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SOME RECENT MATHEMATICAL DEVELOPMENTS IN GENERAL RELATIVITY

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RÉSUMÉ. Nous passons en revue un certain nombre de résultats récents portant sur l'étude analytique des équations d'Einstein en relativité générale. Parmi les questions qui sont considérées figurent le problème du collapse gravitationnel et de la formation de singularités, ainsi que celui du comportement à long terme des solutions de l'équation de Dirac dans la métrique d'un trou noir chargé en rotation.

1. **Introduction.** Einstein's general theory of relativity is widely recognized as one of the outstanding scientific achievements in the history of humanity. It carries a strong aesthetic appeal for physicists and mathematicians alike, and continues to occupy an important place in mathematics as the source of many developments in differential geometry and partial differential equations.

Our purpose in this paper is to give a non-technical overview of some significant recent advances in general relativity dealing with the mathematical study of the interaction of gravity with classical fields. These developments include definitive results due to Christodoulou, [3], [4], [5], on the very important question of gravitational collapse and singularity formation, and striking theorems of Finster, Smoller and S.-T. Yau on the existence of regular solutions and the non-existence of black hole solutions when gravity is coupled to spinor and gauge fields, [10], [11], [12], [13]. We will also present the recent advances obtained by Finster, Kamran, Smoller and S.-T. Yau, [7], [8], [9], on the long-term dynamics of spinor fields in the geometry of a charged rotating black hole.

General relativity is geometric and relativistic theory of gravitation. Space-time is thus assumed to be a four-dimensional C^∞ manifold M_4, endowed with a pseudo-Riemannian metric g of Lorentzian signature $(+,-,-,-)$. The tangent space at each point of space-time space-time is therefore isomorphic to Minkowski space-time and space-time is locally Minkowskian. The world lines of massive test particles are time-like geodesic curves in (M_4, g) while light rays propagate along null geodesics. The *Jacobi equation* shows how space-time curvature governs the behavior of a one-parameter family of neighboring geodesics through the *Jacobi field* n connecting two infinitesimally close geodesics,

$$\frac{d^2n^i}{d\lambda^2} + R^i_{uju}n^i = 0. \tag{1}$$

The gravitational field thus manifests itself through space-time curvature, and the requirement that one recover Newtonian gravity in the non-relativistic limit

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leads to the *Einstein field equations of gravitation*, which are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij}, \tag{2}$$

and which reduce to Poisson's equation in the Newtonian limit. The left-hand side of (2) is a symmetric tensor field which depends on the first and second derivatives of the components of the metric, and is divergence-free as a consequence of the Bianchi identities. This tensor field is known as the *Einstein tensor*. The right hand-side of (2) is the *energy-momentum tensor* of the matter fields interacting with gravity, and is symmetric and divergence-free as a consequence of the field equations for the matter fields. The Einstein field equations thus need to be augmented with a set of field equations for the matter fields involved. For example, in the case of the interaction of gravity with electromagnetism, one obtains the source-free *Einstein-Maxwell equations*,

$$R_{ij} = 8\pi\Big(F_{ik}F_j^k - \frac{1}{4}g_{ij}F_{kl}F^{kl}\Big), \tag{3}$$

$$\nabla_l F^{kl} = 0, \quad \nabla_{[i}F_{jk]} = 0. \tag{4}$$

The Einstein field equations (2) are a complicated system of non-linear partial differential equations, which are extremely difficult to analyze in full generality.[1] A great deal of research activity has thus been aimed at finding *exact* solutions of these equations. It is fair to say that one of the most important challenges facing mathematical relativists is to come up with solutions that describe equilibrium configurations or dynamical behavior which are relevant to astrophysical situations requiring a general relativistic treatment. In the remainder of this paper, we will be particularly concerned with and the phenomenon of singularity formation.

From the classical work of Chandrasekhar, it is known that a star which is sufficiently massive should eventually collapse gravitationally and settle into a configuration known as a *black hole*. Amongst the most remarkable properties of black holes, stand the fact they lead to a *singularity* in the geometric structure of space time, and a natural boundary for the outer region of space-time, known as an *event horizon*. It is a very striking fact that the equilibrium configurations of a charged black hole are described by known exact solutions of the Einstein field equations (2). In the next section, we present these exact solutions, and state the uniqueness theorem which characterizes them as black hole equilibrium configurations.

2. The black hole uniqueness theorem.

Soon after the publication by Einstein of the field equations (2), a one-parameter family of exact spherically

[1]The celebrated theorem of Christodoulou and Klainerman on the non-linear stability of Minkowski space [6] is to this day perhaps the most general global theorem pertaining to the Einstein equations.

symmetric solutions of the vacuum field equations

$$R_{ij} = 0, \tag{5}$$

was discovered by by the astronomer Karl Schwarzschild. This solution is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{6}$$

The Schwarzschild solution describes the external gravitational field of a spherically symmetric black hole in equilibrium. The parameter M corresponds to the mass of the black hole measured from infinity and the event horizon is given by the null hypersurface $r = 2M$. The Schwarzschild solution admits a two-parameter family of charged generalizations which are solutions the source-free Einstein-Maxwell equations. These solutions, which are known as the Reissner-Nordström solutions, are given by

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{7}$$

and

$$F_{ij} = \partial_i A_j - \partial_j A_i, \tag{8}$$

where

$$A_i dx^i := -\frac{Q}{r} dt. \tag{9}$$

The parameters M and Q satisfy the inequality[2]

$$M^2 > Q^2 \tag{10}$$

and correspond to the mass and the electric charge of the black hole. We will only consider the non-extreme case in this paper, and will simply refer to these solutions as the Reissner-Nordström solutions. The event horizon is given by the null hypersurface

$$r = r_1 := M + \sqrt{M^2 - Q^2}. \tag{11}$$

The celebrated black hole uniqueness theorems of Carter and Israel, [14], give a precise description of the exterior space-time geometry of a charged rotating black hole in equilibrium:

[2]More precisely, this inequality defines the class of *non-extreme* Reissner-Nordström solutions.

THEOREM 1. *Let (M_4, g) be a time-orientable, causal, pseudo-stationary, axisymmetric and weakly asymptotically simple space-time solving the source-free Einstein-Maxwell equations*

$$R_{ij} = 8\pi\Big(F_{ik}F_j^k - \frac{1}{4}g_{ij}F_{kl}F^{kl}\Big), \tag{12}$$

$$\nabla_l F^{kl} = 0, \quad \nabla_{[i}F_{jk]} = 0. \tag{13}$$

Suppose that the domain of outer communications $\mathcal{E}$ of (M_4, g) is homeomorphic to $\mathbb{R}^2 \times S^2$ and that the boundary of $\mathcal{E}$ is a global past event horizon $\mathcal{H}^+$ homeomorphic to $\mathbb{R} \times S^2$. Then $\mathcal{E}$ admits a global chart (t, r, θ, ϕ) in which the metric is given by the Kerr-Newman solution of the Einstein-Maxwell equations,

$$ds^2 = \frac{\Delta}{U}(dt - a\sin^2\theta d\phi)^2 - U\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) - \frac{\sin^2\theta}{U}\big(adt - (r^2 + a^2)d\phi\big)^2, \tag{14}$$

where

$$U := r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 + Q^2, \tag{15}$$

and where[3]

$$M^2 > a^2 + Q^2. \tag{16}$$

Furthermore, the electromagnetic field is given by

$$F_{ij} = \partial_i A_j - \partial_j A_i, \tag{17}$$

where

$$A_i dx^i := -Q\frac{r}{U}(dt - a\sin^2\theta d\phi). \tag{18}$$

The Kerr-Newman geometry thus depends on three essential parameters M, a and Q, which correspond to the mass, the angular momentum per unit mass and charge of the black hole measured from infinity. The null hypersurface

$$r = r_1 := M + \sqrt{M^2 - a^2 - Q^2} \tag{19}$$

is the global past event horizon $\mathcal{H}^+$ for the Kerr-Newman metric.[4] In the limit $a = 0$, the Kerr-Newman solution reduces to the spherically symmetric Reissner-Nordström solution (7).

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[3]Likewise, this inequality defines the class of *non-extreme* Kerr-Newman solutions.

[4]The hypersurface $r = r_0 := M - \sqrt{M^2 - a^2 - Q^2}$ is the Cauchy horizon for the Kerr-Newman metric.

3. Gravitational collapse and singularity formation. The celebrated singularity theorem of Penrose, and Hawking, [14], give necessary conditions for the formation of a singularity. We state Penrose's theorem, where the necessary conditions involve the important concept a *trapped surface*:

THEOREM 2. *A space-time (M_4, g) cannot be null geodesically complete if:*

(i) $R_{ij}k^i k^j \geq 0$ *for all null vectors* k.
(ii) *There is a non-compact Cauchy surface in* (M_4, g).
(iii) *There is a closed trapped surface in* (M_4, g)*, that is a space-like two-surface* $\mathcal{T}$ *such that the two families of null geodesics orthogonal to* $\mathcal{T}$ *are converging at* $\mathcal{T}$.

Penrose's theorem does not give any details on the dynamical aspects of the formation of singularities and event horizons in terms of the Cauchy data. Our objective in this section is to briefly review some important results of Christodoulou [3], [4], [5] on the dynamics of relativistic gravitational collapse and singularity formation. From a classical theorem of G. D. Birkhoff, [14], we know that all the spherically symmetric solutions of the *vacuum* Einstein field equations

$$R_{ij} = 0, \tag{20}$$

are necessarily *static*, and given by the one-parameter family of Schwarzschild metrics,

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{21}$$

describing the external gravitational field of a spherically symmetric black hole in equilibrium. It follows that any study of the dynamics of gravitational collapse and singularity formation will require the coupling of the gravitational field to some additional matter field. In a series of extremely deep papers, Christodoulou has given definitive answers to these questions in the case in which the gravitational field is coupled to a massless scalar field. We now proceed to describe one of Christodoulou's earlier results, [3].

