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TWISTED ALEXANDER POLYNOMIAL FOR SL(2, ℂ)-REPRESENTATIONS AND FIBERED KNOTS

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Presented by Vlastimil Dlab, FRSC

ABSTRACT. In this short note, we study a relation between monicness of twisted Alexander polynomials and fiberedness of knots.

RÉSUMÉ. Dans cet article, nous considérons le polynôme Alexander tordu et les nœuds fibrés.

1. **Introduction.** A knot in the 3-sphere is called *fibered* if its exterior has the structure of a surface bundle over the circle (see [4], [5], [8]). In our previous paper [2], we showed that the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ of a fibered knot K associated to a unimodular representation $\rho\colon \pi_1 K \to \mathrm{SL}(2n, \mathbb{F})$, where $\mathbb{F}$ denotes a field, is expressed as a rational function of monic polynomials. If ρ is the trivial 1-dimensional representation $\mathbf{1}$, then $\Delta_{K,\mathbf{1}}(t)$ is essentially equal to the original Alexander polynomial $\Delta_K(t)$ (see [9] for details). Thus the above result can be regarded as a natural generalization of the property on the Alexander polynomial of fibered knots. In fact, this framework is more powerful than $\Delta_K(t)$, because we can detect a nonfibered knot K with the monic Alexander polynomial of degree $2g(K)$, where $g(K)$ is the genus of K (see [2, Example 4.3]). However, it is an open problem that the monic twisted Alexander polynomials imply fiberedness of knots. More precisely, we propose:

PROBLEM 1.1. Does every non-fibered knot K have a unimodular representation $\rho\colon \pi_1 K \to \mathrm{SL}(2n, \mathbb{F})$ so that $\Delta_{K,\rho}(t)$ is a rational function of nonmonic polynomials?

This problem has an affirmative answer if a knot K has a nonmonic Alexander polynomial. In fact, we have only to take the trivial representation. Hence the essential problem is for a nonfibered knot with a monic Alexander polynomial.

Now in this note, we consider the following problem:

PROBLEM 1.2. For a knot K with a nonmonic Alexander polynomial, is the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ always expressed as a rational function of nonmonic polynomials?

The purpose of this note is to show the answer is No.

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THEOREM 1.3. *There exists a knot K and a representation $\rho\colon \pi_1 K \to \mathrm{SL}(2,\mathbb{C})$ such that $\Delta_K(t)$ is nonmonic and $\Delta_{K,\rho}(t)$ is a monic polynomial.*

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2. Twisted Alexander polynomial. For the knot group $\Gamma = \pi_1 K$, we choose and fix a Wirtinger presentation

$$P(\Gamma) = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_{u-1} \rangle.$$

Then the abelianization homomorphism

$$\alpha\colon \Gamma \to H_1(S^3 - K, \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$$

is given by

$$\alpha(x_1) = \cdots = \alpha(x_u) = t.$$

In this note, we only consider a representation $\rho\colon \Gamma \to \mathrm{SL}(2,\mathbb{C})$ of a knot group Γ.

These maps naturally induce the ring homomorphisms $\tilde{\rho}$ and $\tilde{\alpha}$ from $\mathbb{Z}[\Gamma]$ to $M(2,\mathbb{C})$ and $\mathbb{Z}[t^{\pm 1}]$ respectively, where $M(2,\mathbb{C})$ denotes the matrix algebra of degree 2 over $\mathbb{C}$. Then $\tilde{\rho} \otimes \tilde{\alpha}$ defines a ring homomorphism

$$\mathbb{Z}[\Gamma] \to M(2, \mathbb{C}[t^{\pm 1}]).$$

Let F_u denote the free group on generators $x_1, \ldots, x_u$ and denote by

$$\Phi\colon \mathbb{Z}[F_u] \to M(2, \mathbb{C}[t^{\pm 1}])$$

the composite of the surjection $\mathbb{Z}[F_u] \to \mathbb{Z}[\Gamma]$ induced by the presentation and the map $\mathbb{Z}[\Gamma] \to M(2, \mathbb{C}[t^{\pm 1}])$ given by $\tilde{\rho} \otimes \tilde{\alpha}$.

Let us consider the $(u-1) \times u$ matrix M whose (i,j) component is the 2×2 matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M(2, \mathbb{C}[t^{\pm 1}]),$$

where $\frac{\partial}{\partial x}$ denotes the free differential calculus. This matrix M is called *the Alexander matrix of the presentation $P(\Gamma)$ associated to the representation ρ*.

For $1 \le j \le u$, let us denote by M_j the $(u-1) \times (u-1)$ matrix obtained from M by removing the j-th column. We regard M_j as a $2(u-1) \times 2(u-1)$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$.

Then Wada's *twisted Alexander polynomial* (see [9]) of a knot K for a representation $\rho\colon \pi_1 K \to \mathrm{SL}(2,\mathbb{C})$ is defined to be a rational function

$$\Delta_{K,\rho}(t) = \frac{\det M_j}{\det \Phi(x_j - 1)}$$

and moreover well-defined up to a factor t^{2k} $(k \in \mathbb{Z})$. In particular, we note here that the notion of monic makes sense for $\Delta_{K,\rho}(t)$. See [2, Section 2] for details.

3. **Proof of Theorem 1.3.** Let K be the 5_2-knot. This is the first example of a prime knot with nonmonic Alexander polynomial in Reidemeister's table [6] (see also [1]). In fact, we have $\Delta_K(t) = 2t^2 - 3t + 2$. The knot group $\pi_1 K$ has a presentation

$$P(\pi_1 K) = \langle x, y \mid xwy^{-1}w^{-1} \rangle,$$

where $w = yxy^{-1}x^{-1}yx$. We define a noncommutative representation $\rho_0 \colon \pi_1 K \to \mathrm{SL}(2, \mathbb{C})$ by

$$\rho_0(x) = X = \begin{pmatrix} s_0 & 1 \\ 0 & s_0^{-1} \end{pmatrix} \quad \text{and} \quad \rho_0(y) = Y = \begin{pmatrix} s_0 & 0 \\ 3/2 & s_0^{-1} \end{pmatrix},$$

where s_0 denotes a complex number

$$s_0 = \frac{\sqrt{2}}{4}(1 - \sqrt{-7}) \in \mathbb{C}.$$

As pointed out by Wada in [9], it is convenient to use relations instead of relators for the computation of the Alexander matrix. Thereby applying free differential calculus to the relation $r = xw - wy$, we obtain

$$\frac{\partial r}{\partial x} = 1 + xy - xyxy^{-1}x^{-1} + xyxy^{-1}x^{-1}y - y + yxy^{-1}x^{-1} - yxy^{-1}x^{-1}y.$$

Thus we have the matrix

$$\begin{aligned} M_2 &= \Phi\left(\frac{\partial r}{\partial x}\right) \\ &= I + t^2 XY - tXYXY^{-1}X^{-1} + t^2 XYXY^{-1}X^{-1}Y \\ &\qquad - tY + YXY^{-1}X^{-1} - tYXY^{-1}X^{-1}Y, \end{aligned}$$

where I denotes the identity matrix.

First the denominator of $\Delta_{K,\rho_0}(t)$ is given by

$$\begin{aligned} \det \Phi(y - 1) &= \det(tY - I) \\ &= \det \begin{pmatrix} s_0 t - 1 & 0 \\ 3t/2 & s_0^{-1}t - 1 \end{pmatrix} \\ &= t^2 - \frac{\sqrt{2}}{2}t + 1. \end{aligned}$$

On the other hand, a straightforward computation shows that the numerator of $\Delta_{K,\rho_0}(t)$ is

$$\det M_2 = t^4 - \frac{3\sqrt{2}}{2}t^3 + 3t^2 - \frac{3\sqrt{2}}{2}t + 1.$$

Hence the twisted Alexander polynomial of the 5_2-knot K for the representation $\rho_0\colon \pi_1 K \to \mathrm{SL}(2,\mathbb{C})$ is given by

$$\begin{aligned}\Delta_{K,\rho_0}(t) &= \frac{\det M_2}{\det \Phi(y-1)}\\ &= t^2 - \sqrt{2}t + 1.\end{aligned}$$

This completes the proof of Theorem 1.3.

4. Concluding remarks. Finally, we briefly explain how to construct an $\mathrm{SL}(2,\mathbb{C})$-representation satisfying the property in Theorem 1.3.

For two matrices $A, B \in \mathrm{SL}(2,\mathbb{C})$ which are conjugate in $\mathrm{SL}(2,\mathbb{C})$ and not commutative, there exists a matrix $C \in \mathrm{SL}(2,\mathbb{C})$ such that

$$CAC^{-1} = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad CBC^{-1} = \begin{pmatrix} s & 0 \\ -r & s^{-1} \end{pmatrix},$$

where $r, s \in \mathbb{C} - \{0\}$ (see [7]). Thus to construct an irreducible representation of the fundamental group of the 5_2-knot K, we may assume that

$$\rho(x) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} s & 0 \\ -r & s^{-1} \end{pmatrix}.$$

By an elementary computation, we see that the relation $\rho(x)\rho(w) = \rho(w)\rho(y)$ holds if and only if $f(r,s) = 0$, where

$$f(r,s) = s^4r^3 - (2s^6 - 3s^4 + 2s^2)r^2 + (s^8 - 3s^6 + 6s^4 - 3s^2 + 1)r - (2s^6 - 3s^4 + 2s^2).$$

Hence the space of conjugacy classes of $\mathrm{SL}(2,\mathbb{C})$-irreducible representations of $\pi_1 K$ is described via

$$\mathcal{R} = \{(r,s) \in \mathbb{C}^2 \mid f(r,s) = 0,\ r \neq 0,\ s \neq 0\}.$$

Then each coefficient of the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ can be regarded as a function on $\mathcal{R}$. In particular, the coefficient of the highest degree term of the numerator of $\Delta_{K,\rho}(t)$ is given by

$$g(r,s) = r^2 - (s^2 - 2 + s^{-2})r + 4.$$

Here we have to remark that the denominator of $\Delta_{K,\rho}(t)$ is always a monic polynomial. Consequently the representation $\rho_0 = (-3/2, s_0) \in \mathcal{R}$ in the previous section is one of the solutions of the equations

$$f(r,s) = 0 \quad \text{and} \quad g(r,s) = 1.$$

In principle, similar computations can be made for other 2-bridge knots (see [3], [7] for example), although they will be more complicated.

