirable conditions. We can similarly construct an isometry $\tilde{\varphi}^-$ and get the conclusion. ∎

For a tube M around totally geodesic $\mathbb{C}H^k$ $(1 \le k \le n-2)$ in $\mathbb{C}H^n$, we consider the Hopf fibration $\varpi\colon H_1^{2n+1} \to \mathbb{C}H^n$ of an anti-de Sitter space $H_1^{2n+1} = \{w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1} \mid -|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2 = -1\}$ in $\mathbb{C}^{n+1}$. Since the hypersurface $\varpi^{-1}(M)$ in H_1^{2n+1} is isometric to a hypersurface given by

$$\left\{ w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1} \;\middle|\; \begin{array}{l} |w_0|^2 + \cdots + |w_k|^2 = \cosh^2 r, \\ |w_{k+1}|^2 + \cdots + |w_n|^2 = \sinh^2 r \end{array} \right\},$$

by just the same argument as in the proof of Lemma 1, we can obtain the same result for this tube M.

LEMMA 2. *Let M be a tube around totally geodesic $\mathbb{C}H^k$ $(1 \le k \le n-2)$ in $\mathbb{C}H^n$. For arbitrary principal vectors $u \in U_x$, $v \in V_x$ and $u' \in U_y$, $v' \in V_y$ with $\|u\| = \|v\| = \|u'\| = \|v'\| = 1$ at arbitrary points x, y, there exist isometries φ^+, φ^- of $\mathbb{C}H^n$ with*

(i) *$\tilde{\varphi}^+(M) = \tilde{\varphi}^-(M) = M$ and $\tilde{\varphi}^+(x) = \tilde{\varphi}^-(x) = y$,*
(ii) *$d\tilde{\varphi}_x^{\pm}(u) = u'$, $d\tilde{\varphi}_x^{\pm}(v) = v'$ and $d\tilde{\varphi}_x^+(\xi_x) = \xi_y$, $d\tilde{\varphi}_x^-(\xi_x) = -\xi_y$.*

We here give the relationship between the normal curvature and the structure torsion of a geodesic on a hypersurface of type A_2 and the decomposition of its tangent vectors into principal curvature vectors.

LEMMA 3. *Let M be either a tube of radius r around totally geodesic $\mathbb{C}P^k$ $(1 \le k \le n-2)$ in $\mathbb{C}P^n(4)$ or a tube of radius r around totally geodesic $\mathbb{C}H^k$ $(1 \le k \le n-2)$ in $\mathbb{C}H^n(-4)$. For a geodesic γ on M, we denote by $\dot\gamma(s) = u(s) + v(s) + \rho_\gamma \xi_{\gamma(s)} \in U_{\gamma(s)} \oplus V_{\gamma(s)} \oplus \mathbb{R}\xi_{\gamma(s)}$ the decomposition of the tangential vector into principal vectors. We then find the norms $\|u(s)\|$ and $\|v(s)\|$ are constant along γ, and these only depend on normal curvature, structure torsion of a geodesic and the radius of tube.*

PROOF. For a geodesic γ on a tube of radius r around $\mathbb{C}P^k$, as we have

$$\kappa_\gamma = \langle A\dot\gamma(s), \dot\gamma(s)\rangle = \rho_\gamma^2(\cot r - \tan r) + \cot r\|u(s)\|^2 - \tan r\|v(s)\|^2$$

and $1 = \|\dot\gamma(s)\|^2 = \rho_\gamma^2 + \|u(s)\|^2 + \|v(s)\|^2$, we find $\|u(s)\|^2 = \sin^2 r - \rho_\gamma \cos^2 r + \kappa_\gamma \sin r \cos r$ and $\|v(s)\|^2 = \cos^2 r - \rho_\gamma \sin^2 r - \kappa_\gamma \sin r \cos r$.

Similarly, for a geodesic γ on a tube of radius r around $\mathbb{C}H^k$, we have $\|u(s)\|^2 = -\sinh^2 r - \rho_\gamma \cosh^2 r + \kappa_\gamma \sinh r \cosh r$ and $\|v(s)\|^2 = \cosh^2 r + \rho_\gamma \sinh^2 r - \kappa_\gamma \sinh r \cosh r$. Hence we obtain the assertion. ∎

PROOF OF THEOREM 2. Since M is an orbit of some subgroup of $\mathrm{Iso}(\tilde{M}(c))$, for every $\varphi \in \mathrm{Iso}(M)$ there is an isometry $\tilde{\varphi}$ of $\tilde{M}(c)$ with $\tilde{\varphi}|_M = \varphi$. Therefore it is trivial that the first condition implies the second. We hence show the converse. If a hypersurface M is of type A_0 or A_1, it was already shown in [AMY] that geodesics are congruent if and only if they have the same absolute values of structure torsion. If M is of type A_2, we see by Lemma 3 that the decompositions $\dot{\gamma}_i(s) = u_i(s) + v_i(s) + \rho_\gamma \xi_{\gamma_i(s)}$ into principal vectors satisfy $\|u_1\| = \|u_2\|$, $\|v_1\| = \|v_2\|$. Hence, by Lemmas 1 and 2, there is an isometry φ of M with $\dot{\gamma}_1(0) = d\varphi(\dot{\gamma}_2(0))$, which guarantees that $\gamma_1 = \varphi \circ \gamma_2$. ■

REMARK. We say that two geodesics γ_1, γ_2 parameterized by their arc-length s are congruent in the strong sense if there exists an isometry φ of $\tilde{M}_n(c)$ satisfying $\gamma_1(s) = (\varphi \circ \gamma_2)(s)$ for all s. For geodesics on a hypersurface of type A in a nonflat complex space form, congruency in the strong sense is equivalent to congruency.

REFERENCES

[AMY] T. Adachi, S. Maeda and M. Yamagishi, *Length spectrum of geodesic spheres in a non-flat complex space form.* J. Math Soc. Japan **54**(2002), 373–408.

[N] S. Nagai, *The classification of naturally reductive homogeneous real hypersurfaces in complex projective space.* Arch. Math. (Basel) **69**(1997), 523–528.

[NR] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms.* In: Tight and Taut Submanifolds, (eds., T. E. Cecil and S. S. Chern), Cambridge University Press, 1998, pp. 233–305.

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ON THE CLASSIFICATION OF TAI ALGEBRAS

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ABSTRACT. It is shown that unital simple separable nuclear TAI algebras which satisfy UCT (the Universal Coefficient Theorem) can be classified by their ordered K-group and space of tracial states.

RÉSUMÉ. Nous classifions la classe des algèbres nuclaires sparables simples unital de TAI qui satisfont UCT (le théorème universel de coefficient) par leur K-groupe de order et espace des états tracial.