Using advanced null coordinates (u, r, θ, φ), every spherically symmetric space-time metric can be written in the form

$$ds^2 = e^{-2\nu}du^2 - 2e^{\nu+\lambda}dudr - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{22}$$

where

$$\lambda = \lambda(u, r), \quad \nu = \nu(u, r). \tag{23}$$

The hypersurfaces $u = \text{const}$ are future-pointing null geodesic cones, and the cross-sections $r = \text{const} > 0$ of these cones are diffeomorphic to two-spheres.

We consider the Einstein field equations describing the interaction of a gravitational field with a massless scalar field,

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi\Big(\partial_i\phi\partial_j\phi - \frac{1}{2}g_{ij}\partial_k\phi\partial_k\phi\Big), \tag{24}$$

$$g^{ij}\nabla_i\nabla_j\phi = 0, \tag{25}$$

in the spherically symmetric geometry (22). The Cauchy data for this system can be expressed in terms of a single scalar mass function $m(r)$ on the future-pointing null geodesic cone C_0^+ given by $u = 0$. We have:

THEOREM 3. *Let $r_2 > r_1 > 0$ be such that*

$$0 < \delta := \frac{r_2}{r_1} - 1 < \frac{1}{2}, \tag{26}$$

and suppose that

$$2\big(m(r_2) - m(r_1)\big) \geq \eta(\delta)(1+\delta)^{-1}(r_2 - r_1), \tag{27}$$

where

$$\eta(\delta) := \log\Big(\frac{1}{2\delta}\Big) + 5 - \delta. \tag{28}$$

Then:

(i) *A trapped region forms in the future, terminating at a strictly space-like singular boundary $\mathcal{B}$.*
(ii) *Near $\mathcal{B}$, we have*

$$R_{ijkl}R^{ijkl} \geq \frac{c}{r^6}. \tag{29}$$

More recently, [4], [5], Christodoulou has shown that naked singularities can indeed occur when one considers more general Cauchy data, but that these singularities are unstable in the sense that the set of Cauchy data leading to this type of behavior is of codimension at least two in the space of initial data of bounded variation and of codimension at least one in the space of absolutely continuous initial data.

4. The coupling of gravity to spinor and gauge fields. In a series of ground-breaking papers, [10], [11], Finster, Smoller and S.-T. Yau have obtained definitive results on the difficult questions of the existence of particle-like solutions and black hole solutions to the spherically symmetric Einstein-Dirac-Maxwell and Einstein-Dirac-Yang Mills equations, as well as the qualitative behavior of these solutions in terms of the various coupling constants that appear in the theory. We state a few key results in the Einstein-Dirac-Maxwell case, and refer the reader to the paper [13] and the references therein for the Einstein-Dirac-Yang Mills case.

Consider the Einstein-Dirac-Maxwell equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij}, \tag{30}$$

$$(i\gamma^j D_j - m)\psi = 0, \quad \nabla_k F^{jk} = e\bar{\psi}\gamma^j\psi. \tag{31}$$

where T_{ij} is the sum of the energy-momentum tensor for a Dirac spinor field and the Maxwell energy-momentum tensor, for a spherically symmetric and static metric

$$ds^2 = T^{-2}dt^2 - A^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{32}$$

We first consider the question of the existence of global singularity-free solutions of these equations.

THEOREM 4. *The Einstein-Dirac-Maxwell equations with boundary conditions under which the metric (32) is asymptotically Minkowskian*

$$\lim_{r\to\infty}\big(A(r), T(r)\big) = (1,1), \tag{33}$$

and has finite ADM mass,

$$\lim_{r\to\infty} r\big(1 - A(r)\big) < \infty, \tag{34}$$

admits stable regular spherically symmetric and static solutions for $(\frac{e}{m})^2 << 1$.

The following theorem shows that there are essentially no black hole solutions of the Einstein-Dirac-Maxwell equations for which the Dirac spinor is not identically zero:

THEOREM 5. *Consider the Einstein-Dirac-Maxwell equations with boundary conditions given as follows:*

(i) *Asymptotic flatness,*

$$\lim_{r\to\infty}(T, A) = (1,1). \tag{35}$$

(ii) *There exists* $\rho > 0$ *such that the hypersurface* $r = \rho$ *is a regular event horizon,*

$$\lim_{r\to\rho}(T, A) = (\infty, 0), \tag{36}$$

$$T^{-2}A^{-1}, T^2A^1 \in C^\infty\big([\rho,\infty)\big). \tag{37}$$

(iii) *The electromagnetic field is bounded near the event horizon,*

$$F_{ij}F^{ij} = 2\phi^2 AT^2 \tag{38}$$

where

$$|\phi(r)| < c_1, \quad \textit{for } \rho < r < \rho + \epsilon. \tag{39}$$

Suppose furthermore that near the even horizon, the metric function A has a power law behavior,

$$A(r) = c(r-\rho)^s + O\big((r-\rho)^{s+1}\big), \quad r > \rho, \tag{40}$$

where $c > 0$, $s > 0$. Then

(i) *If $0 < s < 2$, the only solutions of the Einstein-Dirac-Maxwell equations are the two-parameter family of Reissner-Nordström solutions (7) of the Einstein-Maxwell equations, that is the Dirac spinor ψ is identically zero.*
(ii) *If $s > 2$, then the Einstein-Dirac-Maxwell equations have no solution.*

5. The Dirac equation in Kerr-Newman geometry. Our objective in this section is to present a few recent results on the dynamical behavior of Dirac spinor fields in the space-time geometry of a charged rotating black hole, which give a strong indication that singularity formation and the presence of Dirac spinor fields are in some sense mutually exclusive. The scope of our results is limited by the fact that we are not considering the fully coupled Einstein-Dirac Maxwell equations, but rather the Dirac equation in the background of the Kerr-Newman geometry. However, our results are now valid in an axisymmetric background, and not just a spherically symmetric one.

A most remarkable feature of the Dirac equation in Kerr geometry is that it admits a complete separation of variables into ordinary differential equations. This very important discovery was made by Chandrasekhar in 1976, [2], and subsequently interpreted by Carter and McLenaghan [1] in terms of the particular first-order generalized symmetry of the Dirac operator arising from the complete separability of the Hamilton-Jacobi equation for the geodesic flow of the Kerr metric.[5] We briefly recall the form of the ordinary differential equations that arise from the separation of variables.

After performing the regular and time-independent transformation

$$\hat{\psi} = S\psi, \tag{41}$$

where

$$S = \Delta^{\frac{1}{4}} \operatorname{diag}\big((r - ia\cos\vartheta)^{\frac{1}{2}}, (r - ia\cos\vartheta)^{\frac{1}{2}}, (r + ia\cos\vartheta)^{\frac{1}{2}}, (r + ia\cos\vartheta)^{\frac{1}{2}}\big), \tag{42}$$

[5]This separability property also holds true for the charged Dirac operator in the Kerr-Newman geometry.

the Dirac equation can be written as

$$(\mathcal{R}+\mathcal{A})\hat{\psi}=0 \tag{43}$$

with

$$\mathcal{R}=\begin{pmatrix} imr & 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 \\ 0 & -imr & 0 & \sqrt{\Delta}\mathcal{D}_- \\ \sqrt{\Delta}\mathcal{D}_- & 0 & -imr & 0 \\ 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 & imr \end{pmatrix}$$

$$\mathcal{A}=\begin{pmatrix} -am\cos\vartheta & 0 & 0 & \mathcal{L}_+ \\ 0 & am\cos\vartheta & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & -am\cos\vartheta & 0 \\ -\mathcal{L}_- & 0 & 0 & am\cos\vartheta \end{pmatrix}$$

and the differential operators

$$\mathcal{D}_\pm=\frac{\partial}{\partial r}\mp\frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial}{\partial t}+a\frac{\partial}{\partial\varphi}-ieQr\right]$$

$$\mathcal{L}_\pm=\frac{\partial}{\partial\vartheta}+\frac{\cot\vartheta}{2}\mp i\left[a\sin\vartheta\frac{\partial}{\partial t}+\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right].$$

Employing the ansatz

$$\hat{\Psi}(t,r,\vartheta,\varphi)=e^{-i\omega t}e^{-i(k+\frac{1}{2})\varphi}\begin{pmatrix} X_-(r)Y_-(\vartheta) \\ X_+(r)Y_+(\vartheta) \\ X_+(r)Y_-(\vartheta) \\ X_-(r)Y_+(\vartheta) \end{pmatrix}, \quad k\in\mathbf{Z}, \tag{44}$$

we obtain the eigenvalue problems,

$$\mathcal{R}\hat{\Psi}=\lambda\hat{\Psi}, \quad \mathcal{A}\hat{\Psi}=-\lambda\hat{\Psi}, \tag{45}$$

under which the Dirac equation (43) decouples into the system of ODEs

$$\begin{pmatrix} \sqrt{\Delta}\mathcal{D}_+ & imr-\lambda \\ -imr-\lambda & \sqrt{\Delta}\mathcal{D}_- \end{pmatrix}\begin{pmatrix} X_+ \\ X_- \end{pmatrix}=0 \tag{46}$$

$$\begin{pmatrix} \mathcal{L}_+ & -am\cos\vartheta+\lambda \\ am\cos\vartheta+\lambda & -\mathcal{L}_- \end{pmatrix}\begin{pmatrix} Y_+ \\ Y_- \end{pmatrix}=0, \tag{47}$$

where $\mathcal{D}_\pm$ and $\mathcal{L}_\pm$ are the radial and angular operators

$$\mathcal{D}_\pm=\frac{\partial}{\partial r}\pm\frac{i}{\Delta}\left[\omega(r^2+a^2)+\left(k+\frac{1}{2}\right)a+eQr\right] \tag{48}$$

$$\mathcal{L}_\pm=\frac{\partial}{\partial\vartheta}+\frac{\cot\vartheta}{2}\mp\left[a\omega\sin\vartheta+\frac{k+\frac{1}{2}}{\sin\vartheta}\right]. \tag{49}$$

We now consider the Cauchy problem for the Dirac equation with Cauchy data supported outside the event horizon, [8].