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UNSTEADY EQUIPARTITION SOLUTIONS TO THE VISCOUS MHD EQUATIONS

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ABSTRACT. The unsteady equipartition solutions are introduced for the viscous magnetohydrodynamics equations.

RÉSUMÉ. Des solutions vacillantes équipartitionelles sont présentées pour les équations magnétohydrodynamiques visqueux.

The viscous magnetohydrodynamics equations have the form:

$$\rho\frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla P + \tilde{\mu}\Delta\mathbf{V} + \frac{1}{\mu}\operatorname{curl}\mathbf{B}\times\mathbf{B} + \mathbf{f}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V}\times\mathbf{B}) + \frac{1}{\mu\sigma}\Delta\mathbf{B}, \tag{2}$$

$$\operatorname{div}\mathbf{V} = 0, \quad \operatorname{div}\mathbf{B} = 0, \tag{3}$$

where $\mathbf{B}$ is the magnetic vector field, $\mathbf{V}$ is the plasma velocity and P is the pressure. We suppose that the plasma density ρ, its viscosity $\tilde{\mu}$, the magnetic permeability μ and the electric conductivity σ are constant and assume that $\mathbf{f}$ is the vector field of external gravitational forces, $\mathbf{f} = -\rho\nabla\Phi$, where $\Phi(t,\mathbf{x})$ is a gravitational potential and denote $\mathbf{x} = (x, y, z)$. For $\mathbf{B} = 0$, the MHD equations turn into the Navier-Stokes equations (NSE) [3], [5]:

$$\rho\frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla P + \tilde{\mu}\Delta\mathbf{V} + \mathbf{f}, \quad \operatorname{div}\mathbf{V} = 0. \tag{4}$$

In this paper we introduce the unsteady equipartition solutions to the viscous MHD equations (1)–(3), which describe a plasma relaxation in the regime of a strong interaction between the plasma velocity $\mathbf{V}(t,\mathbf{x})$ and the magnetic field $\mathbf{B}(t,\mathbf{x})$. The total plasma kinetic and magnetic energies are finite and equal. The generic equipartition solutions have no geometrical symmetries and rapidly decrease at infinity. The solutions exist only when the kinematic viscosity $\nu = \tilde{\mu}\rho^{-1}$ is equal to the magnetic diffusitivity $\nu_m = (\mu\sigma)^{-1}$, $\nu = \nu_m \neq 0$ or $\rho = \mu\sigma\tilde{\mu}$. In the limit $\nu = \nu_m = 0$ or $\tilde{\mu} = 0$, $\sigma = \infty$, the solutions turn into the Chandrasekhar equipartition equilibria of the ideal incompressible plasma [2].

For the unsteady field-aligned equipartition solutions with $\nu = \nu_m \neq 0$, the viscous MHD equations are reduced to the linear system of the vector diffusion

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equation and the incompressibility equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nu_m \Delta \mathbf{B}, \quad \operatorname{div} \mathbf{B} = 0. \tag{5}$$

The exact global solutions under investigation are different from the small Alfvénic fluctuations or waves [1], [4], [6], [7] that satisfy the equation $\delta\mathbf{V} = \pm\delta\mathbf{B}/\sqrt{\rho\mu}$. In the papers [1], [4], [6], [7] no exact solutions were obtained and the necessary condition $\nu = \nu_m$ was not noticed.

We derive exact non-symmetric equipartition solutions to the boundary value problem which model the unsteady ball lightnings, construct the non-equipartition solutions describing a plasma relaxation to an equilibrium with constant magnetic field $\mathbf{B}$ and present the translationally invariant exact solutions with the non-collinear vector fields $\mathbf{V}(t,\mathbf{x})$ and $\mathbf{B}(t,\mathbf{x})$.

THEOREM 1. *The viscous MHD equations* (1)–(3) *for* $\nu = \nu_m$ *or* $\rho = \mu\sigma\tilde{\mu}$ *have the following exact field-aligned equipartition solutions*

$$\mathbf{V}(t,\mathbf{x}) = \pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}(t,\mathbf{x}), \tag{6}$$

$$\mathbf{B}(t,\mathbf{x}) = \frac{1}{(4\pi\nu_m t)^{3/2}} \operatorname{curl} \int_{R^3} \mathbf{A}(\mathbf{x}')\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\nu_m t}\right) dx'\,dy'\,dz', \tag{7}$$

where $\mathbf{A}(\mathbf{x}')$ *is the vector potential for the initial magnetic field:* $\operatorname{curl}\mathbf{A}(\mathbf{x}') = \mathbf{B}(0,\mathbf{x}')$ *and the pressure* P *has the form*

$$P(t,\mathbf{x}) = C - \rho\mathbf{V}^2(t,\mathbf{x})/2 - \rho\Phi(t,\mathbf{x}). \tag{8}$$

PROOF. As is known, the viscous MHD equations (1) have also the form

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{V}\times\operatorname{curl} V + \nu\Delta\mathbf{V} - \nabla(P/\rho + \mathbf{V}^2/2 + \Phi) + \frac{1}{\rho\mu}\operatorname{curl}\mathbf{B}\times\mathbf{B}. \tag{9}$$

Substituting formula (6) we transform equations (9) and (2) to the form

$$\frac{\partial \mathbf{V}}{\partial t} = \nu\Delta\mathbf{V} - \nabla(P/\rho + \mathbf{V}^2/2 + \Phi), \quad \frac{\partial \mathbf{B}}{\partial t} = \nu_m\Delta\mathbf{B}. \tag{10}$$

Applying operator curl to equations (6) and (10) we find the conditions of their compatibility: $\nu = \nu_m$, $P(t,\mathbf{x})/\rho + \mathbf{V}^2(t,\mathbf{x})/2 + \Phi(t,\mathbf{x}) = \text{const}$. For this case, the equations (1)–(3) are reduced to the linear equations (5). The solutions to the scalar diffusion equation $\partial u/\partial t = \nu_m\Delta u$ have the form [5]

$$u(t,\mathbf{x}) = \frac{1}{(4\pi\nu_m t)^{3/2}} \int_{R^3} f(\mathbf{x}')\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\nu_m t}\right) dx'\,dy'\,dz',$$

where $f(\mathbf{x}') = u(0, \mathbf{x}')$ are the initial data. Hence the solutions to the vector equations (5) have the form (7). ∎

REMARK 1. The exact solutions (6), (7) are the unsteady equipartition solutions because the densities of the kinetic and magnetic energies are equal: $\rho\mathbf{V}^2(t,\mathbf{x}) = \mathbf{B}^2(t,\mathbf{x})/\mu$. It is evident that if the vector field $\mathbf{A}(\mathbf{x}')$ is square integrable then the solutions (6), (7) are square integrable either; hence the total kinetic and magnetic energies are finite and equal.

REMARK 2. The derived exact solutions (6)–(8) describe a plasma relaxation in a regime of a strong interaction between the plasma velocity and the magnetic field. Indeed, the very fact of the equality of the densities of the plasma kinetic and magnetic energies and the proportionality (6) mean a strong interaction between the magnetic field $\mathbf{B}(t,\mathbf{x})$ and the plasma velocity $\mathbf{V}(t,\mathbf{x})$. The solutions exist only for the viscous MHD equations (1)–(3) and do not exist for the NSE (4) with the vanishing magnetic field.

BOUNDARY VALUE PROBLEM. The boundary condition for the viscous fluid flow in a domain $\mathcal{D} \subset \mathbb{R}^3$ is the "no-slip" condition: $\mathbf{V}(t,\mathbf{x})|_{\partial\mathcal{D}} = 0$. Therefore the boundary value problem for the equipartition solutions (6)–(8) takes the form:

$$\frac{\partial \mathbf{B}}{\partial t} = \nu_m \Delta \mathbf{B}, \quad \operatorname{div} \mathbf{B} = 0, \quad \mathbf{B}(t,\mathbf{x})|_{\partial\mathcal{D}} = 0. \tag{11}$$

We present some solutions to this problem that model the unsteady ball lightning. Let the domain $\mathcal{D}$ be a ball of radius R. Function $F(\mathbf{x}) = \sin(\alpha|\mathbf{x}|)/(\alpha|\mathbf{x}|)$ satisfies the Helmholtz equation $\Delta F(\mathbf{x}) = -\alpha^2 F(\mathbf{x})$ where α is an arbitrary parameter. Hence for an arbitrary vector $\mathbf{a}$ the vector field $\mathbf{A}(\mathbf{x},\mathbf{a}) = F(\mathbf{x})\mathbf{a}$ satisfies the vector Helmholtz equation $\Delta\mathbf{A}(\mathbf{x}) = -\alpha^2\mathbf{A}(\mathbf{x})$. Therefore the vector field

$$\mathbf{B}(t,\mathbf{x},\mathbf{a}) = e^{-\alpha^2\nu_m t}\operatorname{curl}\big(F(\mathbf{x})\mathbf{a}\big) = e^{-\alpha^2\nu_m t}h(\mathbf{x})\mathbf{x}\times\mathbf{a} \tag{12}$$

satisfies the first two equations (11). Here

$$h(\mathbf{x}) = F'(|\mathbf{x}|)|\mathbf{x}|^{-1} = \cos(\alpha|\mathbf{x}|)|\mathbf{x}|^{-2} - \sin(\alpha|\mathbf{x}|)\alpha^{-1}|\mathbf{x}|^{-3} \tag{13}$$

is a smooth function.