1. **Introduction** TAI algebras were first introduced by Huaxin Lin [Lin2]. Roughly speaking, this is the class of C*-algebras which can be approximated by interval algebras in trace (or in "measure"). This class has been shown to contain many examples, for instance, the unital simple AH algebras with very slow dimension growth, and the unital simple AH algebras with slow dimension growth and with real rank zero (these are in fact TAF algebras).

Recall that a unital simple separable C*-algebra A is a TAI algebra if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$, and any $a \in A^+ \setminus \{0\}$, there exist a nonzero projection $p \in A$ and a sub-C*-algebra $B \cong \bigoplus_{i=1}^{k} M_{n_i}(C([0,1]))$, with $p = 1_B$, such that

1. $\|xp - px\| < \varepsilon, \quad x \in \mathcal{F}$,
2. $\mathrm{dist}(pxp, B) < \varepsilon, \quad x \in \mathcal{F}$, and
3. $1 - p$ is equivalent to a projection in $(aAa)^-$.

In [Lin3], Lin showed that if a unital simple separable C*-algebra satisfies the properties (1) and (2) above, then it has property (SP), which states that every nonzero hereditary sub-C*-algebra contains a nonzero projection.

Concerning the classification of this class of C*-algebras, much work has been done by Huaxin Lin in [Lin2]. Lin introduced the concept of KK-attainability for TAI algebras: Denote by $\mathbf{P}(A)$ the set of projections in

$$\bigoplus_{k=1} M_\infty(\widetilde{A \otimes C_k}) \oplus \bigoplus_{k=1} M_\infty(\widetilde{A \otimes SC_k})$$

where C_k is a fixed commutative C*-algebra with $\mathrm{K}_0(C_k) = \mathbb{Z}/k\mathbb{Z}$ and $\mathrm{K}_1(C_k) = 0$. A C*-algebra A is said to be KK-attainable (for simple TAI algebras) if for any unital separable simple TAI algebra B, any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$ and any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ with $[1_A] \in \mathcal{P}$, there exists a sequence of completely

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positive linear contractions $L_n: A \to B \otimes \mathcal{K}$ such that

$$\|L_n(a)L_n(b) - L_n(ab)\| \to 0$$

for all $a, b \in A$ as $n \to \infty$, and

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for all sufficiently large n. Lin also gave many interesting examples of C*-algebras which are KK-attainable; for instance, every separable locally continuous trace C*-algebra, and hence every AH-algebra, is KK-attainable. For this class of C*-algebras, Lin showed in [Lin2] that:

THEOREM 1.1. *Let A and B be two KK-attainable unital simple separable nuclear* TAI *algebras which satisfy* UCT. *Suppose that there is an isomorphism*

$$(K_0(A), K_0^+(A), [1_A], K_1(A), T(A)) \cong (K_0(B), K_0^+(B), [1_B], K_1(B), T(B)).$$

Then $A \cong B$.

Therefore, an interesting task left for the classification of the general TAI algebra is to see which ones are KK-attainable. This is the main purpose of this paper. We will show that every unital simple separable nuclear TAI algebra which satisfies UCT is KK-attainable, and thus, all such algebras can be classified by their invariant. The basic idea is similar to that of Huaxin Lin's work for TAF algebras [Lin1]. In that paper, he showed that every simple separable nuclear TAF algebra which satisfies UCT is KK-attainable (for simple TAF algebras), by constructing a suitable completely positive linear contraction from a given such TAF algebra to the unital simple AH algebra of real rank zero with slow dimension growth which has the same K-group. In this paper we shall show that we can also construct such a map from a unital simple separable nuclear TAI algebra to the corresponding AH algebra of real rank zero with slow dimension growth which has the same K-group. Since by [EGL] or [Lin2], such an AH algebra is KK-attainable (for simple TAI algebras), we can see that KK-attainability will also hold for this class of TAI algebras. To be precise, we have:

THEOREM 1.2. *Let A be a unital separable simple nuclear* TAI *algebra which satisfies* UCT, *and suppose that B is a unital simple* AH *algebra of real rank zero with slow dimension growth such that*

$$(K_0(A), K_0^+(A), [1_A], K_1(A)) \cong (K_0(B), K_0^+(B), [1_B], K_1(B)).$$

Choose $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))^+$ giving rise to this isomorphism (see [DL]). Then, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there is a sequence of completely positive linear contractions $L_n: A \to B$ such that

1. $\|L_n(a)L_n(b) - L_n(ab)\| \to 0$ *for all $a, b \in A$ as $n \to \infty$,*
2. $[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ *for all sufficiently large n.*

2. The proof of the main result First we give a lemma which has a conclusion similar to the conclusion of Lemma 2.2 of [Lin1] but applies to more general C*-algebras.

LEMMA 2.1. *Let A be a separable unital C*-algebra that has property* (SP), *and let A' be a sub-C*-algebra which contains the unit of A. Suppose that there is a surjective homomorphism h' from A' to some finite dimensional C*-algebra F. Then for any finite subset $\mathcal{F} \subset A'$, and any $\varepsilon > 0$, there is a monomorphism $h\colon F \to A$ with $e = h(1_F)$ such that for all $x \in \mathcal{F}$,*

1. $\|ex - xe\| < \varepsilon$, *and*
2. $\|h \circ h'(x) - exe\| < \varepsilon$.

PROOF. Write $F = \bigoplus_{k=1}^{d} M_{n_k}(\mathbb{C})$ and set $I = \ker h'$. Choose an approximate unit $\{E_n\}$ of I. Let $\{f_{ij}^{(k)}\}$ be a system of matrix units of F, choose $u_{1j}^{(k)}$ in the preimage of $f_{1j}^{(k)}$ for each j and k, and set $x_{ij}^{(k)} = (u_{1i}^{(k)})^* u_{1j}^{(k)}$. For any $x \in \mathcal{F}$, we can write $x = \sum_{i,j,k} \lambda_{ij}^{k}(x) x_{ij}^{(k)} + o_x$ for some $\lambda_{ij}^{k}(x) \in \mathbb{C}$ and $o_x \in I$. Fix these coefficients, and choose E_n large enough that $(1 - E_n)o_x$ and $(1 - E_n)u_{1l}^{(m)} o_x$ are sufficiently small, and for each k, $(1 - E_n)x_{11}^{(k)}$ is sufficiently close to every $(1 - E_n)u_{1i}^{(k)}(u_{1i}^{(k)})^*$, $(1 - E_n)u_{1i}^{(k)} x_{ii}^{(k)}$ is sufficiently close to $(1 - E_n)u_{1i}^{(k)}$, and $(1 - E_n)u_{1i}^{(l)}(u_{1j}^{(k)})^*$ are sufficiently small if $i \neq j$ or $l \neq k$. What is more, we may assume that E_n is quasi-central (page 83, [Ped]), so that $u_{1i}^{(k)}(1 - E_n)u_{1i}^{(k)}$ and $(u_{1i}^{(l)})^*(1 - E_n)u_{1j}^{(k)}$ are almost zero for each i, j, and $l \neq k$. (We stipulate all the errors to be within $\varepsilon/64(\sum n_i)^2$.)