THEOREM 6. *Consider the Cauchy problem for the Dirac equation in the Kerr-Newman geometry,*

$$(i\gamma^j D_j - m)\psi(t,x) = 0, \quad \psi(0,x) = \psi_0(x), \tag{50}$$

where the initial data ψ_0 is in $L^2\left((r_1,\infty)\times S^2\right)^4$ and in L^∞_{loc} near the horizon, i.e., $|\psi_0(x)| < c$ for $x \in (r_1, r_1+\epsilon)\times S^2$. Let $\delta > 0$ be given, let $R > r_1 + \delta$ and consider the compact space-like hyper-surface $K_{\delta,R}$ of $\mathcal{E}$ given by

$$K_{\delta,R} := \{(t,r,\theta,\phi) \mid r_1 + \delta \le r \le R, \ t = \text{const.}\}. \tag{51}$$

Then the probability for the Dirac particle to be inside $K_{\delta,R}$ tends to zero as $t \to \infty$, that is

$$\lim_{t\to\infty} \int_{K_{\delta,R}} (\bar\psi\gamma^j\psi)(t,x)\nu_j d\mu = 0, \tag{52}$$

where ν_j denotes the future-pointing normal to $K_{\delta,R}$ and $d\mu$ denotes the induced volume element on $K_{\delta,R}$.

The proof of theorem is based on a representation of the Dirac propagator which was established in [8], and which is given by:

THEOREM 7. *For every Cauchy data $\psi_0 \in C_0^\infty(\mathbb{R}\times S^2)^4$ we have*

$$\psi(x,t) = \frac{1}{\pi}\sum_{k,n\in\mathbb{Z}} \int_{-\infty}^{+\infty} e^{-i\omega t} \sum_{a,b=1}^{2} t_{ab}^{k\omega n} \psi_a^{k\omega n}(x) \langle \psi_b^{k\omega n} \mid \psi_0\rangle \, d\omega. \tag{53}$$

The integer k is the quantum number corresponding to the projection of the angular momentum on the axis of symmetry and n is a generalized total angular momentum quantum number arising from the separation of variables. The coefficients $t_{ab}^{k\omega n}$ are generalized transmission coefficients and the $\psi_a^{k\omega n}$ are solutions of the Dirac equation which arise from the separation of variables and behave for $|\omega| > m$ like incoming spherical waves for $a = 1$ and outgoing spherical waves for $a = 2$. For $|\omega| < m$, the $\psi_a^{k\omega n}$ are linear combinations of both incoming and outgoing spherical waves near the event horizon.

Theorem 6 implies that the Dirac spinor ψ decays to zero in L^∞_{loc}, or equivalently that the Dirac particle must eventually either disappear into the black hole, or escape to infinity. One would like to determine the likelihood of these possibilities in terms of the energy spectrum of the Cauchy data, as well as the actual rate of decay of the Dirac spinor in a compact region of space. Both of these questions were addressed in [9], under two assumptions. First, we require that the charge e of the Dirac particle be small in the sense that

$$mM > |eQ|. \tag{54}$$

This condition is equivalent to saying that far from the black hole, the gravitational attraction between the the Dirac particle and the the black hole will dominate the electromagnetic interaction. Indeed, since the Kerr-Newman metric is asymptotically flat, the gravitational and electromagnetic interactions can be approximated far from the black hole by their Newtonian limits, given by mM/r^2 and eQ/r^2 respectively.

Next, we assume that only finitely many angular modes are present in the initial data, meaning that

$$|k| \leq k_0, \quad |l| \leq l_0, \tag{55}$$

in the Fourier expansion (53).[6]

We now summarize the main results, beginning with the rates of decay.

THEOREM 8. *Consider the Cauchy problem as in Theorem 6, with initial data normalized by* $\langle \psi_0 \mid \psi_0 \rangle = 1$. *Suppose that the small charge condition (54) and the boundedness condition (55) on the angular modes of the Cauchy data are satisfied. Then we have:*

(i) *If for any* k *and* n,

$$\limsup_{\omega \searrow m} |\langle \psi_2^{k\omega n} \mid \psi_0 \rangle| \neq 0, \tag{56}$$

or

$$\liminf_{\omega \nearrow -m} |\langle \psi_2^{k\omega n} \mid \psi_0 \rangle| \neq 0, \tag{57}$$

then, as $t \to \infty$,

$$|\psi(x,t)| = ct^{-5/6} + O(t^{-5/6-\epsilon}), \tag{58}$$

where $c = c(x) \neq 0$, *and* $\epsilon < 1/30$.

(ii) *If for all* k, n *and* $a = 1, 2$, $\langle \psi_a^{k\omega n} \mid \psi_0 \rangle = 0$ *for all* ω *is a neighborhood of* $+/-m$, *then for any fixed* x, $\psi(x,t)$ *decays rapidly in* t.

Contrary to what one would have expected at first, the rate of decay obtained in Theorem 8 is *slower* than the rate of decay of $t^{-3/2}$ one obtains for the solutions of the Dirac equation in Minkowski space, corresponding to compactly supported initial data, [9]. Indeed, in the Kerr-Newman geometry, the Dirac spinor should behave near infinity like a solution of the Dirac equation in Minkowski space, where pointwise decay occurs at the rate of $t^{-3/2}$. On the other hand, near the event horizon, the Dirac particle should behave like a massless particle, and its wave function should have rapid decay. One would thus have expected the rate

[6]Note that this finiteness assumption is not required in Theorem 6.

of decay in the Kerr-Newman geometry to interpolate the two rates, and thus to be *at least* as fast as $t^{-3/2}$.

The probability p for the Dirac particle to escape to infinity is given by

$$p = \lim_{t\to\infty} \int_{r>R} \bar{\psi}\gamma^j \psi(t,x)\nu_j \, d\mu. \tag{59}$$

It is easy to see that the probability p is independent of R. Indeed, let $R_2 > R_1 > r_1$ and let

$$p_a := \lim_{t\to\infty} \int_{r>R_a} \bar{\psi}\gamma^j \psi(t,x)\nu_j \, d\mu, \quad a = 1,2. \tag{60}$$

By Theorem 6, we have

$$p_1 - p_2 = \lim_{t\to\infty} \int_{R_1}^{R_2} \bar{\psi}\gamma^j \psi(t,x)\nu_j \, d\mu = 0, \tag{61}$$

which proves our claim.

In [9], it is shown that p is given by

$$p = \frac{1}{\pi} \sum_{|k|\le k_0} \sum_{|l|\le l_0} \int_{\mathbb{R}[-m,m]} \left(\frac{1}{2} - 2|t_{12}^{k\omega n}|^2\right) |\langle \psi_2^{k\omega n} \mid \psi_0\rangle|^2 \, d\omega. \tag{62}$$

Using this expression, the following theorem is then shown to be true:

THEOREM 9. *Consider the Cauchy problem as in Theorem 6, with initial data normalized by* $\langle \psi_0 \mid \psi_0\rangle = 1$. *We have:*

(i) *If the outgoing initial energy distribution satisfies* $\langle \psi_2^{k\omega n} \mid \psi_0\rangle \neq 0$ *for some* ω *such that* $|\omega| > m$, *then* $p > 0$.
(ii) *If the initial energy distribution satisfies for* $a = 1$ *or* $a = 2$, $\langle \psi_a^{k\omega n} \mid \psi_0\rangle \neq 0$ *for some* ω *such that* $|\omega| > m$, *then* $p < 1$.
(iii) *If the initial energy distribution is supported in the interval* $[-m, m]$, *then* $p = 0$.
(iv) *If for any* k *and* n,

$$\limsup_{\omega\searrow m} |\langle \psi_2^{k\omega n} \mid \psi_0\rangle| \neq 0, \tag{63}$$

or

$$\liminf_{\omega\nearrow -m} |\langle \psi_2^{k\omega n} \mid \psi_0\rangle| \neq 0, \tag{64}$$

then $0 < p < 1$.

(v) *We have $p = 1$ if and only if for all k and n the following conditions hold:*

(65) $$\langle \psi_1^{k\omega n} \mid \psi_0 \rangle = 0, \quad \text{if} \quad |\omega| \le m,$$

(66) $$\langle \psi_1^{k\omega n} \mid \psi_0 \rangle = -2t_{12}^{k\omega n} \langle \psi_2^{k\omega n} \mid \psi_0 \rangle \quad \text{if} \quad |\omega| > m.$$

Finally, we will mention some results, [12], [7], on the non-existence of time-periodic solutions of the Dirac equation:

THEOREM 10. *The Dirac equation does not admit any time-periodic L^2 solutions in the none-extreme Reissner-Nordström or Kerr-Newman geometries.*

These theorems are of a different nature from the preceding ones, in the sense that they deal with solutions which are supported across the event horizon, and thus defined in a weak sense.

6. Linear stability. The linear perturbations of a Kerr black hole by a massless spin s field are governed by the *Teukolsky* equation, [2],

(67)
$$\left[\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \phi} - (r-M)s\right]^2 - 4s(r + ia\cos\theta)\frac{\partial}{\partial t}\right.$$
$$\left. + \frac{\partial}{\partial \cos\theta}\sin^2\theta\frac{\partial}{\partial\cos\theta} + \frac{1}{\sin^2\theta}\left(a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\phi} + is\cos\theta\right)^2\right]\Psi_s = 0.$$

Just like the Dirac equation, this equation can be reduced to ordinary differential equations by separation of variables. It is a tantalizing problem to try to prove rigorous decay results analogous to the ones given in the preceding section for the Dirac equation.