Let $z_1,\ldots,z_k,\ldots$ be the positive roots of the equation $\tan z = z$. As is known, $z_k \approx k\pi + \pi/2$ as $k \longrightarrow \infty$ and $k\pi - \pi/2 < z_k < k\pi + \pi/2$. Formula (13) yields that $h(z_k\alpha^{-1}) = 0$. Therefore the vector field $\mathbf{B}(t,\mathbf{x},\mathbf{a})$ (12) vanishes on the spheres $|\mathbf{x}| = z_k\alpha^{-1}$. Hence for the discrete values of α: $\alpha_1 = z_1R^{-1}$, $\alpha_2 = z_2R^{-1},\ldots$ and for the arbitrary vectors $\mathbf{a} = \mathbf{a}_k$ the vector fields (12) vanish on the sphere $|\mathbf{x}| = R$. Thus the vector fields

$$\mathbf{B}_k(t,\mathbf{x},\mathbf{a}_k) = e^{-z_k^2R^{-2}\nu_m t}\left(\frac{\cos(z_kR^{-1}|\mathbf{x}|)}{|\mathbf{x}|^2} - R\frac{\sin(z_kR^{-1}|\mathbf{x}|)}{z_k|\mathbf{x}|^3}\right)\mathbf{x}\times\mathbf{a}_k \tag{14}$$

satisfy the boundary value problem (11) in the ball $|\mathbf{x}| \le R$. The vector fields (14) have two first integrals $f_1(t,\mathbf{x}) = |\mathbf{x}|$ and $f_2(t,\mathbf{x}) = \mathbf{x} \cdot \mathbf{a}_k$; hence their magnetic field lines are closed circles in the planes orthogonal to the vectors $\mathbf{a}_k$. A more general solution to the boundary value problem (11) is represented by the convergent series

$$\text{(15)} \quad \mathbf{B}(t,\mathbf{x}) = \sum_{k=1}^{\infty} e^{-z_k^2 R^{-2} \nu_m t} \left(\frac{\cos(z_k R^{-1}|\mathbf{x}|)}{|\mathbf{x}|^2} - R\frac{\sin(z_k R^{-1}|\mathbf{x}|)}{z_k |\mathbf{x}|^3} \right) \mathbf{x} \times \mathbf{a}_k,$$

where we suppose that the series $\sum_{k=1}^{\infty} |\mathbf{a}_k|$ is convergent. The unsteady equipartition solutions (6), (8), (15) satisfy the boundary value problem (11) in the ball $|\mathbf{x}| \le R$ and model the ball lightning. The pressure is $P(t,\mathbf{x}) = P_0 - \mathbf{B}^2(t,\mathbf{x})/(2\mu)$ (8) where P_0 is the atmospheric pressure. The solution is extended to the outer space by the trivial solution $P = P_0$, $\mathbf{B} = 0$, $\mathbf{V} = 0$.

A direct calculation using the equations $\tan z_k = z_k$ proves that the total magnetic energy of the field (15) inside the ball $\mathcal{D}$ has the simple form

$$E = \frac{1}{2\mu} \int_{\mathcal{D}} \mathbf{B}^2 \, dx \, dy \, dz = \frac{2\pi R}{3\mu} \sum_{k=1}^{\infty} e^{-2z_k^2 R^{-2} \nu_m t} |\mathbf{a}_k|^2 \frac{z_k^2}{1+z_k^2},$$

for arbitrary vectors $\mathbf{a}_k$. The total plasma kinetic plus magnetic energy for the solutions (6), (15) is equal $2E$.

The solutions (6), (15) possess invariant magnetic surfaces that are the spheres $|\mathbf{x}| = r = \text{const}$. However the solutions have no geometrical symmetries if some of the vectors $\mathbf{a}_k$ were non-collinear. We have $\mathbf{B}(t,\mathbf{x}) = \mathbf{x} \times \boldsymbol{\Omega}(t,|\mathbf{x}|)$ where

$$\boldsymbol{\Omega}(t,r) = \sum_{k=1}^{\infty} e^{-z_k^2 R^{-2} \nu_m t} \left(\frac{\cos(z_k R^{-1} r)}{r^2} - R\frac{\sin(z_k R^{-1} r)}{z_k r^3} \right) \mathbf{a}_k.$$

Therefore on each sphere $|\mathbf{x}| = r$ the vector field $\mathbf{B}(t,\mathbf{x})$ (15) describes a rotation around the vector $\boldsymbol{\Omega}(t,r)$ that depends on t and r. The boundary value problem (11) is transformed to that for the vector $\boldsymbol{\Omega}(t,r)$: $\boldsymbol{\Omega}_t = \nu_m(4r^{-1}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_{rr})$, $\boldsymbol{\Omega}(t,0) = 0$, $\boldsymbol{\Omega}(t,R) = 0$, where $\boldsymbol{\Omega}_t = \partial\boldsymbol{\Omega}/\partial t$, $\boldsymbol{\Omega}_r = \partial\boldsymbol{\Omega}/\partial r$.

PLASMA RELAXATION TO AN EQUILIBRIUM WITH A CONSTANT MAGNETIC FIELD. For any square integrable vector field $\mathbf{B}(t,\mathbf{x})$ (7) we consider the equipartition solition

$$\text{(16)} \quad \mathbf{B}_1(t,\mathbf{x}) = \mathbf{B}(t,\mathbf{x}) + \mathbf{B}_0, \quad \mathbf{V}_1(t,\mathbf{x}) = \pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}(t,\mathbf{x}) \pm \frac{1}{\sqrt{\rho\mu}}\mathbf{B}_0,$$

where the pressure is $P_1(t,\mathbf{x}) = C - \rho\mathbf{V}_1^2(t,\mathbf{x})/2 - \rho\Phi(t,\mathbf{x})$ (8). The formulae (16) define a solution to the equations (1)–(3) because the equations (5), (6) and (8) are satisfied. As is known, the viscous MHD equations (1)–(3) are invariant with respect to the Galilean transform

$$\mathbf{V}_2(t,\mathbf{x}) = \mathbf{V}_1(t,\mathbf{x}-\mathbf{u}t)+\mathbf{u}, \quad \mathbf{B}_2(t,\mathbf{x}) = \mathbf{B}_1(t,\mathbf{x}-\mathbf{u}t), \tag{17}$$

$P_2(t,\mathbf{x}) = P_1(t,\mathbf{x}-\mathbf{u}t)$, where $\mathbf{u}$ is a constant vector. Applying to the exact solutions (16) the Galilean transform (17) with the vector $\mathbf{u} = \mp\mathbf{B}_0/\sqrt{\rho\mu}$ we obtain the non-equipartition exact solutions

$$\mathbf{V}_2(t,\mathbf{x}) = \pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}\left(t,\mathbf{x}\pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}_0t\right), \quad \mathbf{B}_2(t,\mathbf{x}) = \mathbf{B}\left(t,\mathbf{x}\pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}_0t\right)+\mathbf{B}_0,$$

$$P_2(t,\mathbf{x}) = C-\frac{1}{2\mu}\left(\mathbf{B}\left(t,\mathbf{x}\pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}_0t\right)+\mathbf{B}_0\right)^2-\rho\Phi\left(t,\mathbf{x}\pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}_0t\right).$$

For these solutions, the plasma velocity $\mathbf{V}_2(t,\mathbf{x})$ according to the formulae (6), (7) rapidly tends to zero as $|\mathbf{x}| \longrightarrow \infty$ and is square integrable while the magnetic field $\mathbf{B}_2(t,\mathbf{x}) \longrightarrow \mathbf{B}_0$. Therefore the derived non-collinear and non-equipartition solutions describe a plasma relaxation to an equilibrium with a constant magnetic field.

UNSTEADY TRANSLATIONALLY-INVARIANT EXACT SOLUTIONS. The field-aligned equipartition condition (6) can be relaxed for the z-translationally invariant exact solutions that depend on the three variables t, x, y; we suppose $\mathbf{f} = 0$ and denote $\tilde{\mathbf{x}} = (x, y)$.