Take $a = (1 - E_n)x_{11}^{(1)}(1 - E_n)$ in A', and $b = f(a)$ where f is a positive continuous function which is 1 at 1 and 0 outside the small interval $[1 - \varepsilon, 1 + \varepsilon]$. Since the image of a is a nonzero projection, the spectrum of a must contain 1; hence $b \neq 0$. Since the algebra A has the property (SP), we may choose a nonzero projection p in $(bAb)^-$. We claim that pa is almost p. Indeed without loss of generality, we may assume that $p = bcb$ for some c in A. Then $pa - a = bc(ba - b)$, but $(ba - b)(ba - b)^* = (f(x)(1 - x))^2(a) \leq (f(x)\varepsilon)^2(a) = \varepsilon^2 b^2$. Hence we get $(pa - p)(pa - p)^* = bc(ba - b)(ba - b)c^*b \leq \varepsilon^2(bcbbc^*b) = \varepsilon^2 p$, and therefore $\|pa - p\| \leq \|\varepsilon p\| = \varepsilon$. Setting $w_{12} = p(1 - E_n)u_{12}^{(1)}$, we see that $w_{12}w_{12}^* = p(1 - E_n)u_{12}^{(1)}(u_{12}^{(1)})^*(1 - E_n)p \approx_\varepsilon pap \approx_\varepsilon p$ (where $\approx_\varepsilon$ means close to within ε), and hence we can find a partial isometry v_{12}' which is close to w_{12} to within 6ε such that $v_{12}'{v_{12}'}^* = p$.

Continuing in this way, setting $w_{1i} = p(1 - E_n)u_{1i}^{(1)}$, we get $w_{1i}w_{1i}^* \approx_\varepsilon p$, and hence we get n_1 partial isometries $\{v_{1i}'\}_{i=1}^{n_1}$ such that $p = v_{1i}'(v_{1i}')^*$ and $\|v_{1i}' - w_{1i}\| < 6\varepsilon$ for each i. Write $p_{ii}' = (v_{1i}')^* v_{1i}'$ for $i \geq 2$ and set $p_{11}' = p$. If

$i \neq j$ and neither i or j is 1, then

$$\begin{aligned} p'_{ii}p'_{jj} &= (v'_{1i})^* v'_{1i}(v'_{1j})^* v'_{1j} \\ &\approx_{13\varepsilon} (u_{1i}^{(1)})^*(1-E_n)(p(1-E_n)u_{1i}^{(1)}(u_{1j}^{(1)})^*(1-E_n)p)(1-E_n)u_{1j}^{(1)} \\ &\approx_{16\varepsilon} 0, \end{aligned}$$

and

$$\begin{aligned} p'_{11}p'_{ii} &\approx_\varepsilon pap'_{ii} \\ &\approx_{7\varepsilon} p(1-E_n)u_{1i}^{(1)}((u_{1i}^{(1)})^*(1-E_n)(u_{1i}^{(1)})^*(1-E_n))p(1-E_n)u_{1i}^{(1)} \\ &\approx_{8\varepsilon} 0. \end{aligned}$$

In other words, $\{p'_{ii}\}_{i=1}^{n_1}$ are almost mutually orthogonal. Therefore we can find mutually orthogonal projections p_{ii} such that p_{ii} is very close to p'_{ii}, and so there exist partial isometries $\{v_{1i}^{(1)}\}_{i=2}^{n_1}$ which are close to $\{v'_{1i}\}_{i=1}^{n_1}$ and hence to $\{w_{1i}\}_{i=1}^{n_1}$ such that $(v_{1i}^{(1)})^* v_{1i}^{(1)} = p_{ii}$ and $v_{1i}^{(1)}(v_{1i}^{(1)})^* = p$ for each i. We can also see that the error can be made smaller than any given positive number by choosing ε suitably.

We can carry out the same argument for the all components of F, to find partial isometries $\{v_{1i}^{(k)}\}_{i,k}$ such that the projections $p_{ii}^{(k)} = (v_{1i}^{(k)})^* v_{1i}^{(k)}$ are mutually orthogonal and $v_{1i}^{(k)}$ is close to $p_{11}^{(k)}(1-E_n)u_{1i}^{(k)}$. Without loss of generality, we may assume that the distance between them is less than $\varepsilon/4(\sum_i n_i)$. Set $e_{ij}^{(k)} = (v_{1i}^{(k)})^* v_{1j}^{(k)}$ for each i, j, k; then

$$\begin{aligned} e_{ij}^{(k)} e_{jl}^{(k)} &= (v_{1i}^{(k)})^* v_{1j}^{(k)} (v_{1j}^{(k)})^* v_{1l}^{(k)} = (v_{1i}^{(k)})^* p_{11}^{(k)} v_{1l}^{(k)} \\ &= (v_{1i}^{(k)})^* v_{1l}^{(k)} (v_{1l}^{(k)})^* v_{1l}^{(k)} = (v_{1i}^{(k)})^* v_{1l}^{(k)} \\ &= e_{il}^{(k)}. \end{aligned}$$

What is more, if $j \neq m$ or $k \neq l$, we have

$$e_{ij}^{(k)} e_{mn}^{(l)} = (v_{1i}^{(k)})^* v_{1j}^{(k)} (v_{1m}^{(l)})^* v_{1n}^{(l)} = (v_{1i}^{(k)})^* v_{1j}^{(k)} p_{jj}^{(k)} p_{mm}^{(l)} (v_{1m}^{(l)})^* v_{1n}^{(l)} = 0.$$

Thus, we get a system of matrix units $e_{ij}^{(k)}$ in A of the same type as F, and the map h: $f_{ij}^{(k)} \mapsto e_{ij}^{(k)}$ will give a monomorphism from F to A.

In what follows, we shall show that h satisfies the requirements of the lemma. Set $e = h(1_F)$. Notice that by our choice of E_n and $e_{ii}^{(k)}$, $o_x e$ and eo_x are arbitrarily small, and so we only need to verify the inequalities for each $x_{ij}^{(k)}$. We have

$$\begin{aligned}
ex_{ij}^{(k)} &= (\sum p_{mm}^{(l)})x_{ij}^{(k)} = (\sum (v_{1m}^{(l)})^* v_{1m}^{(l)})x_{ij}^{(k)} \\
&\approx_{\varepsilon/4} (\sum (u_{1m}^{(l)})^*(1-E_n)p_{11}^{(l)}(1-E_n)u_{1m}^{(l)})(u_{1i}^{(k)})^* u_{1j}^{(k)} \\
&\approx_{\varepsilon/64} (u_{1i}^{(k)})^*(1-E_n)p_{11}^{(k)}(1-E_n)u_{1i}^{(k)}(u_{1i}^{(k)})^* u_{1j}^{(k)} \\
&\approx_{\varepsilon/64} (u_{1i}^{(k)})^*(1-E_n)p_{11}^{(k)}(1-E_n)u_{1j}^{(k)}(u_{1j}^{(k)})^* u_{1j}^{(k)} \\
&\approx_{\varepsilon/64} (u_{1i}^{(k)})^*(1-E_n)p_{11}^{(k)}(1-E_n)u_{1j}^{(k)}.
\end{aligned}$$

The same argument shows that $x_{ij}e \approx_{\varepsilon/2} (u_{1i}^{(k)})^*(1-E_n)p_{11}^{(k)}(1-E_n)u_{1j}^{(k)}$. Hence the first inequality holds.