REFERENCES

1. B. Carter and R. G. McLenaghan, *Generalized total angular momentum operator for the Dirac equation in curved space-time.* Phys. Rev. D (3) **19**(1979), 1093–1097.
2. S. Chandrasekhar, *The mathematical theory of black holes.* Revised reprint of the 1983 original. Internat. Ser. Monographs Phys. **69**, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1992.
3. D. Christodoulou, *The formation of black holes and singularities in spherically symmetric gravitational collapse.* Comm. Pure Appl. Math. **44**(1991), 339–373.
4. ———, *Examples of naked singularity formation in the gravitational collapse of a scalar field.* Ann. of Math. (2) **140**(1994), 607–653.
5. ———, *The instability of naked singularities in the gravitational collapse of a scalar field.* Ann. of Math. (2) **149**(1999), 183–217
6. D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space.* Princeton Math. Ser. **41**, Princeton University Press, Princeton, NJ, 1993.
7. F. Finster, N. Kamran, J. Smoller and S.-T. Yau, *Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry.* Comm. Pure Appl. Math. **53**(2000), 902–929.
8. ———, *The long-time dynamics of Dirac particles in the Kerr-Newman black hole geometry.* Preprint.

9. ———, *Decay Rates and Probability Estimates for Massive Dirac Particles in the Kerr-Newman Black Hole Geometry.* Comm. Math. Phys. **230**(2002), 201–244.
10. F. Finster, J. Smoller and S.-T. Yau, *Particle-like solutions of the Einstein-Dirac-Maxwell equations.* Phys. Lett. A **259**(1999), 431–436.
11. ———, *Non-existence of black hole solutions for a spherically symmetric, static Einstein-Dirac-Maxwell system.* Comm. Math. Phys. **205**(1999), 249–262.
12. ———, *Non-existence of time-periodic solutions of the Dirac equation in a Reissner-Nordström black hole background.* J. Math. Phys. **41**(2000), 2173–2194.
13. ———, *The interaction of Dirac particles with non-abelian gauge fields and gravity—bound states.* Nuclear Phys. B **584**(2000), 387–414.
14. S. Hawking and G. F. R. Ellis, *The large scale structure of space-time.* Cambridge Monographs Math. Phys. **1**, Cambridge University Press, London-New York, 1973.

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A NEW CONSTRUCTION OF SALEM POLYNOMIALS

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ABSTRACT. An earlier result of the author on the zeros of reciprocal polynomials is applied to give a new construction of Salem numbers.

RÉSUMÉ. Cet article applique un résultat précédent de l'auteur sur les zéros des polynomes réciproques pour répondre à la question de la construction des nouveaux nombres Salem.

1. Definitions and preliminary results. A monic reciprocal polynomial with integer coefficients having exactly one zero (of multiplicity 1) outside the unit circle is called a *Salem polynomial* [2].

A *Salem number* is a real algebraic integer $\alpha > 1$ of degree ≥ 4, all of whose conjugates, apart from α and α^{-1}, lie on the unit circle.

Here we give a new construction of Salem polynomials and therefore (in case of their real zeros are positive) a construction of Salem numbers. This is based on the following result of the author [4].

All zeros of the (real reciprocal) polynomial

$$(1)\qquad u_m(z) = u_{(m,l,\mathbf{a})}(z) = l(z^m + z^{m-1} + \cdots + z + 1) + \sum_{k=1}^{[\frac{m}{2}]} a_k(z^{m-k} + z^k)$$

of degree m where $l \in \mathbb{R}$, $l \neq 0$, $m \in \mathbb{N}$, $m \geq 2$, $\mathbf{a} = (a_1, \ldots, a_{[\frac{m}{2}]}) \in \mathbb{R}^{[\frac{m}{2}]}$ are on the unit circle if

$$(2)\qquad |l| \geq 2\sum_{k=1}^{[\frac{m}{2}]} |a_k|.$$

Write $v_m(z) := z^m + z^{m-1} + \cdots + z + 1$. It has been proved in [2] that for any sequence of positive integers $m_1, m_2, \ldots, m_s$, $s \geq 3$, the polynomial

$$(3)\qquad g(z) = (z+1)\prod_{i=1}^{s} v_{m_i}(z) - z\sum_{k=1}^{s}\Big(v_{m_k-1}(z) \prod_{i=1, i\neq k}^{s} v_{m_i}(z)\Big)$$

is either cyclotomic or Salem.

The polynomial (3) is cyclotomic only if $s = 3$ and $(m_1+1)^{-1} + (m_2+1)^{-1} + (m_3+1)^{-1} \geq 1$ or $s = 4$ and $m_1 = m_2 = m_3 = m_4 = 1$.

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To determine the location of zeros of reciprocal polynomials we apply Chebyshev transformation. This transformation projects the unit circle on the real interval $[-2,2]$.

A polynomial p of the form $p(z)=\sum_{j=0}^{2n} a_j z^j$ $(z \in \mathbb{C})$ where $n \in \mathbb{N}$, $a_0,\dots,a_{2n} \in \mathbb{R}$ and $a_j = a_{2n-j}$ $(j=0,\dots,n-1)$ is called a *real semi-reciprocal polynomial* of degree at most $2n$. If $a_{2n} \neq 0$ we call p a *real reciprocal polynomial* of degree $2n$. Denote by $\mathcal{R}_{2n}$ the set of all real semi-reciprocal polynomials of degree at most $2n$.

If $p \in \mathcal{R}_{2n}$, $p \neq o$ (o = the zero polynomial), then there is an integer k, $0 \le k \le n$, such that $a_{2n} = a_{2n-1} = \dots = a_{n+k+1} = 0 = a_{n-k-1} = \dots = a_0$ but $a_{n+k} = a_{n-k} \neq 0$ hence

$$p(z) = z^n\Big[a_{n+k}\Big(z^k+\frac{1}{z^k}\Big)+\dots+a_{n+1}\Big(z+\frac{1}{z}\Big)+a_n\Big]. \tag{4}$$

Let T_j be the j-th Chebyshev polynomial of the first kind, and C_j be the j-th normalized Chebyshev polynomial of the first kind defined by $C_0(x) = T_0(x)$, $C_j(x) = 2T_j(\frac{x}{2})$ for $j > 0$.

With $z+\frac{1}{z} = x$, we have $z^j + \frac{1}{z^j} = C_j(x)$ $(j=1,2,\dots)$, and hence by (4)

(5)
$$p(z) = z^n \sum_{j=0}^{k} a_{n+j}C_j(x) = a_{n+k}z^n \prod_{j=1}^{k}(x-\alpha_j) = a_{n+k}z^{n-k}\prod_{j=1}^{k}(z^2-\alpha_j z+1),$$

where $\alpha_j \in \mathbb{C}$ $(j=1,\dots,k)$ are the zeros of the polynomial $\sum_{j=0}^{k} a_{n+j}T_j(x)$.

The *Chebyshev transform* of a non-zero polynomial $p \in \mathcal{R}_{2n}$ having the factorization (5) is defined by

$$\mathcal{T}p(x) = a_{n+k}\prod_{j=1}^{k}(x-\alpha_j),$$

while for the zero polynomial p we put $\mathcal{T}p(x) = 0$.

It is clear that $\mathcal{T}$ maps $\mathcal{R}_{2n}$ into the set $\mathcal{P}_n$ of all polynomials of degree $\le n$ with real coefficients. In fact, we have the following:

PROPOSITION 1. *The Chebyshev transform $\mathcal{T}$ is an isomorphism between the (real) vector spaces $\mathcal{R}_{2n}$ and $\mathcal{P}_n$.*

Our basic tool is the following lemma (*cf.* [2]).

LEMMA 1. *Let $f(z)$ be a monic integral reciprocal polynomial and let $\bar{f}(z)$ be defined by*

$$\bar{f}(z) = \begin{cases} f(z) & \text{if the degree of } f \text{ is } 2n,\\ f(z)(z+1) & \text{if the degree of } f \text{ is } 2n-1.\end{cases}$$

Then $f(z)$ is a Salem polynomial if and only if the Chebyshev transform $\mathcal{T}\bar{f}(x)$ of $\bar{f}(z)$ has $n-1$ zeros in the interval $[-2,2]$.

2. **The construction.** Write $\tilde{v}_{2n}(z) = \frac{v_{2n+1}(z)}{z+1}$. Similarly, if $u_m(z)$ is a reciprocal polynomial of the form (1) and $m = 2n+1$, we write $\tilde{u}_{2n}(z) = \frac{u_{2n+1}(z)}{z+1}$.

LEMMA 2. *Supose that* (2) *holds with strict inequality.*

(i) *The values of the polynomial* $\mathcal{T}u_{2n}(x)$ *alternate sign at the zeros of* $\mathcal{T}\tilde{v}_{2n}(x)$.
(ii) *The values of the polynomial* $\mathcal{T}\tilde{u}_{2n-2}(x)$ *alternate sign at the zeros of* $\mathcal{T}v_{2n}(x)$.

PROOF. It is easy to see that $\mathcal{T}\tilde{v}_{2n}(x) = U_n(\frac{x}{2}) = \prod_{j=0}^{n}(x-\beta_j)$ and $\mathcal{T}v_{2n}(x) = U_n(\frac{x}{2}) + U_{n-1}(\frac{x}{2}) = \prod_{j=1}^{n}(x-\gamma_j)$ where $\beta_j = 2\cos\frac{2j\pi}{2n+2}$, $\gamma_j = 2\cos\frac{2j\pi}{2n+1}$ for $j = 1, 2, \ldots, n$ and U_n is the n-th Chebyshev polynomial of the second kind (for the details see [4]).