THEOREM 2. *The viscous MHD equations* (1)–(3) *for* $\nu = \nu_m$ *have an infinite-dimensional family of exact non-collinear solutions*

$$\mathbf{V}(t,\tilde{\mathbf{x}}) = -a\psi_y\hat{\mathbf{e}}_x + a\psi_x\hat{\mathbf{e}}_y + (b\psi+c)\hat{\mathbf{e}}_z, \tag{18}$$

$$\mathbf{B}(t,\tilde{\mathbf{x}}) = -k\psi_y\hat{\mathbf{e}}_x + k\psi_x\hat{\mathbf{e}}_y + (\ell\psi+m)\hat{\mathbf{e}}_z, \tag{19}$$

where $\psi_x = \partial\psi/\partial x$, $\psi_y = \partial\psi/\partial y$, a, b, c, k, ℓ, m *are arbitrary parameters satisfying the equation* $a^2 = k^2(\rho\mu)^{-1}$ *and the flux function* $\psi(t,\tilde{\mathbf{x}})$ *is an arbitrary solution to the 2-dimensional diffusion equation*

$$\psi_t = \nu(\psi_{xx}+\psi_{yy}). \tag{20}$$

The pressure P *is*

$$P = C - a^2\rho(\psi_x^2+\psi_y^2)/2 - \ell^2\psi^2/(2\mu) - \ell m\psi/\mu. \tag{21}$$

PROOF. Indeed, the formulae (18)–(19) imply

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla a^2(\psi_x^2+\psi_y^2)/2 - a^2\Delta\psi\nabla\psi, \tag{22}$$

$$\operatorname{curl}\mathbf{B}\times\mathbf{B} = -\nabla(\ell^2\psi^2/2+\ell m\psi) - k^2\Delta\psi\nabla\psi, \tag{23}$$

$$\mathbf{V}\times\mathbf{B} = \nabla\big((a\ell-bk)\psi^2/2+(am-ck)\psi\big), \tag{24}$$

$$\operatorname{curl}\mathbf{V}\times\mathbf{V} = -\nabla(b^2\psi^2/2+bc\psi) - a^2\Delta\psi\nabla\psi. \tag{25}$$

A substitution of the formulae (22)–(24) reduces the equations (1) and (2) for $a^2 = k^2(\rho\mu)^{-1}$ to the form

$$\frac{\partial \mathbf{V}}{\partial t} = \nu\Delta\mathbf{V} - \nabla P_1, \quad \frac{\partial \mathbf{B}}{\partial t} = \nu_m \Delta\mathbf{B}, \tag{26}$$

where $P_1 = P/\rho + a^2(\psi_x^2 + \psi_y^2)/2 + \ell^2\psi^2/(2\rho\mu) + \ell m\psi/(\rho\mu)$. Applying operator curl to equations (26) we find the conditions of their compatibility with equations (18), (19): $\nu = \nu_m$, $P_1 = \text{const}$. Hence the pressure P is given by the formula (21) and equations (18), (19) and (26) yield that the flux function $\psi(t, \tilde{\mathbf{x}})$ satisfies the diffusion equation (20). The square integrable solutions to the equation (20) have the form [5]

$$\psi(t, \tilde{\mathbf{x}}) = \frac{1}{4\pi\nu t}\int_{R^2} f(\tilde{\mathbf{x}}')\exp\left(-\frac{|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'|^2}{4\nu t}\right) dx'\,dy',$$

where $f(\tilde{\mathbf{x}}') = \psi(0, \tilde{\mathbf{x}}')$ is an arbitrary square integrable function in $\mathbb{R}^2$. ∎

REMARK 3. The z-invariant solutions (18)–(21) are rather generic for they do not belong to any simpler class of exact solutions. Indeed, formula (23) implies that the solutions are not force-free, formula (24) shows that they are not field-aligned and formula (25) yields that the flows $\mathbf{V}(t, \tilde{\mathbf{x}})$ are not the Beltrami flows. The solutions are not the equipartition ones either because $\rho\mathbf{V}^2(t, \tilde{\mathbf{x}}) - \mathbf{B}^2(t, \tilde{\mathbf{x}})/\mu = \rho(b\psi + c)^2 - (\ell\psi + m)^2/\mu \neq 0$.

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THE RIESZ PROPERTY FOR THE K_*-GROUP OF A C^*-ALGEBRA OF MINIMAL STABLE AND REAL RANK

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RÉSUMÉ. Il est démontré que le groupe ordonné somme directe des groupes de K-théorie C^*-algébriques paire et impaire d'une C^*-algèbre et de rang réel zéro et de rang stable un possède la propriété de Riesz, sous l'assomption technique que le groupe ordonné pair soit faiblement imperforé.

1. Introduction. The equivalent decomposition and interpolation properties of Riesz and Birkhoff ([11], [1], [8]) for ordered abelian groups play an important role in the classification theory of C^*-algebras. In particular these notions have been pivotal in describing exactly which collections of K-groups are attained when the C^*-algebras range in the classifiable classes.

Examples of K-groups without the Riesz property are known—even just for the K_0-group, and even in the setting of minimal real rank (*cf.* [9]). As pointed out in Theorem 3.2 of [6], under the hypothesis of both minimal real rank and minimal stable rank, the K_0-group has the Riesz property. (This is an immediate consequence of the corresponding result of Zhang in [12] for the semigroup of Murray-von Neumann equivalence classes of projections, which uses minimal real rank alone, together with cancellation in this semigroup, which, in the presence of minimal real rank, is known to be equivalent to minimal stable rank.)

It was asserted in Theorem 3.2 of [6] that also the ordered group $K_*(\mathcal{A})$ ($= K_0(\mathcal{A}) \oplus K_1(\mathcal{A})$) defined there has the Riesz property, assuming again only minimal real and stable rank. Regrettably, as was discovered when trying to generalize this statement to the case of K-groups with torsion coefficients, the proof given in [6] is incorrect.

An alternative approach in [5] can be used to prove that the K_*-group of any C^*-algebra given as an approximately homogeneous algebra, and with minimal ranks, has the Riesz property. It is the purpose of the present note to demonstrate how to fill the gap in the *a priori* much more general case that $K_0(\mathcal{A})$ is weakly unperforated.

Our approach is based on checking certain properties of the natural map associating to each (order) ideal I of $K_0(\mathcal{A})$ the (unique) subgroup $\lfloor I \rfloor$ of $K_1(\mathcal{A})$ such that

$$I \oplus \lfloor I \rfloor$$

is an ideal of $K_*(\mathcal{A})$. This point of view is valuable in its own right, although for instance when discussing range questions, a direct characterization of the property of the order in question is often more desirable. A link between results

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of these two types, which will be our main tool, was discovered by the second-named author and G. Gong [7]. The result is repeated as Theorem 2.8 below.

After this work was completed, but before it had been written up, L. G. Brown in [3], using completely different methods, found a way to prove Riesz interpolation in the ordered K_*-group of any C^*-algebra with minimal ranks. This generalizes our result and verifies the original assertion of [6, Theorem 3.2]. Although our proof is less general, we believe its focus on the interplay between the graded ideal structure of $K_*(\mathcal{A})$ and the Riesz property may be of independent interest.

2. Ideals of $K_*(\mathcal{A})$.

Following [4], for each $i = 0, 1$ and each (closed, two-sided) ideal $\mathcal{I}$ of a C^*-algebra $\mathcal{A}$ we denote by $K_i(\mathcal{I}\|\mathcal{A})$ the image of the i-th K-group of $\mathcal{I}$ in the K-group of $\mathcal{A}$ under $\iota_*\colon K_i(\mathcal{I}) \longrightarrow K_i(\mathcal{A})$, where ι denotes the inclusion map. In most of what follows, we could identify this with $K_i(\mathcal{I})$, but we shall refrain from doing so to avoid confusion.

The following definition is based on the observation (from [12, 2.3]) that the lattices of ideals of $K_0(\mathcal{A})$ and of $\mathcal{A}$ are naturally isomorphic for C^*-algebras with minimal ranks.

DEFINITION 2.1. Let $\mathcal{A}$ be a C^*-algebra of real rank zero and stable rank one. When I is an ideal of $K_0(\mathcal{A})$, we define

$$\lfloor I \rfloor = K_1(\mathcal{I}\|\mathcal{A})$$

where $\mathcal{I}$ is the unique ideal of $\mathcal{A}$ with $I = K_0(\mathcal{I}\|\mathcal{A})$.

As in [6], we define the following order structure on $K_*(\mathcal{A})$. We say that $(x, y) \geq 0$ in $K_*(\mathcal{A})$ when $(x, y) = ([p], [u])$ with $p \in M_n(\mathcal{A})$ and $u \in M_n(\mathcal{A})^\sim$ such that

$$u(1-p) = (1-p)u = (1-p).$$

This defines a cone $K_*(\mathcal{A})_+$ in $K_*(\mathcal{A})$.

We reserve the term *ordered group* for those abelian groups equipped with positive cones such that the group is positively generated and such that only the neutral element is simultaneously positive and negative. The following was noted in [6, Theorem 3.2]:

LEMMA 2.2. *When $\mathcal{A}$ is a C^*-algebra of real rank zero and stable rank one,*

$$(K_*(\mathcal{A}), K_*(\mathcal{A})_+)$$

is an ordered group. In fact, $K_*(\mathcal{A}) = K_*(\mathcal{A})_+ - \big(K_0(\mathcal{A})_+ \oplus 0\big)$.

If x is an element of an ordered group, let us denote by $I(x)$ the smallest ideal containing x.

LEMMA 2.3. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero and stable rank one, and let I be an ideal of* $\mathrm{K}_0(\mathcal{A})$. *We then have*

$$\lfloor I \rfloor = \{y \in \mathrm{K}_1(\mathcal{A}) \mid \exists x \in I\colon (x,y) \geq 0\}.$$

Furthermore,

$$\mathrm{K}_*(\mathcal{A})_+ = \{(x,y) \in \mathrm{K}_*(\mathcal{A}) \mid x \geq 0,\ y \in \lfloor I(x) \rfloor\}.$$

PROOF. Given $y \in \lfloor I \rfloor$, choose $y' \in \mathrm{K}_1(\mathcal{I})$ with $\iota_* y' = y$. We may by Lemma 2.2 choose $x' \in \mathrm{K}_0(\mathcal{I})$ with $(x', y') \geq 0$, and since ι induces a positive map also on K_*, we have that $(x,y) \geq 0$ with $x = \iota_* x'$. In the other direction, if $(x,y) \geq 0$ with $x \in I$, we have the pair represented by $p \in \mathrm{M}_n(\mathcal{I})$ and $u \in \mathrm{M}_n(\mathcal{A})^\sim$ with $(1-p)u = u(1-p) = 1-p$; clearly then also p is a unit for $u-1$, and $u \in \mathrm{M}_n(\mathcal{I})^\sim$.