To see the second inequality, note that $h \circ h'(x_{ij}^{(k)}) = e_{ij}^{(k)}$ and that

$$\begin{aligned}
ex_{ij}^{(k)}e &\approx_{\varepsilon/16} (u_{1i}^{(k)})^*(1-E_n)p_{11}^{(k)}(1-E_n)u_{1j}^{(k)}(u_{ij}^{(k)})^*(1-E_n)p_{11}^{(k)}(1-E_n)u_{1j}^{(k)} \\
&\approx_{\varepsilon/4} (u_{1i}^{(k)})^*(1-E_n)p_{11}^{(k)}p_{11}^{(k)}(1-E_n)u_{1j}^{(k)} \\
&\approx_{\varepsilon/2} (v_{1i}^{(k)})^* v_{1j}^{(k)} = e_{ij}^{(k)}.
\end{aligned}$$

This completes the proof of the lemma. ■

Let us return to the proof of Theorem 1.2. By [Lin3], every separable simple TAI algebra A can be embedded into the asymptotic sequence algebra

$$\prod_i M_{n_i}(\mathbb{C}) / \bigoplus_i M_{n_i}(\mathbb{C})$$

for some sequence of natural numbers $\{n_i\}$. Hence by [BE1, BE2], as A is nuclear, it is strong NF. In other words, A is the closure of $\bigcup A_n$ where A_n is an increasing sequence of residually finite dimensional (RFD) sub-C*-algebras of A. For any given finite subset $\mathcal{P}$ of $\mathbf{P}(A)$, we may assume that A_1 is sufficiently large that $\mathcal{P}$ is in $P(A_1)$. Choose a finite subset $\mathcal{F} \subset A_1$ and $\delta > 0$ such that for any completely positive linear contraction L from A_1 to B which is $\mathcal{F}$-δ-multiplicative, $[L]|_{\mathcal{P}}$ is well defined. By Theorem 4.1 of [Lin1], there exist two completely positive linear contractions $\Phi, \Phi^{(0)}: A \to B \otimes \mathcal{K}$ such that Φ is $\mathcal{F}$-δ-multiplicative, the image of $\Phi^{(0)}$ is contained in a finite dimensional sub-C*-algebra, $\Phi^{(0)}|_{A_1}$ is a homomorphism, and

$$[\Phi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} + [\Phi^{(0)}]|_{\mathcal{P}}.$$

Using Lemma 2.1, we get a homomorphism h from the image of $\Phi^{(0)}$ to A. Denote the composed map $h \circ \Phi^{(0)}: A_1 \to A$ by H. H will play a role similar to that played by map H of the §2 and §4 of [Lin1].

By a construction similar to that of 2.4 of [Lin1], we may choose an increasing sequence of finite subsets $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ (with $\mathcal{F} \subset \mathcal{F}_1$) in the unit ball of A with union dense in this unit ball, a decreasing sequence of positive numbers $\{\delta_n\}$ (with $\delta_n < \delta_{n-1}/4$ and $\delta_1 < \delta/2$), a sequence of projections $\{q_n\}$ in A, a sequence of sub-C^*-algebras F_n of A with $1_{F_n} = q_n$ where F_n is an interval C*-algebra, and a sequence of homomorphisms $h_{n+1}\colon F_n \to F_{n+1}$ such that

1. $\|q_n x - x q_n\| < \delta_n/4$ for each $x \in \mathcal{F}_n$,
2. $\mathrm{dist}(q_n x q_n, F_n) < \delta_n/16$ for each $x \in \mathcal{F}_n$,
3. $\tau(1 - q_n) < 1/2^{n+1}$ for every tracial state τ on A,
4. $\mathcal{F}_{n+1} \supset S_n$, where S_n is a set of standard generators of F_n, and
5. $\|L_{n+1}(a) - h_{n+1}(a)\| < \delta_1/16^n$ for all $a \in \{L_n(\mathcal{F}_n)\} \cup \{S_n\}$, where $L_n\colon A \to F_n$ is a completely positive linear contraction such that $L_n|_{F_n} = \mathrm{id}_{F_n}$.

With the same argument and the same notation as in the §2 of [Lin1], we have for any $p \in \mathcal{P}$,

$$\tau([p]) = \lim_{k\to\infty} \sum_{j=1}^{s(k)} a_j^{(k)} \mathrm{tr}([\psi_j]([p])) \quad \text{uniformly on} \quad \mathrm{T}(A),$$

where tr is the standard normalized trace on projections in interval algebras. What is more, we have $a_j^{(k)} > 0$ and $a_j^{(k)} \to a_j > 0$ uniformly on $\mathrm{T}(A)$ (as $k \to \infty$), in other words, $a_j^{(k)}$ converges to a positive element a_j in $\mathrm{Aff}(\mathrm{T}(A))$. Since A is nuclear, every state on $\mathrm{K}_0(A)$ comes from a trace on A, and therefore we can look at both $a_j^{(k)}$ and a_j as positive elements in $\mathrm{Aff}(\mathrm{S}_{[1]}(\mathrm{K}_0(A)))$, such that $a_j^{(k)} \to a_j > 0$ in $\mathrm{Aff}(\mathrm{S}_{[1]}(\mathrm{K}_0(A)))$.

The image of $\rho\colon \mathrm{K}_0 \to \mathrm{Aff}(\mathrm{S}_{[1]}(\mathrm{K}_0))$ by evaluation is dense [Bla], so Lemma 3.4 of [Lin1] still applies, as by the same argument as the proof of Theorem 4.3 of [Lin1] with $\mathrm{T}(A)$ replaced by $\mathrm{S}_{[1]}(\mathrm{K}_0(A))$, we can eliminate $\Phi^{(0)}$ and fix Φ to get a completely positive linear contraction from A to $B \otimes \mathcal{K}$ which is $\mathcal{F}$-δ-multiplicative and acts in the same way as α on $\mathcal{P}$ at the level of K-theory. Since B is KK-attainable ([Lin1], [Lin2]) and α is invertible, it is easy to see that A is also KK-attainable (for simple TAI algebras) by the definition. Hence by Theorem 1.1, the unital simple nuclear separable TAI algebras which satisfy UCT can be classified by their K-group together with their simplex of tracial states.