In [4], we found that

$$\mathcal{T}u_{2n}(x) = l\left[U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)\right] + \sum_{k=1}^{n} 2a_k T_{n-k}\left(\frac{x}{2}\right), \tag{6}$$

$$\mathcal{T}\tilde{u}_{2n-2}(x) = lU_{n-1}\left(\frac{x}{2}\right) + \sum_{k=1}^{n-1} a_k\left[U_{n-k-1}\left(\frac{x}{2}\right) - U_{n-k-2}\left(\frac{x}{2}\right)\right]. \tag{7}$$

Substituting the corresponding zeros into (6), (7) we have, after some calculations, that

$$\mathcal{T}u_{2n}(\beta_j) = 2\left(\frac{l}{2}(-1)^{j+1} + \sum_{k=1}^{n} a_k \cos\frac{2j(n-k)\pi}{2n+2}\right) \tag{8}$$

$$\mathcal{T}\tilde{u}_{2n-2}(\gamma_j) = 2\frac{\frac{l}{2}(-1)^{j+1} + \sum_{k=1}^{n-1} a_k \cos\frac{2j(2n-2k-1)\pi}{2(2n+1)}}{2\cos\frac{j\pi}{2n+1}}.$$

By the strict inequality (2), we get that $\operatorname{sgn}\mathcal{T}\tilde{u}_{2n-2}(\gamma_j) = \operatorname{sgn} l \operatorname{sgn}(-1)^{j+1}$ and $\operatorname{sgn}\mathcal{T}u_{2n}(\beta_j) = \operatorname{sgn} l \operatorname{sgn}(-1)^{j+1}$ proving Lemma 2. ■

THEOREM 1. *Let* $m_1, m_2, \ldots, m_s$ *and* $l_1, l_2, \ldots, l_s$ *be arbitrary positive integers and for* $i = 1, 2, \ldots, s$; $m_i \geq 2$ *let* $\mathbf{a}_i = (a_{i1}, \ldots, a_{i[\frac{m_i}{2}]})$ *be vectors with non-negative integer components such that*

$$l_i > \sigma_i \quad \textit{where} \quad \sigma_i = 2\sum_{k=1}^{[\frac{m_i}{2}]} a_{ik} \tag{9}$$

holds while for $m_i = 1$ *let* $\mathbf{a}_i$ *be the zero vector. Further let*

$$u_{m_i}(z) := u_{(m_i,l_i,\mathbf{a}_i)}(z) = l_i(z^{m_i} + z^{m_i-1} + \cdots + z + 1) + \sum_{k=1}^{[\frac{m_i}{2}]} a_{ik}(z^{m_i-k} + z^k) \quad \text{if } m_i \geq 2$$

$$u_{m_i}(z) := u_{(m_i,l_i,\mathbf{a}_i)}(z) = l_i(z+1) \quad \text{if } m_i = 1.$$

Define the polynomials

$$f(z) = f_{(m_1,\dots,m_s),(l_1,\dots,l_s),(\mathbf{a}_1,\dots,\mathbf{a}_s)}(z) = (z+1)\prod_{i=1}^{s} v_{m_i}(z) - z\sum_{k=1}^{s}\Big(u_{m_k-1}(z)\prod_{i=1,i\neq k}^{s} v_{m_i}(z)\Big) \tag{10}$$

where

$$u_{m_i-1}(z) := u_{(m_i-1,l_i,\mathbf{a}_i)}(z) \quad \text{if } m_i \geq 2$$

$$u_{m_i-1}(z) := l_i \quad \text{if } m_i = 1.$$

Then $f(x)$ *is a Salem polynomial unless one of the following conditions hold, in which case the polynomial is cyclotomic:*

(a) $s = 1$, $l_1 = 1$ *or* $l_1 = 2$ *and* $m_1 \geq 1$
(b) $s = 2$, $l_1 = l_2 = 1$, $m_1 \geq 1$ *and* $m_2 \geq 1$
$s = 2$, $l_1 = 2$ *or* $l_1 = 3$, $l_2 = 1$ *and* $m_1 = m_2 = 1$
$s = 2$, $l_1 = 2$, $l_2 = 1$ *and* $m_1 = 1$, $m_2 = 2$
$s = 2$, $l_1 = l_2 = 2$ *and* $m_1 = m_2 = 1$
(c) $s = 3$, $l_1 = l_2 = l_3 = 1$ *and* $(m_1+1)^{-1} + (m_2+1)^{-1} + (m_3+1)^{-1} \geq 1$
$s = 3$, $l_1 = 2$, $l_2 = l_3 = 1$ *and* $m_1 = m_2 = m_3 = 1$
(d) $s = 4$, $l_1 = l_2 = l_3 = l_4 = 1$ *and* $m_1 = m_2 = m_3 = m_4 = 1$.

PROOF. We may assume that $m_1, m_2, \dots, m_s$ are arranged such that all odd m_i's are listed first, *i.e.*, $m_i = 2n_i + 1$, $n_i \geq 0$ for $1 \leq i \leq r$ and $m_i = 2n_i$, $n_i \geq 1$ if $r + 1 \leq i \leq s$ for some $0 \leq r \leq s$. Taking out the factor $z + 1$ corresponding to the zero -1 of $f(z)$, we have

$$\begin{aligned} f(z) &= (z+1)^{r+1}\prod_{i=1}^{r}\tilde{v}_{2n_i}\prod_{i=r+1}^{s} v_{2n_i} \\ &\quad - z(z+1)^{r-1}\sum_{k=1}^{r}\Big(u_{2n_k}\prod_{i=1,i\neq k}^{r}\tilde{v}_{2n_i}\prod_{i=r+1}^{s} v_{2n_i}\Big) \\ &\quad - z(z+1)^{r+1}\sum_{k=r+1}^{s}\Big(\tilde{u}_{2n_k-2}\prod_{i=1}^{r}\tilde{v}_{2n_i}\prod_{i=r+1,i\neq k}^{s} v_{2n_i}\Big) \\ &= (z+1)^{r-1}\big((z+1)^2 f_1(z) - z f_2(z) - z(z+1)^2 f_3(z)\big) \end{aligned}$$

with suitable polynomials f_1, f_2, f_3.

Write $d = \sum_{i=1}^{s} n_i$ then the degree of f is $N = \sum_{i=1}^{s} m_i + 1 = 2d + r + 1$. Let

$$F(x) = \begin{cases} \frac{\mathcal{T}f(x)}{(x+2)^{\frac{r-1}{2}}} = (x+2)\mathcal{T}f_1(x) - \mathcal{T}f_2(x) - (x+2)\mathcal{T}f_3(x) & \text{if } r \text{ is odd} \\ \frac{\mathcal{T}\bar{f}(x)}{(x+2)^{\frac{r}{2}}} = (x+2)\mathcal{T}f_1(x) - \mathcal{T}f_2(x) - (x+2)\mathcal{T}f_3(x) & \text{if } r \text{ is even.} \end{cases}$$

Applying Proposition 1 we can rewrite F as

$$F(x) = (x+2)\prod_{i=1}^{r}\mathcal{T}\tilde{v}_{2n_i}\prod_{i=r+1}^{s}\mathcal{T}v_{2n_i} - \sum_{k=1}^{r}\Big(\mathcal{T}u_{2n_k}\prod_{i=1,i\neq k}^{r}\mathcal{T}\tilde{v}_{2n_i}\prod_{i=r+1}^{s}\mathcal{T}v_{2n_i}\Big)$$
$$- (x+2)\sum_{k=r+1}^{s}\Big(\mathcal{T}\tilde{u}_{2n_k-2}\prod_{i=1}^{r}\mathcal{T}\tilde{v}_{2n_i}\prod_{i=r+1,i\neq k}^{s}\mathcal{T}v_{2n_i}\Big).$$

By Lemma 1 we have to show that $\mathcal{T}\bar{f}$ has $[\frac{N+1}{2}]-1$ zeros in $[-2,2]$, or that F has $[\frac{N+1}{2}]-1-[\frac{r}{2}] = d$ zeros in the interval $[-2,+2]$ apart from the exceptional cases (a)–(d) when each of the $d+1$ zeros of F are in $[-2,+2]$.

To determine the number of zeros of $\mathcal{T}\bar{f}$ we use:

LEMMA 3. *Let Φ_i, Ψ_i, h_i ($i = 1, 2, \ldots, s$) and h_0 be polynomials with real coefficients such that $h_0, h_1, \ldots, h_s$ are positive on the interval $(a, b]$, $h_0(a) = 0$ and for each $i = 1, 2, \ldots, s$*

(A) *all zeros of Φ_i are single, and $\Phi_i(b) > 0$,*
(B) *Ψ_i alternates sign at the zeros of Φ_i, and Ψ_i is positive at the largest zero of Φ_i,*
(C) *for each $i = 1, 2, \ldots, s$ such that $h_i(a) \neq 0$ and $\Psi_i(a) \neq 0$; Ψ_i alternates sign on $\{a\} \cup \{\text{zeros of } \Phi_i\}$, and if Φ_i has no zeros on $(a, b]$, then $\Psi_i > 0$.*

Let M be the number of zeros of the product $\Phi_1 \cdots \Phi_s$ on (a, b), counted with multiplicity. Then the function

$$\Phi = h_0\prod_{i=1}^{s}\Phi_i - \sum_{k=1}^{s}\Big(h_k\Psi_k\prod_{i=1,i\neq k}^{s}\Phi_i\Big)$$

has M zeros in $[a, b]$.

Lemma 3 can be proved following the arguments of Proposition 4.1, Corollaries 1, 2 of [2], pp. 246–249. Condition (7) of Corollary 1 was changed to (C) to accomodate our situation, this again causes just a small deviation from the arguments of [2].