When $x \geq 0$ is represented by the projection p, $I(x) = \mathrm{K}_0(\mathcal{I})$, where $\mathcal{I}$ is generated by p. When $y \in \lfloor I(x) \rfloor$, we can write $y = \iota_* y'$ with $y' \in \mathrm{K}_1(\mathcal{I})$. Say y' is represented by the unitary v; then we can apply [13] to a matrix algebra over $\mathcal{I}$ as p is a full projection there. We get that $u \sim_h v$ with $vp = vp = p$, and thus $y' = [v]$ with

$$v(1-p) = (1-p)v = (1-p),$$

proving that $(x,y) \geq 0$. ■

The argument below, which replaces our less elegant original proof, was communicated to us by L. G. Brown.

PROPOSITION 2.4. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero and stable rank one, and let I and J be ideals of* $\mathrm{K}_0(\mathcal{A})$. *Then*

$$\lfloor I \cap J \rfloor = \lfloor I \rfloor \cap \lfloor J \rfloor,$$
$$\lfloor I + J \rfloor = \lfloor I \rfloor + \lfloor J \rfloor.$$

PROOF. Note that $I \cap J$ and $I + J$ are ideals by the Riesz property of $\mathrm{K}_0(\mathcal{A})$. Also note that since the other inclusions are obvious by monotonicity of $\lfloor \cdot \rfloor$, we only need to establish $\lfloor I \rfloor \cap \lfloor J \rfloor \subseteq \lfloor I \cap J \rfloor$ and $\lfloor I \rfloor + \lfloor J \rfloor \supseteq \lfloor I + J \rfloor$.

Denote by $\mathcal{I}$, $\mathcal{J}$ the (unique) ideals of $\mathcal{A}$ satisfying $I = \mathrm{K}_0(\mathcal{I} \| \mathcal{A})$ and $J = \mathrm{K}_0(\mathcal{J} \| \mathcal{A})$. Considering the map $S(a,b) = a + b$ we obtain the C^*-algebra extension

$$0 \hookrightarrow \ker S \longrightarrow \mathcal{I} \oplus \mathcal{J} \xrightarrow{S} \mathcal{I} + \mathcal{J} \longrightarrow 0.$$

Note that the projection $\pi\colon \mathcal{I} \oplus \mathcal{J} \longrightarrow \mathcal{J}$ restricts to an isomorphism π_S from $\ker S$ to $\mathcal{I} \cap \mathcal{J}$.

Repeating the construction with $\mathcal{A}$ replacing both $\mathcal{I}$ and $\mathcal{J}$, and passing to the six-term exact sequences of Bott periodicity, we end up with the diagram

$$\begin{array}{ccccccccc}
\xrightarrow{\partial} & \mathrm{K}_1(\mathcal{I}\cap\mathcal{J}) & \xrightarrow{\pi_{S_*}^{-1}} & \mathrm{K}_1(\mathcal{I})\oplus\mathrm{K}_1(\mathcal{J}) & \xrightarrow{S_*} & \mathrm{K}_1(\mathcal{I}+\mathcal{J}) & \xrightarrow{\partial} \\
 & \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow[\partial]{} & \mathrm{K}_1(\mathcal{A}) & \xrightarrow[\pi_{S_*}^{-1}]{} & \mathrm{K}_1(\mathcal{A})\oplus\mathrm{K}_1(\mathcal{A}) & \xrightarrow[S_*]{} & \mathrm{K}_1(\mathcal{A}) & \xrightarrow[\partial]{}
\end{array}$$

Clearly,

$$\pi_{S_*}^{-1}x = (x,-x), \quad S_*(x,y) = x+y,$$

and since the index maps vanish (see, *e.g.*, [10, Theorem 5]) and the images of the vertical maps are $\lfloor I\cap J\rfloor$, $\lfloor I\rfloor \oplus \lfloor J\rfloor$, and $\lfloor I+J\rfloor$ respectively, a diagram chase gives the desired inclusions. ■

We recall that a *quotient* of an ordered group G has the form G/I, where I is an ideal of G, and that this again is an ordered group by the induced order. Furthermore, we recall that an ideal of an ordered group is *prime* if, in the quotient by this ideal, any two nonzero ideals have nonzero intersection.

DEFINITION 2.5. ([7, 4.28]) An ordered group G is said to be *liminary* if every quotient of G by a prime ideal is isomorphic to $\mathbb{Z}$.

PROPOSITION 2.6. *Let $\mathcal{A}$ be a separable C^*-algebra with real rank zero and stable rank one. If $I \subseteq J$ are ideals of $\mathrm{K}_0(\mathcal{A})$ with J/I liminary, then $\lfloor I\rfloor = \lfloor J\rfloor$.*

PROOF. Recall (see *e.g.* [10, Theorem 5]) that as $\mathcal{A}$ has minimal real and stable rank, $\mathrm{K}_*(\mathcal{A}/\mathcal{I})$ is equal to $\mathrm{K}_*(\mathcal{A})/\mathrm{K}_*(\mathcal{I}\|\mathcal{A})$ for any ideal $\mathcal{I}$ of $\mathcal{A}$. Hence in particular, it suffices to prove that $\mathrm{K}_1(\mathcal{A}) = 0$ when $\mathrm{K}_0(\mathcal{A})$ is liminary. Let $\mathcal{I}$ be a prime ideal of $\mathcal{A}$. Then, by definition, any two nonzero ideals of $\mathcal{A}/\mathcal{I}$ have nonzero intersection, whence we conclude that $\mathrm{K}_0(\mathcal{I}\|\mathcal{A})$ is a prime ideal of $\mathrm{K}_0(\mathcal{A})$. Hence by our assumption,

$$\mathrm{K}_0(\mathcal{A}/\mathcal{I}) \cong \mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{I}\|\mathcal{A}) \cong \mathbb{Z},$$

so since the stable rank of $\mathcal{A}/\mathcal{I}$ is one, cancellation applies to prove that the class $1 \in \mathbb{Z}$ is represented by a minimal projection p. Invoking the assumption of real rank zero, we conclude that $\dim\big(p(\mathcal{A}/\mathcal{I})p\big) = 1$, and hence $\mathcal{A}/\mathcal{I}$ is elementary. We have proved that every prime quotient of $\mathcal{A}$ is elementary, *i.e.*, that $\mathcal{A}$ is of type I.

Now we may apply [2, Theorem 7], noting that the primitive ideal space has the required property as all ideals are generated by projections. This yields that $\mathcal{A}$ is an AF algebra, and hence that $\mathrm{K}_1(\mathcal{A}) = 0$. ■

Let G be an ordered abelian group with the Riesz interpolation property, and let H be an abelian group. Let $\langle\cdot\rangle$ denote a map from the set of ideals of G to the set of subgroups of H. Let us enumerate the following relevant properties that such a map may have, where I and J are ideals of G:

(I) $I \subseteq J \Rightarrow \langle I\rangle \subseteq \langle J\rangle$;
(II) $(I_\alpha)_A$ upward directed $\Rightarrow \langle \bigcup_A I_\alpha\rangle = \bigcup_A \langle I_\alpha\rangle$;
(III) $\langle 0\rangle = 0$, $\langle G\rangle = H$;
(IV) $\langle I+J\rangle = \langle I\rangle + \langle J\rangle$;
(V) $\langle I\cap J\rangle = \langle I\rangle \cap \langle J\rangle$;
(VI) If J/I is liminary, then $\langle I\rangle = \langle J\rangle$.

The identity map will clearly satisfy (I)–(V). By contrast, the identity map can only meet (VI) when G has no liminary subquotients.

PROPOSITION 2.7. *The map* $\lfloor\cdot\rfloor$ *has the properties* (I)–(VI).

PROOF. (I)–(III) follow by definition and by continuity of K-theory. For (IV) and (V) we need Proposition 2.4, and (VI) follows from Proposition 2.6. ■

THEOREM 2.8. ([7, 4.28]) *Let G be an ordered group with the Riesz property, and let H be an abelian group admitting a map $\langle\cdot\rangle$ associating subgroups of H to order ideals of G as above. Equip $G\oplus H$ with the order*

$$(x,y)\succeq 0 \Longleftrightarrow x\geq 0 \quad \text{and} \quad y\in\langle I(x)\rangle .$$

If G is weakly unperforated, then $(G\oplus H,\succeq)$ has the Riesz interpolation property precisely when $\langle\cdot\rangle$ has the properties (I)–(VI).

It is now clear from the results above that $(\mathrm{K}_*(\mathcal{A}),\succeq)$ has the Riesz property. And by Lemma 2.3, this proves:

COROLLARY 2.9. *If $\mathcal{A}$ is a separable C^*-algebra with real rank zero and stable rank one, and if $\mathrm{K}_0(\mathcal{A})$ is weakly unperforated, then $\mathrm{K}_*(\mathcal{A})$ has the Riesz interpolation property.*

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COMPLEX RANDOM MATRICES AND APPLICATIONS

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Presented by Miklos Csörgo, FRSC

ABSTRACT. The eigenvalue and condition number distributions of complex Wishart matrices are investigated and applied to open problems in information theory.

RÉSUMÉ. On précise la distribution des valeurs propres et du conditionnement de matrices de Wishart complexes appliquées à des problèmes ouverts en théorie de l'information.

1. Introduction. Let an $n \times m$ complex Gaussian random matrix $\mathbf{A}$ be distributed as $\mathbf{A} \sim CN(M, I_n \otimes \Sigma)$ with mean $\mathcal{E}\{\mathbf{A}\} = M$ and covariance $\text{cov}\{\mathbf{A}\} = I_n \otimes \Sigma$. The matrix $\mathbf{W} = \mathbf{A}^H\mathbf{A}$ is called a noncentral Wishart matrix. If $M = \mathbf{0}$, then $\mathbf{W}$ is called a central Wishart matrix. The central and noncentral Wishart distributions are denoted by $CW_m(n, \Sigma)$ and $CW_m(n, \Sigma, \Omega)$, respectively, where $\Omega = \Sigma^{-1}M^HM$.