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References

[BE1] Bruce Blackadar and Eberhard Kirchberg *Generalized inductive limits of finite-dimensional C*-algebras,* Math. Ann. **307** (1997), 343–380.

[BE2] Bruce Blackadar and Eberhard Kirchberg *Inner quasidiagonality and strong NF algebras,* Pacific J. Math. **198** (2001), 307–329.

[Bla] Bruce Blackadar, *K-Theory for Operator Algebras,* M.S.R.I. Monographs, Vol. 5, Cambridge University Press, Second Edition, 1998.

[DL] M. Dadarlat and T. Loring *A universal multi-coefficient theorem for the Kasparov groups,* Duke J. Math. **84** (1996), 355–377.

[EGL] G. A. Elliott, G. Gong and L. Li *On the classification of simple inductive limit C*-algebras, II: The isomorphism theorem,* Preprint.

[Lin1] Huaxin Lin, *Classification of simple C*-algebras of tracial topological rank zero,* Preprint. (arXiv:math.OA/0011079)

[Lin2] Huaxin Lin, *A Classification theorem for nuclear simple C*-algebras of stable rank one I, II,* Preprints 1999.

[Lin3] Huaxin Lin, *The tracial topological rank of C*-algebras,* Proc. London Math. Soc. **83** (2001), 199–234.

[Ped] G. K. Pedersen, *C*-algebras and their automorphism group,* Academic Press, London/New York/San Francisco, 1979.

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UNIFORM BANACH GROUPS AND STRUCTURES

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Presented by George A. Elliot, FRSC

ABSTRACT. If a Banach space is uniformly homeomorphic to a locally bounded linear space, then the uniform homeomorphism induces a group structure on the Banach space. Such group structures provide examples of uniform Banach groups and were first studied by Per Enflo.

This paper focuses on uniform Banach groups induced by uniform homeomorphisms with two questions in mind. Are the resulting group structures uniformly dissipative with respect to the norm on the Banach space? Can a Banach space be uniformly homeomorphic to a non-normable locally bounded linear space? Our main result answers these questions Yes and No (respectively) if it is assumed that the Banach space has non-trivial (metric) type. The second answer generalizes a theorem of Enflo.

RÉSUMÉ. Si un espace de Banach est uniformément homéomorphe à un espace linéaire localement borné, alors l'homéomorphisme uniforme induit une structure de groupe sur l'espace de Banach. De telles structures de groupe fournissent des examples de groupes de Banach uniformes ont été étudiées pour la première fois par Per Enflo.

Dans cet article nous étudions deux problèmes concernant les groupes de Banach uniformes induit par des homéomorphismes uniformes. Les structures de groupe induites de telle fac(cedilla) on sont-elles uniformément resultantes dissipatives par rapport à la norm sur l'espace de Banach? Un espace de Banach peut-il être uniformément homéomorphe à un espace linéaire localement borné qui n'est pas normable? Oui et notre résultat principal répond à ces questions Oui et Non (respectivement) si on suppose que l'espace de Banach soit de type (métrique) non trivial. La deuxième résponse généralise un théorèm d'Enflo.

1. Introduction The topology of a linear topological space is locally bounded if and only if it is induced by a quasi-norm. A prominent open problem in the uniform theory of metric linear spaces asks whether a non-normable quasi-Banach space can be uniformly homeomorphic to a Banach space. Examples of quasi-Banach spaces include the $L_p(\mu)$-spaces, $0 < p < 1$, endowed with the usual quasi-norm. Enflo [E3] showed that $L_p[0,1]$ is not uniformly homeomorphic to $L_q[0,1]$ if $0 < p < 1 < q < \infty$. Enflo obtained this result by introducing an invariant he called "roundness" and by showing that a non-normable quasi-normed space cannot be uniformly homeomorphic to a Banach space with roundness greater than one.

This paper obtains new results in the spirit of Enflo [E3] by treating roundness as a prototype of the Banach space notion of type via the intermediary of

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metric type, and by executing a detailed analysis of the structure of uniform Banach groups that are induced by a uniform homeomorphism. The advantage of (metric) type is that it has been studied very extensively over the past thirty years and is a considerably more general notion than that of roundness. Our main result shows that if a Banach space X with (metric) type greater than one is uniformly homeomorphic to a locally bounded linear space Y, then the uniform Banach group structure induced on X by the uniform homeomorphism is uniformly dissipative and Y is normable. Consequently, a non-normable quasi-normed space cannot be uniformly homeomorphic to a Banach space with type greater than one. Since many linear spaces are known to be non-normable or to have type greater than one, our result implies new theorems on uniform non-equivalence. The seminal reference on the non-linear classification of Banach spaces is Benyamini and Lindenstrauss [BL].

2. **Roundness and (Metric) Type** The notion of roundness was introduced in Enflo [E1] where it is applied to prove that an infinite-dimensional $L_p(\mu)$-space is not uniformly homeomorphic to $L_q(\eta)$ if $1 \leq p, q \leq 2$ and $p \neq q$.

DEFINITION 2.1. Let $p \in [0, \infty]$. A metric space (X, d) has *roundness* p if p is the supremum of the set of all qs with the following property: for any four points $x_{00}, x_{01}, x_{11}, x_{10} \in X$ we have

$$d(x_{00}, x_{11})^q + d(x_{01}, x_{10})^q \leq d(x_{00}, x_{01})^q + d(x_{01}, x_{11})^q + d(x_{11}, x_{10})^q + d(x_{10}, x_{00})^q.$$

Definition 2.1 can easily be rephrased in terms of 2-cubes. Recall that the unit cube in $\mathbb{R}^n$ ($n \in \mathbb{N}$) is the set of n-vectors $\{0,1\}^n$. An *n-cube* N in an arbitrary metric space (X, d) is a collection of 2^n (not necessarily distinct) points in X where each point in the collection is indexed by a distinct n-vector $\varepsilon \in \{0,1\}^n$ from the unit cube. A *diagonal* in N is a pair of vertices $(x_\varepsilon, x_\delta)$ such that ε and δ differ in all coordinates. An *edge* in N is a pair of vertices $(x_\varepsilon, x_\delta)$ such that ε and δ differ in precisely one coordinate. An n-cube N has 2^{n-1} diagonals and $n2^{n-1}$ edges. If $f = (x, y)$ is an edge or diagonal in N, we will let $l(f)$ denote the d-length of f in X. In other words, $l(f) = d(x, y)$.

Enflo [E1] showed that for any n-cube N in a metric space (X, d) with roundness p we have $l(d_{\min}) \leq n^{1/p} \cdot l(e_{\max})$, where $d_{\min}$ denotes a diagonal of minimal d-length in N and $e_{\max}$ denotes an edge of maximal d-length in N.