In the sequel let $\Phi = F$, $h_0 = x + 2$,

$$\Phi_i = \begin{cases}\mathcal{T}\tilde{v}_{2n_i} \\ \mathcal{T}v_{2n_i}\end{cases} \quad h_i = \begin{cases}1 \\ x+2\end{cases} \quad \Psi_i = \begin{cases}\mathcal{T}u_{2n_i} & \text{if } 1 \le i \le r \\ \mathcal{T}\tilde{u}_{2n_i-2} & \text{if } r+1 \le i \le s.\end{cases}$$

The number of zeros (counted with multiplicity) of the product $\Phi_1 \cdots \Phi_s$ is easily seen to be $\sum_{i=1}^{s} n_i = d$. All these zeros are in $(-2,2)$, denote by α_d the largest one.

Choose $[a,b]$ to be $[-2, \alpha_d + \epsilon]$ where $0 < \epsilon < 2 - \alpha_d$. We show that the conditions of Lemma 3 are satisfied. Obviously h_i are positive on $(-2, \alpha_d + \epsilon]$ and $h_0(-2) = 0$.

(A) All zeros of $\mathcal{T}v_{2n_i}$ and $\mathcal{T}\tilde{v}_{2n_i}$ in $(-2, \alpha_d + \epsilon)$ are single. Since $\mathcal{T}v_{2n_i}(2) = U_{n_i}(1) + U_{n_i-1}(1) = 2n_i + 1 > 0$, $\mathcal{T}\tilde{v}_{2n_i}(2) = U_{n_i}(1) = n_i + 1 > 0$ and $\mathcal{T}v_{2n_i}$, $\mathcal{T}\tilde{v}_{2n_i}$ have no zeros in $[\alpha_d + \epsilon, 2]$ they are positive at the point $\alpha_d + \epsilon$.

(B) is ensured by Lemma 2, and also by Lemma 2 we have $\operatorname{sgn} \mathcal{T}u_{2n_i} = 1 > 0$ at the largest zero of $\mathcal{T}\tilde{v}_{2n_i}$ and $\operatorname{sgn} \mathcal{T}\tilde{u}_{2n_i-2} = 1 > 0$ at the largest zero of $\mathcal{T}v_{2n_i}$.

To show that (C) is also satisfied we remark first that $\Psi_i(-2) = \mathcal{T}u_{2n_i}(-2) = 0$ $(1 \le i \le r)$ if and only if $x+2$ is a factor of it, *i.e.*, if and only if -1 is a double zero of u_{2n_i}, and this holds if only if (see [4, Theorem 1])

(11)
$$l_i = 2\sum_{k=1}^{n_i} a_{ik} \quad \text{and} \quad \operatorname{sgn}(-1)^{k+1} = 1 \quad \text{for all } k = 1, \dots, n_i \text{ for which } a_{ik} \neq 0.$$

By $l_i > \sigma_i$, (11) cannot be satisfied. Thus $h_i(-2) \neq 0$ and $\Psi_i(-2) \neq 0$ holds if and only if $l \le i \le r$.

Let us fix a subscript i with $l \le i \le r$.

CASE 1. $\Phi_i = \mathcal{T}\tilde{v}_{2n_i}$ has no zeros on $(-2, \alpha_d + \epsilon]$. This holds if and only if $m_i = 1$, $n_i = 0$. Then we have $\Psi_i = \mathcal{T}u_{2n_i} = l_i > 0$ as required by (C).

CASE 2. $m_i > 1$. By Lemma 2, Ψ_i alternates sign at the zeros of Φ_i. By Lemma 1 the smallest zero of Φ_i is $\beta_{n_i} = 2\cos\frac{2n_i\pi}{2n_i+2}$. To show that (C) holds in this case it is enough to prove that

$$\operatorname{sgn} \Psi_i(-2) \neq \operatorname{sgn} \Psi_i(\beta_{n_i}). \tag{12}$$

From (8) $\operatorname{sgn} \Psi_i(\beta_{n_i}) = \operatorname{sgn} \mathcal{T}u_{2n_i}(\beta_{n_i}) = (-1)^{n_i+1}$, and by (6)

$$\begin{aligned}\Psi_i(-2) = \mathcal{T}u_{2n_i}(-2) &= l_i[U_{n_i}(-1) + U_{n_i-1}(-1)] + \sum_{k=1}^{n_i} 2a_{ik}T_{n_i-k}(-1) \\ &= l_i(-1)^{n_i} + \sum_{k=1}^{n_i} 2a_{ik}(-1)^{n_i-k} = (-1)^{n_i}\Big(l_i + 2\sum_{k=1}^{n_i} a_{ik}(-1)^k\Big)\end{aligned}$$

and the expression in the last parenthesis is positive by $l_i > \sigma_i$, proving that (12) is satisfied.

Since $\epsilon > 0$ was arbitrary we have proved that F has d zeros in the interval $[-2, \alpha_d]$. The polynomial F is of degree $d+1$ its $d+1$-th zero can only be outside this interval.

The exceptional cases of the statement are the cases, when F has $d+1$ zeros in the interval $[-2,2]$, *i.e.*, when $F(2) \geq 0$ since $\operatorname{sgn} F(\alpha_d) = -1$. A simple calculation shows that

$$F(2) = 4\prod_{i=1}^{r}(n_i+1) \prod_{i=r+1}^{s} (2n_i+1)\Bigg[1 - \sum_{k=1}^{r} \frac{(2n_k+1)l_k + 2\sum_{i=1}^{n_k} a_{ik}}{4(n_k+1)} - \sum_{k=r+1}^{s} \frac{n_k l_k + \sum_{i=1}^{n_k} a_{ik}}{2n_k+1}\Bigg].$$

The first sum in the bracket is $\geq \sum_{k=1}^{r} l_k/4$ the second $\geq \sum_{k=r+1}^{s} l_k/3$. Using this, one can prove that if $s \geq 4$ then $F(2) \geq 0$ implies that $l_1 = \cdots = l_s = 1$. However, $s \geq 5$ and $l_1 = \cdots = l_s = 1$ is not possible since then two sums are at least $s/4 > 1$ making $F(2) \geq 0$ impossible. This is how we get the exceptional case (d). Similarly, if $s = 3$ then $F(2) \geq 0$ implies $l_k \leq 2$ $(k = 1,2,3)$, if $s = 2$ then $F(2) \geq 0$ implies that $l_k \leq 3$ $(k = 1,2)$, finally if $s = 1$ then $F(2) \geq 0$ implies that $l_1 \leq 4$. For a given s $(= 3,2,1)$ we list the possible values of $l_1, \ldots, l_s$ and using the condition $F(2) \geq 0$ we can select the suitable cases (a)–(c). ∎

REMARK 1. We remark that the cases $s \geq 3$ and $l_k = 1$ $(k = 1, \ldots, s)$ are known for example from [5]. In this way our result gives a new method for the description of trees with maximum eigenvalue ≤ 2.

REMARK 2. The sets defined in [5], [3] and [6] are very likely proper subsets of the set defined by polynomials of Theorem 1. For example, the Salem number $\simeq 1.673324849$ defined by the polynomial

$$\begin{aligned} &z^{14} - z^{12} - z^{11} - z^{10} - z^9 - 2z^8 - 3z^7 - 2z^6 - z^5 - z^3 - z^2 + 1 \\ &\quad = (z+1)v_1(z)v_{12}(z) - z\big(2v_{11}(z) + z^5 + z^6\big)(z+1) - zv_{12}(z) \end{aligned}$$

which is of the form (10) of Theorem 1 (with equality in the first condition of (9)), is not in the previously known sets. This holds since previous sets were defined by help of graphs. Due to the discrete nature of graphs there are "gaps" in these sets. The Salem number above is in such a gap.

REMARK 3. In some cases the polynomials of the form (10) are Salem polynomials even if u_{m_i-1} do not satisfy condition (9) but their zeros are on the unit circle. For example let $s = 1$, $m_1 = 16$, $l_1 = 2$ and

$$u_{15}(z) = 2v_{15}(z) + z^5 + z^6 + z^9 + z^{10}; \quad f_1(z) = (z+1)v_{16}(z) - zu_{15}(z).$$

Then $f_1(z) = (z+1)\Phi_4(z)\Phi_{12}(z)g_1(z)$ where $g_1(z)$ is the Salem polynomial of the smallest known Salem number.

This holds since the Chebyshev transform of $f_1(z)/(z+1)$ has 7 zeros on the interval $[-2,2]$. The polynomial g_1 is an irreducible factor of many Salem polynomials of the form

$$(z+1)v_{m-1}(z) - z\big(2v_{m-2}(z) + z^i + z^j + z^{m-2-i} + z^{m-2-j}\big), \tag{13}$$

for example if $m = 20$, $i = 3$, $j = 6$; $m = 21$, $i = 2$, $j = 9$; $m = 25$, $i = 1$, $j = 10$. Also, the majority of the small Salem numbers listed in parameter set for the Salem polynomials of the 8 smallest Salem numbers

#	1	2	3	4	5	6	7	8
m	17	19	17	16	15	19	13	25
i	5	3	3	3	3	1	4	0
j	6	5	5	6	5	7	5	8.

It may be of interest to show that the construction defined in Theorem 1 yields all small Salem numbers.

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REFERENCES

1. D. W. Boyd, *Small Salem numbers*. Duke Math. J. **44**(1977), 315–328.
2. J. W. Cannon and P. Wagreich, *Growth function of surface groups*. Math. Ann. **293** (1992), 239–257.
3. P. Lakatos, *Salem numbers, PV numbers and spectral radii of Coxeter transformations*. C. R. Math. Acad. Sci. Soc. R. Can. **23**(2001), 71–77.
4. ———, *On zeros of reciprocal polynomials*. Publ. Math. Debrecen **61**(2002), 645–661.
5. J. F. McKee, P. Rowlinson and C. J. Smyth, *Salem numbers and Pisot numbers from stars*. In: Number theory in progress, Vol. 1 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, 309–319.
6. C. J. Smyth, *Salem numbers of negative trace*. Math. Comp. **69**(2000), 827–838.