The joint eigenvalue distributions of complex central Wishart matrices are represented in [4] by complex hypergeometric functions of a matrix argument, which can be expressed in terms of complex zonal polynomials. The distribution of complex noncentral Wishart matrix can also be represented by complex hypergeometric functions; however, in this case, the eigenvalue distributions cannot be solved in terms of zonal polynomials. We derive these distributions using invariant polynomials of two matrix arguments [1], [2], which extend the single matrix argument of zonal polynomials. We also derive the distributions of the largest, the smallest and the single unordered eigenvalues of central and noncentral complex Wishart matrices [7].

The condition number, cond(A), of a matrix A is defined as the positive square root of the ratio of the largest to the smallest eigenvalues of the positive definite Hermitian matrix $W = A^HA$. We assume that the eigenvalues of W are ordered in strictly decreasing order, $\lambda_{\max} = \lambda_1 > \cdots > \lambda_m = \lambda_{\min} > 0$, since the probability that any two eigenvalues of A be equal is zero.

The distributions of $\lambda_{\max}$ and $\lambda_{\min}$ and the condition number distribution of random matrices are studied in [3] (and the references therein) for $\Sigma = I$. In [5] the largest and the smallest eigenvalue distributions of complex Wishart matrices are studied for $\Sigma = \sigma^2 I$. In this report, we derive the eigenvalue distributions of complex central and noncentral Wishart matrices and the condition number

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distribution of complex random matrices for arbitrary Σ. The theory of these random matrices is used to evaluate the capacity of multiple-input, multiple-output (MIMO) wireless communication systems. Note that the capacity of the communication channel expresses the maximum rate at which information can be reliably conveyed by the channel [8].

2. **Complex Wishart matrices.** The joint eigenvalue density of a complex central Wishart matrix is given in [4].

THEOREM 1. *Let* $\mathbf{W} \sim CW_m(n, \Sigma)$ *with* $n > m - 1$. *The joint density of the eigenvalues of* $\mathbf{W}$ *is*

$$(1) \qquad f(\Lambda) = \frac{\pi^{m(m-1)}(\det \Sigma)^{-n}}{C\Gamma_m(m) C\Gamma_m(n)} \prod_{k=1}^{m} \lambda_k^{n-m} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 {}_0F_0(-\Sigma^{-1}, \Lambda)$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ *and the complex multivariate gamma function is*

$$C\Gamma_m(n) = \pi^{m(m-1)/2} \prod_{k=1}^{m} \Gamma(n - k + 1), \quad n > m - 1.$$

The following theorems give the distributions of the largest and smallest eigenvalues of a central Wishart matrix.

THEOREM 2. *If* $W\mathbf{W} \sim CW_m(n, \Sigma)$ $(n \geq m)$ *and* $\lambda_{\max}$ *is the largest eigenvalue of* W, *then*

$$(2) \qquad P(\lambda_{\max} < x) = \frac{C\Gamma_m(m)}{C\Gamma_m(n+m)} \frac{x^{mn}}{(\det \Sigma)^n} {}_1F_1(n, n+m, -x\Sigma^{-1}).$$

THEOREM 3. *If* $\mathbf{W} \sim CW_m(n, \Sigma)$ *and* $\lambda_{\min}$ *is the smallest eigenvalue of* $\mathbf{W}$, *then*

$$(3) \qquad P(\lambda_{\min} > x) = \operatorname{etr}(-x\Sigma^{-1}) \sum_{k=0}^{m(n-m)} \widehat{\sum_{\kappa}} \frac{C_\kappa(x\Sigma^{-1})}{k!},$$

where $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$ *and* $\widehat{\sum}_\kappa$ *denotes summation over the partitions* $\kappa = (k_1, \ldots, k_m)$ *of* k *with* $k_1 \leq n - m$.

The condition number density is given by the following theorem.

THEOREM 4. *Let* $\mathbf{W} \sim CW_m(n, \Sigma)$. *Then the density of* $y = 1 - 1/\operatorname{cond}(W)$ *is*

(4)

$$f(y) = \frac{\pi^{m(m-1)}(\det \Sigma)^{-n}}{C\Gamma_m(m) C\Gamma_m(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma(mn+k) C_\kappa(\Sigma^{-1})}{m^{mn+k} k!\, C_\kappa(I)}$$
$$\times \sum_{t=0}^{\infty} \sum_{\tau,\delta} \frac{(m-n)_\tau g^{\delta}_{\tau,\kappa} y^{(m-1)(m+1)+t+k-1}}{t!} [(m-1)(m+1)+k+t]$$
$$\times \frac{C\Gamma_{m-1}(m-1)}{\pi^{(m-1)(m-2)}} \frac{C\Gamma_{m-1}(m+1,\delta) C\Gamma_{m-1}(m-1)}{C\Gamma_{m-1}(2m,\delta)} C_\delta(I),$$

where κ, τ *and* δ *are the ordered partitions of the nonnegative integers* k, t, *and* $f = k + t$, *respectively, into not more than* m *parts.*

The joint eigenvalue density of a noncentral Wishart matrix is given by following theorems and lemma.

THEOREM 5. *Let* $\mathbf{W} \sim CW_m(n, \Sigma, \Omega)$ *with* $n > m - 1$. *The joint density of the eigenvalues of* $\mathbf{W}$ *is*

$$f(\Lambda) = \frac{\pi^{m(m-1)}(\det \Sigma)^{-n}}{C\Gamma_m(m) C\Gamma_m(n)} \operatorname{etr}(-\Omega) \prod_{k=1}^{m} \lambda_k^{n-m} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2$$
$$\times \sum_{k,t=0}^{\infty} \sum_{\kappa,\tau;\phi\in\kappa.\tau} \frac{C_\phi^{\kappa,\tau}(-\Sigma^{-1}, \Omega\Sigma^{-1}) C_\phi^{\kappa,\tau}(\Lambda, \Lambda)}{k!\, t!\, [n]_\tau C_\phi(I_m)}, \tag{5}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, $C_\phi^{\kappa,\tau}$ *is an invariant polynomial indexed by the ordered partitions* κ, τ *and* ϕ *of the nonnegative integers* k, t, *and* $f = k + t$, *respectively, into not more than* m *parts.*

LEMMA 1. *The following inequality holds:*

$$ {}_0F_1(b; X) < {}_0F_0(X/b), \tag{6}$$

where X *is an* $m \times m$ *complex matrix and* b *is an arbitrary complex number.*

A numerical evaluation shows that this bound is tight. Using Lemma 1, we can express the joint eigenvalue density of a noncentral Wishart matrix as a bounded density function, which is given by the following theorem.

THEOREM 6. *Let* $\mathbf{W} \sim CW_m(n, \Sigma_1, \Omega)$ *with* $n > m - 1$. *The joint density of the eigenvalues of* $\mathbf{W}$ *satisfies the inequality*

$$f(\Lambda) < \frac{\pi^{m(m-1)}(\det \Sigma_1)^{-n}}{C\Gamma_m(m) C\Gamma_m(n)} \operatorname{etr}(-\Omega) \prod_{k=1}^{m} \lambda_k^{n-m} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 {}_0F_0(-\Psi, \Lambda), \tag{7}$$

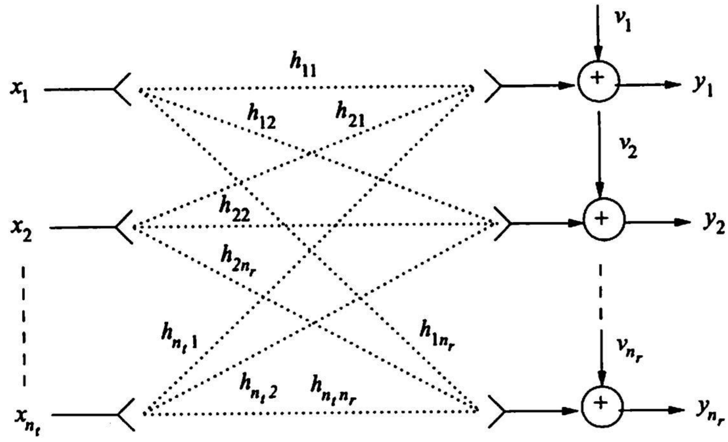

Figure 1: A MIMO communication system.

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$, *the diagonal elements of* $\Psi = \text{diag}(\psi_1, \ldots, \psi_m)$ *are the eigenvalues of the matrix* $(\Sigma_1^{-1} - \Omega\Sigma_1^{-1}/n)$ *and* $\Omega = \Sigma^{-1}M^HM$.

Due to space limitation we only give the distribution of $\lambda_{\max}$.

THEOREM 7. *If* $W \sim CW_m(n, \Sigma, \Omega)$ *and* $\lambda_{\max}$ *is the largest eigenvalue of* W, *then its distribution is given by*

(8) $P(\lambda_{\max} < y)$

$$= \frac{y^{mn} C\Gamma_m(m)\,\text{etr}(-\Omega)}{C\Gamma_m(n+m)(\det\Sigma)^n} \sum_{k,t=0}^{\infty} \sum_{\kappa,\tau;\phi\in\kappa.\tau} \frac{[n]_\phi \theta_\phi^{\kappa,\tau} C_\phi^{\kappa,\tau}(-y\Sigma^{-1}, y\Omega\Sigma^{-1})}{k!\,t!\,[n]_\tau [n+m]_\phi},$$

where $C_\phi^{\kappa,\tau}$ *is an invariant polynomial, indexed by the ordered partitions* κ, τ *and* ϕ *of the nonnegative integers* k, t, *and* $f = k + t$, *respectively, into not more than* m *parts and* $\theta_\phi^{\kappa,\tau} = C_\phi^{\kappa,\tau}(I, I)/C_\phi(I)$.