Enflo [E1] computed directly the roundness of L_p-spaces as being p whenever $1 \leq p \leq 2$. Lennard, Tonge, and Weston [LTW1] used inequalities of Clarkson to show that L_p-spaces have roundness p' (the conjugate exponent of p) whenever $p \geq 2$. The roundness of many other classical Banach spaces has not been computed and it does not present itself as an easy problem to do so. A theme of this paper is that there are situations wherein roundness can be supplanted with the exceptionally well understood Banach space notion of type via the intermediary of metric type. Metric type was introduced by Bourgain, Milman, and Wolfson [BMW] as the correct metric space analogue of type and is defined as follows.

DEFINITION 2.2. A metric space (X, d) has *metric type p* if there is a constant $\mathcal{B} > 0$ such that for all $n \in \mathbb{N}$ and all n-cubes N in X we have $(\sum l(d)^2)^{1/2} \leq \mathcal{B} \cdot n^{\frac{1}{p}-\frac{1}{2}} \cdot (\sum l(e)^2)^{1/2}$, the sums being taken over all diagonals and edges in N (respectively).

REMARK 2.3. For an n-cube N in a metric space (X, d) that has metric type p (with constant $\mathcal{B}$) it follows that $l(d_{\min}) \leq \mathcal{B} \cdot n^{1/p} \cdot l(e_{\max})$ where (as before) $d_{\min}$ is a diagonal of minimal d-length in N and $e_{\max}$ is an edge of maximal d-length in N. It is this similarity to roundness (modulo the constant $\mathcal{B}$) that suggests the possibility of supplanting roundness with metric type in some non-isometric settings.

Obviously any Banach space that has roundness p will have metric type p with constant $\mathcal{B} = 1$. It is basic that a Banach space which has metric type $p \in [1, 2]$ necessarily has type p. A theorem due to Bourgain, Milman, and Wolfson [BMW] shows that the supremum of types of a Banach space cannot exceed the supremum of its metric types. Consequently, a Banach space has type > 1 if and only if it has metric type > 1. A theorem of Maurey and Pisier [MP] shows that a Banach space X is of type p for some $p > 1$ if and only if it does not contain $\ell_1^{(n)}$ s uniformly. For a Banach space X the type supremum p_X can exceed the roundness of X. For example, if $p > 2$, $L_p[0, 1]$ has type 2 with roundness $p' < 2$. In particular, spaces such as $\ell_1^{(2)} \oplus_\infty \ell_2$ are seen to have roundness one and type greater than one.

3. Uniform Banach Groups and Intrinsic Metrics The notion of a uniform Banach group was introduced and studied in Enflo [E2-3].

DEFINITION 3.1. A *uniform Banach group* G is a Banach space $(X, \|\cdot\|)$ equipped with a group operation $X \times X \to X \colon (x, y) \mapsto x.y$ which is uniformly continuous (as a function of two variables) and such that the zero element of the Banach space is the identity element of the group. We say that the group G is *modelled* on the Banach space $(X, \|\cdot\|)$.

EXAMPLE 3.2. If X is a Banach space, F is a locally bounded linear space, and $\phi \colon X \to F$ is a uniform homeomorphism such that $\phi(0) = 0$, then the operation $x.y = \phi^{-1}(\phi(x) + \phi(y))$ $(x, y \in X)$, introduces an abelian uniform Banach group structure G that is modelled on X. We say that G is the "uniform Banach group induced by ϕ".

DEFINITION 3.3. A uniform Banach group G modelled on a Banach space $(X, \|\cdot\|)$ is said to be *dissipative* if $\|x^n\| \xrightarrow{n} \infty$ for all $x \neq 0$, and G is said to be *uniformly dissipative* if for every $\varepsilon > 0$ and every $K > 0$, there is a constant $N = N(\varepsilon, K)$ such that $\|x^n\| > K$ whenever $\|x\| \geq \varepsilon$ and $n \geq N$.

We show in Theorem 4.2 (vii) that if $(X, \|\cdot\|)$ is a Banach space with (metric) type $p > 1$ and if $\phi\colon X \to F$ is a uniform homeomorphism as in Example 3.2, then the uniform Banach group G induced by ϕ is uniformly dissipative. This surprising new result indicates that such uniform homeomorphisms ϕ exhibit similar radial behaviour to linear isomorphisms. (Note, in this context, that $x^n = \phi^{-1}(n\phi(x))$.)

An important property of uniform Banach groups is that they admit G-invariant metrics which are uniformly equivalent to the norm. If G is a uniform Banach group modelled on a Banach space $(X, \|\cdot\|)$, one such metric is given by $d(x,y) = \sup_{w,z\in G} \|w.x.z - w.y.z\|$. This metric d satisfies $d(x,y) \geq \|x - y\|$ for all $x, y \in X$, and it is uniformly equivalent to the norm because multiplication is assumed uniformly continuous in a uniform Banach group. Moreover, for any G-invariant metric d, uniformly equivalent to the norm and satisfying $d(x,y) \geq \|x-y\|$ for all $x, y \in X$, we can introduce an *intrinsic* (or, *chain*) *metric* d_I as follows. An ordered sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ points in G is called a *one-chain* between given points x and y in G if $x = x_0, y = x_n$, and $d(x_i, x_{i+1}) \leq 1$ for all i, $0 \leq i \leq n-1$. We then define the intrinsic distance $d_I(x,y) = \inf \sum d(x_i, x_{i+1})$, where the infimum is taken over all one-chains $\{x_i\}$ between x and y.

REMARK 3.4. Obviously the intrinsic metric d_I is G-invariant and uniformly equivalent to the norm (because d is) and satisfies $d_I(x,y) \geq d(x,y) \geq \|x-y\|$ for all $x, y \in G$. Moreover, as noted by Enflo [E3], there are constants C_1 and K_1 such that

(1) $d_I(x,y) \leq C_1\|x-y\|$ whenever $\|x-y\| \geq 1$, and
(2) $d_I(x,y) \leq K_1\|x-y\|$ whenever $d_I(x,y) \geq 1$.

4. Main Results

We begin with a technical result.

THEOREM 4.1. *Let G be an abelian uniform Banach group modelled on a Banach space $(X, \|\cdot\|)$ with (metric) type $p > 1$ and let d_I be a G-invariant intrinsic metric as in Remark 3.4. Then, there is a constant $K > 0$ such that for each $z \in G$ with $d_I(z,0) \geq 1$ we can find a $u = u(z) \in G$ satisfying $d_I(u^2, z) \leq K$ and $|2d_I(u,0) - d_I(u^2,0)| \leq K \cdot d_I^{1/p}(z,0)$.*

If every element of G has a unique square root and if the square root map is uniformly continuous, then there is a constant C so that every $v \in G$ with $d_I(v,0) \geq 1$ satisfies

$$|2d_I(v,0) - d_I(v^2,0)| \leq C \cdot d_I^{1/p}(v,0).$$

Theorem 4.1 is due to Enflo [E3, Theorem 5.2.1] in the case that the Banach space X has roundness $p > 1$. Enflo's arguments, in light of Remark 2.3, carry over to the more general setting of (metric) type without significant difficulties. No proof of Theorem 4.1 is therefore given.