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SUMMATORY FUNCTIONS OF ELEMENTS IN SELBERG'S CLASS II

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RÉSUMÉ. Soit $F(s)$ une série de Dirichlet, $F(s) = \sum_{n=1} a_n n^{-s}$, $\Re s > 1$. Définissons la fonction de sommation $S(x)$ comme étant $\sum_{n \leq x} a_n$. Supposons que $F(s)$ satisfait les conditions suivantes: tout d'abord, pour tout $\epsilon > 0$, $|a_n| = O(n^\epsilon)$. Ensuite, F admet des continuations analytiques et des équations fonctionelles. Plus précisément, il existe une fonction $\Delta(s) = Q^s \prod \Gamma(\alpha_i s + \gamma_i)$, $Q > 0$, $\alpha_i > 0$, $\Re\gamma_i > 0$, telle que $F(s)\Delta(s) = \omega \overline{F}(1-s)\overline{\Delta}(1-s)$, $|\omega| = 1$. De plus, assumons que $F(s)$ est entière. Cet article donne une estimation de $S(x)$ sans conditions additionelles. L'estimation triviale est $S(x) = O(x^{1+\epsilon})$, $\forall \epsilon > 0$. Soit $\theta = \frac{d-1}{d+1} < 1$, $d = 2\sum \alpha_i$. Assumons que $d \geq 2$. Le théorème principal est $S(x) = O(Q^{1-\theta-\epsilon} x^{\theta+\gamma\epsilon})$, $\forall \epsilon > 0, \gamma > 1$.

1. Introduction. Let $F(s)$ be a Dirichlet series, *i.e.*, $F(s) = \sum_{n=1} a_n n^{-s}$. The summatory function $S(x)$ is defined as follows

$$S(x) = \sum_{n \leq x} a_n, \quad x \geq 1.$$

We suppose that $F(s)$ extends to a meromorphic functions for $\Re(s) \geq 1$. If $F(s)$ has only a simple pole at $s = 1$ on the line of $\Re(s) = 1$ and satisfies some conditions on coefficients, the celebrated Tauberian theorem (see [5]) gives us

$$S(x) = \kappa x + R(x), \quad R(x) = o(x), \quad \text{for some constant } \kappa.$$

To find the exact order of the error term $R(x)$ is, in general, a difficult problem.

It is difficult to deal with general Dirichlet series. However, almost all interesting Dirichlet series admit analytic continuation or meromorphic continuation to the entire complex plane and satisfy a functional equation. More precisely, we assume that there is a function

$$\Delta(s) = Q^s \prod \Gamma(\alpha_i s + \gamma_i), \quad Q > 0, \ \alpha_i > 0,$$

such that

$$F(s)\Delta(s) = \omega \overline{F}(1-s)\overline{\Delta}(1-s),$$

where ω is a complex number with $|\omega| = 1$, $\overline{F}(s) = \overline{F(\overline{s})}$, and $\overline{\Delta}(s) = \overline{\Delta(\overline{s})}$.

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In [1], Chandrasekharan and Narasimhan proved an O-theorem and an Ω-theorem for the error term $R(x)$. In [4], R. Murty used such theorems to get an estimation of eigenvalues of Hecke operators.

Now we assume that $F(s)$ is entire and satisfies Ramanujan hypothesis, *i.e.*, for all $\epsilon > 0$, $|a_n| = O(n^\epsilon)$. The main term in the summatory function disappears. By the Ramanujan hypothesis, the trivial estimation of $S(x)$ is $O(x^{1+\epsilon})$, $\forall \epsilon > 0$. Define τ_F to be the smallest value of ξ such that

$$S(x) = O(x^{\xi+\epsilon}), \quad \forall \epsilon > 0.$$

To determine τ_F is not an easy job. If we can find some upper bound θ of τ_F less than 1, then the Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ will converge in $\Re(s) > \theta$ (*cf.* Lemma 5.) The importance of this fact is that we can estimate the size of $F(s)$ inside the the critical strip, which cannot be inferred from the functional equations.

Let $d = 2\sum \alpha_i$ be the degree of $F(s)$ and Q the conductor appearing in the functional equation. In [2], Ram Murty and the author used Perron's formula, with the necessary estimates for the functions coming from convexity properties of entire functions of finite order, to get that $\theta = d/d + 2$ works. The argument is done very efficiently and clearly. More precisely:

THEOREM ([2, THEOREM 4]). *Keep all assumptions in this section. We have*

$$S(x) = O(Q^{1-\theta'} x^{\theta'+\epsilon}), \quad \theta' = \frac{d}{d+2} = 1 - \frac{2}{d+2} < 1.$$

In the paper, the author use the non-trivial results in [1] to get the better result. The main theorem of this paper is:

THEOREM (THEOREM 4). *Keep all assumptions in this section. Then for $d \geq 2$, we have*

$$\forall \epsilon > 0,\ \forall \gamma > 1, \quad S(x) = O(Q^{1-\theta+\epsilon} x^{\theta+\gamma\epsilon}), \quad \theta = \frac{d-1}{d+1} = 1 - \frac{2}{d+1} < 1.$$

Except for [2], all old literatures, such as [6], concentrate only on the exponents of x for the case of modular forms. Our main contribution is write down the general theorem for Selberg's class, and more importantly, the dependence of the conductor, which was never done before except in [2]. From our work, a totally new phenomenon is observed as follows: whatever the method we use, the sum of two exponents is always equal to $1 + \epsilon$.

The organization of this paper is following. We set up the notations in the second section and state some results and tools in the third section. And then we prove our main theorem in the fourth section. In the final section we make some concluding remarks.

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2. Notations. In this paper, we consider a set $\mathcal{S}'$ of complex functions $F(s)$ satisfying the following conditions:

(i) (Dirichlet series)

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

a Dirichlet series absolutely convergent for $\Re s > 1$;

(ii) (Ramanujan hypothesis) $\forall \epsilon > 0$, $a_n = O(n^\epsilon)$ and the constant only depending on F and ϵ;

(iii) (Analytic continuation) $F(s)$ extends to an entire function of finite order;

(iv) (Functional equation) it has a functional equation of the form:

$$\Phi(s) = \omega \overline{\Phi}(1-s),$$

where $\omega \in \mathbb{C}$, $|\omega| = 1$,

$$\Phi(s) = \Delta(s)F(s), \quad \Delta(s) = Q^s \prod_{i=1}^{m} \Gamma(\alpha_i s + \gamma_i),$$

and $\overline{\Phi}(s) = \overline{\Phi(\overline{s})}$. Here $Q > 0$, $\alpha_i > 0$, $\Re(\gamma_i) \geq 0$, and m is a non-negative integer. The number Q is called the *conductor*. We define

$$\tilde{\Delta}(s) = \prod_{i=1}^{m} \Gamma(\alpha_i s + \gamma_i), \quad \text{and} \quad \Delta(s) = Q^s \prod_{i=1}^{m} \Gamma(\alpha_i s + \gamma_i) = Q^s \tilde{\Delta}(s).$$

REMARK. Here our conditions are the same as the Selberg class $\mathcal{S}$ defined in [7], except for the Euler product condition and the admission of poles at $s = 1$.

Let d_F be *the degree of* F defined by

$$d_F = 2 \sum_{i=1}^{m} \alpha_i.$$

Define the *summatory function* $S_F(x)$ of $F(s)$ by

$$S_F(x) = \sum_{n \leq x} a_n.$$

Sometimes we drop the subscript F if there is no confusion.

We shall adopt the following notation:

$$\int_{(c)} f(s)\, ds := \int_{c-i\infty}^{c+i\infty} f(s)\, ds,$$

where $f(s)$ is any complex valued function on $\mathbb{C}$.

3. Preliminaries. In this section we state the results and tools needed in our proof. The main tool is the method of contour integration.

The following lemma is standard contour integration (see for example, [5]).

LEMMA 1. *If k is any positive integers, $c > 0$, then*

$$\frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s(s+1)\cdots(s+k)}\,ds = \begin{cases} \frac{1}{k!}(1-\frac{1}{x})^k & \text{if } x \geq 1, \\ 0 & \text{if } 0 \leq x \leq 1. \end{cases}$$

DEFINITION 2. Let $F(x)$ be a smooth function in x. For $y > 0$ and $0 \leq r \in \mathbb{Z}$, define the r-th finite difference $\Delta_y^r F(x)$ of $F(x)$

$$\begin{aligned} \Delta_y^r F(x) &= \sum_{v=0}^{r} (-1)^{r-v}\binom{r}{v} F(x+vy) \\ &= \int_x^{x+y} dt_1 \int_{t_1}^{t_1+y} dt_2 \cdots \int_{t_{r-1}}^{t_{r-1}+y} F^{(r)}(t_r)\,dt_r. \end{aligned}$$

It is easy to get the following estimations from the definition of $\Delta_y^r F(x)$:

$$\Delta_y^r F(x) = \begin{cases} O(|F(x)|), \\ O(y^r |F^{(r)}(x)|). \end{cases}$$

The main ingredients of our proof is the following results in [1]:

THEOREM 3. ([1, (4.5), (4,11)]) *We adopt the notations in Section 2 and Section 3. Let $\tilde{\Delta}(s) = \prod_{i=1}^{m} \Gamma(\alpha_i s + \gamma_i)$. Suppose that $d \geq 2$. For $0 \leq r \in \mathbb{Z}$, and $c \in \mathbb{R}$, define*

$$J(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(1-s)\tilde{\Delta}(s)}{\Gamma(r+2-s)\tilde{\Delta}(1-s)} x^{r+1-s}\,ds.$$