3. Channel capacity. An n_t-input (or transmitter) and n_r-output (or receiver) MIMO channel can be represented by an $n_r \times n_t$ complex random matrix $\mathbf{H}$, as shown in Figure 1.

The complex vector signal, $y = \mathbf{H}x + v$, received at the j-th output is

$$y_j = \sum_{i=1}^{n_t} h_{ij} x_i + v_j, \tag{9}$$

where h_{ij} is the complex channel coefficient between input i and output j, x_i is the complex signal at the i-th input and v_j is complex Gaussian noise. The total power of the input is constrained to ρ,

$$\mathcal{E}\{x^H x\} \leq \rho \quad \text{or} \quad \operatorname{tr}\mathcal{E}\{xx^H\} \leq \rho.$$

Assume $\mathbf{H}$ is a complex Gaussian random matrix and its realization is known to the receiver. If $W = \mathbf{H}^H\mathbf{H}$, then the capacity of this MIMO channel is [8]

$$C = \mathcal{E}_{\mathbf{W}}\{\log\det(I_{n_t} + (\rho/n_t)\mathbf{W})\}. \tag{10}$$

The channel is Rician distributed if $\mathbf{W} \sim CW_{n_t}(n_r, \Sigma, \Omega)$ and Rayleigh distributed if $\mathbf{W} \sim CW_{n_t}(n_r, \Sigma)$. These are typical satellite and fixed or mobile communication environments, respectively. The capacity (10) is computed using the complex noncentral and central Wishart distributions, respectively. It can also be computed using the joint eigenvalue density $f(\Lambda)$ or the single unordered eigenvalue density $f(\lambda_1)$, that is,

$$\begin{aligned} C &= \mathcal{E}_\Lambda\Big\{\log\Big(\prod_{k=1}^{n_t}[1 + (\rho/n_t)\lambda_k]\Big)\Big\} = \sum_{k=1}^{n_t}\mathcal{E}_{\lambda_k}\{\log(1 + (\rho/n_t)\lambda_k)\} \\ &= n_t\mathcal{E}_{\lambda_1}\{\log(1 + (\rho/n_t)\lambda_1)\}. \end{aligned} \tag{11}$$

We only give the capacity evaluation of a correlated Rayleigh $n_r \times 2$ channel matrix. Thus, we assume that we have a two-input ($n_t = 2$), n_r-output communication system operating over a correlated Rayleigh fading environment. The joint eigenvalue density of a central Wishart matrix depends on the population covariance matrix Σ only through its eigenvalues $v_1, \ldots, v_{n_t}$, *i.e.*,

$${}_0F_0(-\Sigma^{-1}, \Lambda) = {}_0F_0(-\Upsilon^{-1}, \Lambda),$$

where $\Upsilon = \operatorname{diag}(v_1, \ldots, v_{n_t})$. If $n_t = 2$ and $\Upsilon^{-1} = \operatorname{diag}(a_1, a_2)$, then [6]

$${}_0F_0(-\Upsilon^{-1}, \Lambda) = \frac{1}{(a_2 - a_1)(\lambda_1 - \lambda_2)}\{\exp[-(a_1\lambda_1 + a_2\lambda_2)] - \exp[-(a_1\lambda_2 + a_2\lambda_1)]\}. \tag{12}$$

Substituting (12) into (1) and integrating with respect to λ_2 and dividing by 2, we obtain the single unordered eigenvalue density $f(\lambda_1)$ which can be used to evaluate (11).

THEOREM 8. *Consider the two-input correlated Rayleigh channel, i.e.,* $\mathbf{H} \sim CN(0, I_{n_r} \otimes \Sigma)$, *with* $n_r \geq 2$. *If the input power is constrained by* ρ, *then the*

	SNR ρ in dB							
n_r	0 dB	5 dB	10 dB	15 dB	20 dB	25 dB	30 dB	35 dB
2	1.0326	1.9252	3.1157	4.5641	6.2023	7.9419	9.7221	11.5165
4	1.6408	2.8426	4.4118	6.3154	8.4439	10.6803	12.9577	15.2490
6	2.0685	3.4398	5.1852	7.2250	9.4266	11.6948	13.9863	16.2855
8	2.4033	3.8917	5.7454	7.8540	10.0862	12.3653	14.6604	16.9606
10	2.6804	4.2568	6.1838	8.3330	10.5817	12.8666	15.1635	17.4643
12	2.9179	4.5639	6.5437	8.7196	10.9786	13.2669	15.5650	17.8661
14	3.1265	4.8293	6.8489	9.0437	11.3096	13.6003	15.8992	18.2005
16	3.3129	5.0631	7.1139	9.3226	11.5936	13.8860	16.1853	18.4869

Table 1: Capacity in nats for a two-input, n_r-output communication system operating over a correlated Rayleigh fading channel, where the correlation coefficient is equal to 0.9.

capacity C is given by

$$
\begin{aligned}
(13) \qquad C = {} & \frac{a_1^{n_r} a_2}{(a_2 - a_1)\Gamma(n_r)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r-1} e^{-a_1\lambda_1}\, d\lambda_1 \\
& - \frac{a_1 a_2^{n_r}}{(a_2 - a_1)\Gamma(n_r)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r-1} e^{-a_2\lambda_1}\, d\lambda_1 \\
& - \frac{a_1^{n_r}}{(a_2 - a_1)\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r-2} e^{-a_1\lambda_1}\, d\lambda_1 \\
& + \frac{a_2^{n_r}}{(a_2 - a_1)\Gamma(n_r - 1)} \int_0^\infty \log[1 + (\rho/2)\lambda_1]\lambda_1^{n_r-2} e^{-a_2\lambda_1}\, d\lambda_1,
\end{aligned}
$$

where λ_1 is an unordered eigenvalue of $W = H^H H$ and (a_1, a_2) are the eigenvalues of Σ^{-1}.

In (13), C is measured in nats or bits if we use $\log_e$ or $\log_2$, respectively. Table 1 shows the capacity in nats for an $n_r \times 2$ correlated Rayleigh fading channel matrix with correlation coefficient 0.9. Note that each column represents different levels of input power or signal to noise ratio (SNR) in dB. Figure 2 shows the capacity vs. the correlation coefficient. From the table and figure we note the following:

(i) the capacity is decreasing with increasing channel correlation,
(ii) the capacity is increasing with increasing n_r and SNR.

The covariance matrix is $\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$ and its eigenvalues are 1.9 and 0.1. Hence $\Upsilon = \text{diag}(1.9, 0.1)$ and $a_1 = 1/1.9$, $a_2 = 1/0.1$. The non-diagonal element of Σ, called correlation coefficient, gives the correlation between the channel coefficient from different transmitter antennas to a single receiver antenna.

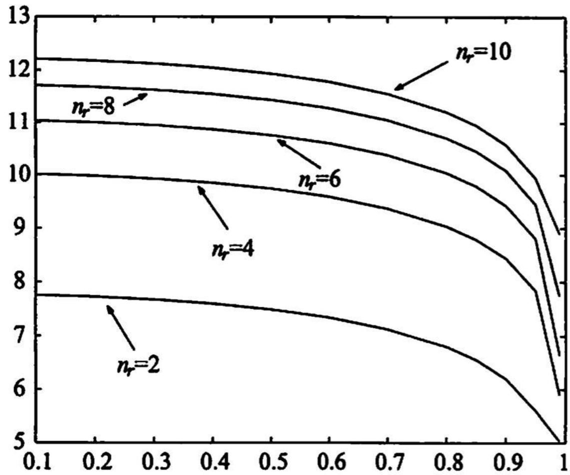

Figure 2: Capacity vs. correlation coefficient for SNR=20dB and $n_t = 2$ and $n_r = 2, 4, 6, 8, 10$, *i.e.*, $\mathbf{H}$ is a $n_r \times 2$ correlated Rayleigh fading channel matrix.

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SPECTRA OF 1-D PERIODIC DIRAC OPERATORS AND SMOOTHNESS OF POTENTIALS

PLAMEN DJAKOV AND BORIS MITYAGIN

Presented by I. Michael Sigal

ABSTRACT. This note presents a series of results on the relationship between the spectra of 1-D Dirac operator with periodic potential (considered with periodic, antiperiodic, or Dirichlet boundary conditions) and the smoothness of its potential.

RÉSUMÉ. Dans cette note nous présentons une série de résultats sur les liens entre un opérateur de Dirac 1-D a potentiel périodique, avec condition au bord périodique, anti-périodique ou de Dirichlet, et la régularité de ce potentiel.

0. We consider Dirac operators

$$L = iJ\frac{d}{dx} + V, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$$

on $I = [0,1]$ with boundary conditions (bc) of three types: for $F = \binom{f_1}{f_2} \in H^1 \times H^1$,

$$\text{Per}^+: \ F(0) = F(1); \quad \text{Per}^-: \ F(0) = -F(1)$$

and

$$\text{Dir}: \ f_1(0) = f_2(0), \quad f_1(1) = f_2(1).$$

See basic facts and further references on 1-D Dirac operators in [7], [9], [10], [14]. A potential V is assumed to be in $L^2(I)$, *i.e.*, *complex valued* functions p, q are in $L^2(I)$. We consider them as periodic functions on $\mathbb{R}$, $V(x+1) = V(x)$, and their smoothness is measured by a weight sequence

$$(1) \qquad \Omega(k) > 0, \quad \Omega(k) = \Omega(-k), \quad \Omega(k) \nearrow \infty \text{ as } k \nearrow \infty,$$

i.e., $V \in H(\Omega)$ if $p = \sum p_k \exp(2\pi ikx)$, $q = \sum q_k \exp(2\pi ikx)$ and

$$\|V|H(\Omega)\|^2 = \sum (|p_k|^2 + |q_k|^2)(\Omega(2k))^2 < \infty.$$

If $\Omega(k) = (1+k^2)^{\alpha/2}$, $\alpha > 0$, then $H(\Omega)$ is a Sobolev space H^α.