THEOREM 4.2. *Let $\phi\colon X \to F$ be a uniform homeomorphism such that $\phi(0) = 0$ where X is a Banach space with (metric) type $p > 1$ and F is a quasi-normed space. Let G denote the uniform Banach group structure on X induced by ϕ. Let d and d_I be the G-invariant metrics introduced prior to Remark 3.4. For $z \in X$ and $t \in \mathbb{R}$ define $z^t = \phi^{-1}(t\phi(z))$ and $N(z) = \limsup_{t\to\infty}\{\frac{d_I(z^t,0)}{t}\}$. Define vector space operations on G as follows: Addition is the group operation $(x,y) \mapsto x.y$ and multiplication of z by the scalar t is given by z^t. Then*

(i) *$(G, N(\cdot))$ is a normed space.*
(ii) *For each $z \in G$ we have $\frac{\|z\|}{2} \leq N(z) \leq d_I(z,0)$.*
(iii) *The identity map $(X, \|\cdot\|) \to (G, N(\cdot))\colon z \mapsto z$ is a uniform homeomorphism.*
(iv) *The normed space $(G, N(\cdot))$ is linearly isomorphic to the quasi-normed space F.*
(v) *The quasi-normed space F is normable. That is to say, there exists a norm $M(\cdot)$ on F (uniformly) equivalent to the given quasi-norm $|||\cdot|||$. The norm is given by the formula $M(y) = N(\phi^{-1}(y))$, $y \in F$.*
(vi) *The uniform Banach group G is uniformly dissipative for large distances, meaning that for every $\varepsilon \geq 2$ and every $K > 0$, there is a constant $N = N(\varepsilon, K)$ such that $\|x^n\| > K$ whenever $\|x\| \geq \varepsilon$ and $n \geq N$.*
(vii) *The uniform Banach group G is uniformly dissipative.*

PROOF. A direct calculation shows that $(z^t)^s = z^{st}$ for all $z \in G$ and $s,t \in \mathbb{R}$. Using this, a routine change of variable argument establishes that $N(z^t) = |t| \cdot N(z)$ for all $z \in G$ and $t \in \mathbb{R}$. Another simple check shows that $(x.y)^t = x^t.y^t$ for all $x, y \in G$ and $t \in \mathbb{R}$. And hence $d_I((x.y)^t, 0) = d_I(x^t.y^t, 0) \leq d_I(x^t, 0) + d_I(y^t, 0)$ (for all $x, y \in G$ and $t \in \mathbb{R}$) by G-invariance. Thus we have $N(x.y) \leq N(x) + N(y)$ for all $x, y \in G$.

The G-invariance of d_I also implies that $d_I(z^t, 0) \leq t d_I(z, 0)$ for all $z \in G$ and $t \in \mathbb{N}$. And, for a fixed $z \in G$, the map $\mathbb{R} \to (G, d_I)\colon t \mapsto z^t = \phi^{-1}(t\phi(z))$ is uniformly continuous. Hence, for each fixed $z \in G$, there is a constant C_z such that $d_I(z^t, 0) \leq t d_I(z, 0) + C_z$ for all $t \in \mathbb{R}$. By definition of $N(\cdot)$, this implies that $N(z) \leq d_I(z, 0)$ for all $z \in G$.

Remembering that X has (metric) type $p > 1$, we may apply Theorem 4.1 to infer there is a constant C so that $|d_I(v^2, 0) - 2d_I(v, 0)| \leq C \cdot (1 \vee d_I(v, 0))^{1/p}$ for all $v \in G$. So, if $d_I(v, 0) \geq 1$, then for every $n \in \mathbb{N}$ we see that

$$\begin{aligned} |d_I(v^{2^n}, 0) - 2^n d_I(v, 0)| &= \left| \sum_{j=1}^{n} 2^{n-j} \left(d_I(v^{2^j}, 0) - 2d_I(v^{2^{j-1}}, 0) \right) \right| \\ &\leq \sum_{j=1}^{n} 2^{n-j} |d_I(v^{2^j}, 0) - 2d_I(v^{2^{j-1}}, 0)| \end{aligned}$$

$$\leq \sum_{j=1}^{n} 2^{n-j} C\big(1 \vee d_I(v^{2^{j-1}}, 0)\big)^{1/p}$$

$$\leq 2^n C \sum_{j=1}^{n} 2^{-j} \left(2^{j-1} d_I(v,0)\right)^{1/p}$$

$$\leq 2^n K d_I(v,0)^{1/p},$$

for some constant K. Since $1 - \frac{1}{p} > 0$ it follows that

$$N(v) \geq \limsup_{n\to\infty} 2^{-n} d_I(v^{2^n}, 0) \geq d_I(v,0) - K d_I^{1/p}(v,0) \geq \frac{d_I(v,0)}{2},$$

provided $d_I(v,0)$ is sufficiently large — say; $d_I(v,0) \geq L$. So if $\|v\| = L$, whence $d_I(v,0) \geq L$, we see that $N(v) \geq \frac{d_I(v,0)}{2} \geq \frac{L}{2} = \frac{\|v\|}{2}$. But then $d_I(v,0) \geq N(v) \geq \frac{\|v\|}{2}$ for all $v \in G$ because $N(\cdot)$ is positive homogeneous. This completes the proof of parts (i) — (iii).

The identity map $X \to G\colon z \mapsto z$ is a uniform homeomorphism by part (iii), and $\phi\colon X \to F$ is a uniform homeomorphism by assumption. Hence the map $\phi\colon (G, N(\cdot)) \to F$ is a uniform homeomorphism that is — moreover — linear by definition of the vector space operations on G. In other words, the map $\phi\colon (G, N(\cdot)) \to F$ is a linear isomorphism. This establishes part (iv).

Denote by $|||\cdot|||$ the quasi-norm on F and define $M(\cdot)$ as above. To derive the triangle inequality for M fix $z_1 = \phi(x_1) \in F$ and $z_2 = \phi(x_2) \in F$, and note that

$$\begin{aligned} M(z_1 + z_2) &= N(\phi^{-1}(z_1 + z_2)) = N\big(\phi^{-1}(\phi(x_1) + \phi(x_2))\big) = N(x_1.x_2) \\ &\leq N(x_1) + N(x_2) = N(\phi^{-1}(z_1)) + N(\phi^{-1}(z_2)) = M(z_1) + M(z_2). \end{aligned}$$

To see that M is positive homogeneous fix $z = \phi(x) \in F$ and $\alpha \in \mathbb{R}$, and note that

$$\begin{aligned} M(\alpha z) &= N(\phi^{-1}(\alpha z)) = N(\phi^{-1}(\alpha\phi(x))) = N(x^\alpha) \\ &= |\alpha| \cdot N(x) = |\alpha| \cdot N(\phi^{-1}(z)) = |\alpha| \cdot M(z). \end{aligned}$$

Also note that for any $y \in F$ we have, by part (iv), the following estimates:

$$M(y) = N(\phi^{-1}(y)) \leq \|\phi^{-1}\| \cdot |||y||| \qquad \text{and}$$

$$|||y||| = |||\phi(\phi^{-1}(y))||| \leq \|\phi\| \cdot N(\phi^{-1}(y)) = \|\phi\| \cdot M(y).$$

Hence the norm $M(\cdot)$ is (uniformly) equivalent to the quasi-norm $|||\cdot|||$. This completes the proof of part (v).

To prove part (vi), let $\varepsilon \geq 2$ and $K > 0$ be given. Consider an arbitrary $x \in G$ such that $\|x\| \geq \varepsilon$. By Part (ii) of this theorem, we have $1 \leq \frac{\|x\|}{2} \leq N(x) \leq d_I(x, 0)$. So for any $m \in \mathbb{N}$,

$$d_I(x^m, 0) \geq N(x^m) = m \cdot N(x) \geq \frac{m\|x\|}{2} \geq m \geq 1.$$

We may choose a constant $K_1 \geq 1$ so that $d_I(z, w) \leq K_1 \cdot \|z - w\|$ whenever $d_I(z, w) \geq 1$. (See Remark 3.4.) This gives, in particular, $d_I(x^m, 0) \leq K_1 \cdot \|x^m\|$ for all $m \in \mathbb{N}$.

Let N be the smallest natural number k such that $k\varepsilon > 2K_1K$, and note that if $n \geq N$ then

$$\|x^n\| \geq \frac{d_I(x^n, 0)}{K_1} \geq \frac{N(x^n)}{K_1} = \frac{n \cdot N(x)}{K_1} \geq \frac{n \cdot \|x\|}{2K_1} \geq \frac{n\varepsilon}{2K_1} > K.$$

To prove (vii), let $\varepsilon > 0$ and $K > 0$ be given. Consider an arbitrary $x \in G$ with $\|x\| \geq \varepsilon$. By Part (ii) of this theorem, $N(x) \geq \frac{\|x\|}{2} \geq \frac{\varepsilon}{2}$. Let M be the smallest natural number k such that $k \cdot \frac{\varepsilon}{2} \geq 2K_1$ where $K_1 \geq 1$ is as in Part (vi) of this theorem. Then $N(x^k) = k \cdot N(x) \geq \frac{M\varepsilon}{2} \geq 2K_1$ for all $k \geq M$. In particular, $2K_1 \leq N(x^k) \leq d_I(x^k, 0) \leq K_1 \cdot \|x^k\|$, and so we have $\|x^k\| \geq 2$ for all $k \geq M$. We now concentrate on M and consider the constant $N = N(2, 2KK_1)$ as given in Part (vi) of this theorem. Then we have $\|(x^M)^n\| = \|x^{Mn}\| > 2KK_1$ whenever $n \geq N$. In particular, we see that for any $k \geq MN$ we have $KK_1 < \frac{\|x^{MN}\|}{2} \leq N(x^{MN}) \leq N(x^k) \leq d_I(x^k, 0) \leq K_1 \cdot \|x^k\|$; and so, $\|x^k\| > K$, establishing uniform dissipativity. ■

COROLLARY 4.3. *If a locally bounded linear space $(F, |||\cdot|||)$ is uniformly homeomorphic to a Banach space $(X, \|\cdot\|)$ with (metric) type $p > 1$, then there exists a norm on F that is uniformly equivalent to the quasi-norm $|||\cdot|||$. In particular, a non-normable quasi-Banach space cannot be uniformly homeomorphic to a Banach space with non-trivial type.*

PROOF. This follows from Theorem 4.2 (v) modulo a translation of the uniform homeomorphism. ■

The (supremal) type of many Banach spaces has been computed. The following families of Banach spaces are all known to have type greater than one: commutative $L_p(\mu)$-spaces with $p \in (1, \infty)$, Schatten-von Neumann classes C_p and other non-commutative L_p-spaces with $p \in (1, \infty)$, Lorentz $L_{p,q}$-spaces if both $p, q \in (1, \infty)$, and Orlicz spaces L_Φ with an appropriate condition on the Orlicz function Φ (see Corollary 4.4 below). Examples of non-normable quasi-normed spaces include the various $L_p(\mu)$ and H_p spaces with $p \in (0, 1)$, and Orlicz spaces L_Ψ with an appropriate condition on the Orlicz function Ψ. Corollary 4.3 in tandem with some of these examples leads to new results such as the following:

COROLLARY 4.4. *Consider locally bounded Orlicz spaces L_Φ and L_Ψ corresponding to (possibly different) non-atomic finite measure spaces. If there is a constant K and a $p > 1$ such that*

$$\Phi(\lambda s) \geq K \cdot \lambda p' \cdot \Phi(s)$$

for $s \geq 0$ whenever $\lambda \geq 1$ and if $\liminf_{t \to \infty} \frac{\Psi(t)}{t} = 0$, then L_Φ is not uniformly homeomorphic L_Ψ.

PROOF. L_Φ has type p by a theorem of Kamińska and Turett [KT]. L_Ψ is non-normable since it has a trivial dual by a theorem of Rolewicz [R]. Hence these spaces are not uniformly homeomorphic by Corollary 4.3. ∎

REMARK 4.5. In relation to Corollary 4.4, it should be noted that an Orlicz space L_Υ is locally bounded if and only if $\lim_{\lambda \to 0} \lim_{t \to \infty} \frac{\Upsilon(\lambda t)}{\Upsilon(t)} = 0$. This is a theorem of Rolewicz [R].

It remains an open problem whether a non-normable quasi-normed space can be uniformly homeomorphic to a Banach space with type supremum one.

REFERENCES

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1.*, Amer. Math. Soc., Providence, RI, 2000.

[BMW] J. Bourgain, V. Milman and H. Wolfson, *On type of metric spaces*, Trans. Amer. Math. Soc. **294** (1986), 295–317.

[E1] P. Enflo, *On the nonexistence of uniform homeomorphisms between L_p-spaces*, Ark. Mat. **8** (1969), 103–105.

[E2] P. Enflo, *Uniform structures and square roots in topological groups I*, Israel J. Math. **8** (1970), 230–252.

[E3] P. Enflo, *Uniform structures and square roots in topological groups II*, Israel J. Math. **8** (1970), 253–272.

[KT] A. Kamińska and B. Turett, *Type and cotype in Musielak-Orlicz spaces*, London Math. Soc. Lect. Notes **158** (1990), 165–180.

[LTW1] C. J. Lennard, A. M. Tonge and A. Weston, *Generalized roundness and negative type*, Mich. Math. J. **44** (1997), 37–45.

[MP] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. **58** (1976), 45–90.

[R] S. Rolewicz, *Some remarks on the spaces $N(L)$ and $N(\ell)$*, Studia Math. **18** (1959), 1–9.

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