Operators L with symmetric (or skew-symmetric) matrix functions V help to solve—after Zaharov-Shabat [15]—non-linear Schrödinger equations. Although our results are new even for selfadjoint V, let us point out again that we consider *any L^2-potentials with complex valued p, q.*

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1. As is shown in [11], if $V \in H(\Omega)$ and $\|V|L^2\| \le m, \|V|H(\Omega)\| \le M$, then the spectrum $\sigma_{bc}(L)$ is discrete, and moreover, for a large enough N it consists of two parts: a finite one, inside the strip $\{|Re z| \le N\pi\}$, and its complement, which lies in the union of disks

$$D_k = D(\pi k, \varepsilon_k) = \{z \in \mathbb{C}: |z - \pi k| < \varepsilon_k\}, \quad \varepsilon_k = \left(1/\Omega(k) + 1/\sqrt{k}\right)^{1/2}.$$

This observation (see [11, Theorems 1 and 2]) improves (and completes) weaker versions of "Counting Lemma" given in [8], [3], [4]. It guarantees that each disk D_k, $|k| > N$, contains three and only three spectral points λ_k^+, λ_k^-, μ_k, where, with

$$P_k = \frac{1}{2\pi i} \int_{\partial D_k} (z - L_{bc})^{-1}\, dz, \quad |k| > N,$$

we have $\{\lambda_k^+, \lambda_k^-\} = \sigma(P_k L_{bc} P_k)$, $bc = \mathrm{Per}^+$ if k is even, $bc = \mathrm{Per}^-$ if k is odd, and $\{\mu_k\} = \sigma(P_k L_{\mathrm{Dir}} P_k)$ for $|k| > N$. (We put $\lambda_k^+ = \lambda_k^-$ if this is a double root of $\det(z - P_k L_{bc} P_k)$, $bc = \mathrm{Per}^\pm$, even if its geometric multiplicity is *one*, *i.e.*, this block $P_k L_{bc} P_k$ is Jordan.) We'll need also the following result from [11, Proposition 7].

LEMMA 1. *Let P_k^0 be defined by the operator L with zero potential; then* $\|P_k - P_k^0 \colon L^2 \to L^\infty\| \le \varepsilon_k$.

We analyze the relationship between the rate of decay of sequences

$$\gamma_k = |\lambda_k^+ - \lambda_k^-|, \quad \delta_k = \left|\mu_k - \frac{1}{2}(\lambda_k^+ + \lambda_k^-)\right| \tag{2}$$

and the smoothness of potential V. In the case of selfadjoint V, λ_k's are real and $\gamma_k = \lambda_k^+ - \lambda_k^-$ are the spectral gaps of Dirac operator L on the real line $\mathbb{R}$.

We will often require that Ω is a *submultiplicative sequence, i.e.*,

$$\exists C > 0 \quad \Omega(k+n) \le C\Omega(k)\Omega(n), \quad k, n \in \mathbb{Z}. \tag{3}$$

A weight $\Omega \in (1)$ is said to be *slowly increasing* if

$$\sup_n \Omega(2n)/\Omega(n) < \infty. \tag{4}$$

We consider also a class of *rapidly increasing* weights of the form

$$\Omega(n) = \exp\big(\varphi(\log|n|)\big), \quad |n| \ge 1, \tag{5}$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ is a twice-differentiable function such that

$$\varphi'(t) \nearrow \infty \text{ as } t \nearrow \infty; \tag{5.a}$$

$$e^t/\varphi'(t) \nearrow \infty \text{ as } t \nearrow \infty; \tag{5.b}$$

$$\liminf_{t\to\infty} [\varphi'(t) - \varphi''(t)]/\log\varphi'(t) > 1. \tag{5.c}$$

2. Our main result is the following statement.

THEOREM 2. *Let Ω satisfy the conditions* (1), (3) *and either* (4) *or* (5.a)–(5.c), *and let $V \in L^2(I)$.*

(i) *If V is selfadjoint, and $V \in L^2(\Omega)$, with Ω satisfying* (1), (3) *and either* (4) *or* (5.a)–(5.c), *then a condition*

$$\sum |\lambda_n^+ - \lambda_n^-|^2 (\Omega(2n))^2 < \infty \tag{6}$$

implies $V \in H(\Omega)$.

(ii) *For any complex-valued $p, q \in L^2$, condition* (6) *together with a condition*

$$\sum \left|\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)\right|^2 (\Omega(2n))^2 < \infty \tag{7}$$

imply $V \in H(\Omega)$.

In the case of Schrödinger operators, similar statements have been proven in [1] and [2]. But, although the formulation of Theorem 2 resembles the main results of these papers, there are new essential difficulties in its proof. Next we give some idea about the proof.

STEP 1. Eigenvalues $\{\lambda_n^{\pm}\}$, $|n| \geq N_*$, $bc = \mathrm{Per}^{\pm}$, are roots of a quasi-quadratic equation

$$\det \begin{pmatrix} \alpha_n^{11}(z) - z & \beta_n^{12}(z) \\ \beta_n^{21}(z) & \alpha_n^{22}(z) - z \end{pmatrix} = 0 \tag{8}$$

(compare [5, (2.14), p. 624], or [1, (14), p. 93]).

This matrix comes from the 2-D operator $P_k L_{bc} P_k$. With $\alpha^{11} = \alpha^{22}$ (compare [5, Lemma 2.2] or [6, Lemma 2.4]) the distance between the two roots of (8) depends on β^{12} and β^{21}. We do not write the explicit formulas for β's but some adjustment and simplification of their structure (with elimination of unessential dependence on z, $|z| \leq \pi/2$) can be seen in formulas (10) and (11) below.

STEP 2. Here we analyze the relationship between the geometry of 2-D blocks generated by the projectors P_k and the structure of the eigenvectors of the matrix in (8). The perturbation type inequality given in Lemma 1 is essential in this analysis that leads to the following crucial claim: there exists $N = N(M)$, $M = \|V|H(\Omega)\|$, such that for $|n| \geq N$ we have

$$A^{-1}(|\beta_n^{12}| + |\beta_n^{21}|) |\lambda_n^+ - \lambda_n^-| + \left|\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)\right| A(|\beta_n^{12}| + |\beta_n^{21}|), \tag{9}$$

where A is an absolute constant.

STEP 3. The estimate (9) allows us to reduce the proof of the theorem to the study of the solution w of the system of non-linear equations

$$\xi_n = w(2n) + \sum_{\nu=1}^{\infty} \Psi_\nu(n, w), \tag{10}$$

$$\Psi_\nu(n, w) = \sum_{j_1,\ldots,j_{2\nu} \neq n} \frac{w(n - j_1)w(j_1 + j_2)(-j_2 - j_3)\cdots w(-j_{2\nu} + n)}{(n - j_1)\cdots(n - j_{2\nu})}. \tag{11}$$

LEMMA 3. *If $w = (w(n)) \in \ell^2(\mathbb{Z})$ and $\xi = (\xi_n) \in \ell^2(\Omega)$, with Ω as in Theorem 2, then $w \in \ell^2(\Omega)$.*

The proof of Lemma 3 is rather long and complicated. In fact, it overcomes the main difficulties in the proof of Theorem 2.

3. In [3] and [4] it is proven for a submultiplicative weight Ω (under some restrictive conditions on Ω) that if $V \in H(\Omega)$ then (6) and (7) hold. By direct estimates of the non-linear operators (11) we can prove that $V \in H(\Omega)$ implies (6) and (7) for any submultiplicative weight Ω, thus in fact conditions (6) and (7) characterize the relation $V \in H(\Omega)$ for weights that satisfy the assumptions of Theorem 2.

Finally, we note that the restrictions on weight Ω in Theorem 2 are essential for validity of the claim. It is easy to see that conditions (5) imply $[\log \Omega(n)]/n \to 0$ as $n \to \infty$. If $\Omega(n) = e^{a|n|}$, then the functions $V \in H(\Omega)$ are analytic in a strip around the real line. In this case, instead of Theorem 2 we have the following proposition:

PROPOSITION 4. *The potential $V \in L^2(I)$ is analytic, i.e.,*

$$|p_n| + |q_n| \le B\exp(-b|n|), \quad B, b > 0, \tag{12}$$

if and only if

$$|\lambda_n^+ - \lambda_n^-| + |\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)| \le A\exp(-a|n|), \quad \exists A, a > 0, \ \forall n \in \mathbb{Z}. \tag{13}$$

This statement is an analog of V. Tkachenko's result [12], [13] about spectral gaps of Hill operators with analytic potentials (see an alternative proof of his result in [2, Section 5.4]). However, as in the Schrödinger case (compare the discussion in [2, Section 5.4]) the choice of b, the type of decay, or the width of the strip of analyticity of an extension of $V(x)$, cannot be determined by a in (13). It depends on the norm $\|V|L^2\|$ as well. This phenomenon leads us almost with necessity to restrictions (4) and (5). Certainly, Gevrey weights

$$\Omega(n) = (1 + n^2)^\alpha \exp(a|n|^\gamma), \quad a > 0, \ \gamma \in (0, 1),$$

satisfy these restrictions because (5.a)–(5.c) hold for $\varphi(t) = \log(\Omega(e^t))$.

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