Then for suitable choices c and r, we have

$$J(x) = O(x^{\frac{1}{2}+\frac{d-1}{d}r-\frac{1}{2}d}), \quad \text{and} \quad J^{(r)}(x) = O(x^{\frac{1}{2}-\frac{1}{2}d}).$$

4. The Main O-Theorem. Now we state the main theorem.

THEOREM 4. *Let $F(s)$ be the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ that satisfies the conditions in Section 2. Suppose that $d \geq 2$. Then*

$$\forall \epsilon > 0,\ \forall \gamma > 1, \quad S(x) = \sum_{n \leq x} a_n = O(Q^{1-\theta-\epsilon} x^{\theta+\gamma\epsilon}).$$

Here

$$\theta = \frac{d-1}{d+1} < 1.$$

PROOF. We follow the line of proof in [1]. Set

$$A_r(x) = \frac{1}{r!}\sum_{n\le x} a_n(x-n)^r, \quad A_0(x) = \sum_{n\le x} a_n = S(x). \tag{1}$$

By Lemma 1, for $c > 1$, we have

$$A_r(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s)}{\Gamma(r+1+s)} F(s) x^{r+s}\, ds.$$

Using the entireness of $F(s)$ and moving the line of integration to $\Re(c) < 0$, we change the variable from s to $1-s$ and apply the functional equation to get

$$A_r(x) = \omega \cdot \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(1-s)\tilde{\Delta}(s)}{\Gamma(r+2-s)\tilde{\Delta}(1-s)} x^{r+1-s} Q^{2s-1} F(s)\, ds.$$

Let

$$I(x) = \frac{1}{2\pi i}\int_{(c)} Q^{2s-1} \frac{\Gamma(1-s)\tilde{\Delta}(s)}{\Gamma(r+2-s)\tilde{\Delta}(1-s)} x^{r+1-s}\, ds.$$

By the Dirichlet series expression of $F(s)$, we can rewrite $A_r(x)$ as

$$A_r(x) = \omega \sum_{n=1}^{\infty} \frac{a_n}{n^{1+r}} I(nx). \tag{2}$$

Two different expressions (1), (2) of $A_r(x)$ are the core of this proof. We apply Δ_y^r on (1).

$$\begin{aligned}\Delta_y^r A_r(x) &= \sum_{n\le x} a_n \frac{\Delta_y^r (x-n)^r}{r!} + \sum_{v=0}^{r} \frac{1}{r!}(-1)^{r-v}\binom{r}{v} \sum_{x<n\le x+vy} a_n(x+vy-n)^r \\ &= A_0(x) y^r + O\Big(y^r \sum_{x<n\le x+vy} |a_n|\Big)\end{aligned}$$

where we use $\Delta_y^r (x-n)^r/r! = y^r$. Since $A_0(x) = S(x)$, we get

$$\Delta_y^r A_r(x) = y^r S(x) + O\Big(y^r \sum_{x<n\le x+vy} |a_n|\Big). \tag{1'}$$

On the other side, apply Δ_y^r on (2),

$$\Delta_y^r A_r(x) = \omega \sum_{n=1}^{\infty} \frac{a_n}{n^{1+r}} \Delta_y^r\big(I(nx)\big).$$

To estimate $\Delta_y^r(I(nx))$, we use Theorem 3. By Theorem 3, for suitable r, we have

$$I(x) = O(Q^{\frac{2}{d}r+\frac{1}{d}} \cdot x^{\frac{1}{2}+\frac{d-1}{d}r-\frac{1}{2}d}), \quad \text{and} I^{(r)}(x) = O(Q^{\frac{1}{d}} \cdot x^{\frac{1}{2}-\frac{1}{2}d}).$$

Therefore,

$$\Delta_y^r I(x) = \begin{cases} O(|I(x)|) \\ O(y^r|I^{(r)}(x)|) \end{cases} = \begin{cases} O(Q^{\frac{2}{d}r+\frac{1}{d}} \cdot x^{\frac{1}{2}+\frac{d-1}{d}r-\frac{1}{2}d}), \\ O(y^r \cdot Q^{\frac{1}{d}} \cdot x^{\frac{1}{2}-\frac{1}{2}d}). \end{cases}$$

We invoke the Ramanujan hypothesis on $F(s)$ and write the summation to two parts; one is from 1 to A and the other is from A to infinity. For $\epsilon > 0$,

$$\sum_{n=1}^{A} \frac{a_n}{n^{1+r}} I(nx) = O\Big(\sum_{n=1}^{A} y^r Q^{\frac{1}{d}} \cdot x^{\frac{1}{2}-\frac{d}{2}} \frac{n^\epsilon}{n^{1+r}} n^r \cdot n^{\frac{1}{2}-\frac{d}{2}}\Big) = O(y^r Q^{\frac{1}{d}} x^{\frac{1}{2}-\frac{d}{2}} A^{\frac{1}{2}-\frac{d}{2}+\epsilon}),$$

and

$$\sum_{n=A}^{\infty} \frac{a_n}{n^{1+r}} I(nx) = O\Big(\sum_{n=A}^{\infty} Q^{\frac{2}{d}r+\frac{1}{d}} \cdot x^{\frac{1}{2}+\frac{d-1}{d}r-\frac{d}{2}} \frac{n^\epsilon}{n^{1+r}} n^{\frac{1}{2}+\frac{d-1}{d}r-\frac{d}{2}}\Big)$$

$$= O(Q^{\frac{2}{d}r+\frac{1}{d}} x^{\frac{1}{2}+\frac{d-1}{d}r-\frac{d}{2}} A^{\frac{1}{2}+\frac{d-1}{d}r-\frac{d}{2}+\epsilon}).$$

Set $A = y^{-d} Q^2 x^{d-1}$. We have

$$(2') \qquad \Delta_y^r A_r(x) = O(y^{r-\frac{d}{2}+\frac{1}{2}-d\epsilon} Q^{1+2\epsilon} x^{\frac{d}{2}-\frac{1}{2}+(d-1)\epsilon}).$$

Combining $(1')$ with $(2')$ and using the Ramanujan hypothesis, for $\epsilon' > 0$, $y = O(x)$, we have

$$S(x) = O(y^{-\frac{d}{2}+\frac{1}{2}-d\epsilon} Q^{1+2\epsilon} x^{\frac{d}{2}-\frac{1}{2}+(d-1)\epsilon}) + O(y \cdot x^{\epsilon'}).$$

Amazingly, r disappears in the end.

To minimize the sum, we equalize $y^{-\frac{d}{2}+\frac{1}{2}-d\epsilon} Q^{1+2\epsilon} x^{\frac{d}{2}-\frac{1}{2}+(d-1)\epsilon}$ and $y \cdot x^{\epsilon'}$. Taking

$$y = x^\alpha \cdot Q^\beta,$$

where

$$\alpha = \frac{\frac{d}{2} - \frac{1}{2} + (d-1)\epsilon - \epsilon'}{\frac{1}{2} + \frac{d}{2} + d\epsilon}, \quad \beta = \frac{1+\epsilon}{\frac{1}{2} + \frac{d}{2} + d\epsilon}.$$

Then it follows that

$$S(x) = O(y \cdot x^{\epsilon'}) = O(x^{\alpha+\epsilon'} \cdot Q^\beta) = O(x^{\frac{d-1}{d+1}+\gamma\epsilon''} Q^{\frac{2}{d+1}-\epsilon''}).$$

Here ϵ'' is any positive number and γ is any positive number greater than 1. This ends the proof. ∎

5. Concluding Remarks.

In this section we discuss some applications.

LEMMA 5. *Let $F(s)$ be a Dirichlet series. Assume that its summatory function $S(x)$ is $O(x^\delta)$. For $\Re(s) > \delta$, the series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges. Note that we do not assume that the $F(s)$ converges absolutely for $\Re(s) > \delta$.

PROOF. Let $F_x(s)$ be $\sum_{n\le x} a_n/n^{-s}$. It suffices to prove that $F_x(s)$ converge uniformly when x tends to infinity. Let $s = c + it$. By definition,

$$F_x(s) = \int_{1-}^{x} \frac{1}{t^s}\, dS(t),$$

where $S(t)$ is the summatory function. By partial summation,

$$\begin{aligned} F_x(s) &= \int_{1-}^{x} \frac{1}{t^s}\, dS(t) = \frac{S(x)}{x^s} + s\int_{1-}^{x} \frac{S(t)}{t^{s+1}}\, dt \\ &= s\int_{1-}^{\infty} \frac{S(t)}{t^{s+1}}\, dt + O(x^{\delta-c}) - \int_{x}^{\infty} \frac{S(t)}{t^{s+1}}\, dt \\ &= A(s) + O(x^{\delta-c}) + O\left(\int_{x}^{\infty} \frac{t^\delta}{t^{s+1}}\, dt\right) = A(s) + O(x^{\delta-c}), \end{aligned}$$

where

$$A(s) = s\int_{1-}^{\infty} \frac{S(t)}{t^{s+1}}\, dt.$$

Therefore, $F_x(s)$ converges to $A(s)$ uniformly when x tends to infinity. This proves our assertion. ■

Immediately, we have the following corollary.

COROLLARY 6. *If $F(s) \in S'$ and $d \ge 2$, then for $\Re s > \theta = (d-1)/(d+1)$, the series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges.

We have several remarks in the end.

(i) In [6], Rankin got the better result in the case of L-functions attached to cusp forms. More precisely, for cusp forms, whose d is 2, for any positive ϵ, he proved

$$S(x) = O\big(x^{(2-1)/(2+1)}(\log x)^{-\delta+\epsilon}\big) = O\big(x^{1/3}(\log x)^{-\delta+\epsilon}\big)$$

where $\delta = (8 - 3\sqrt{6})/10 = 0.065153\ldots$. Unfortunately, Rankin's method cannot be applied on Selberg class because it relies on a better estimate of coefficients, namely

$$a_n = O\big(d(n)\big), \quad d(n) = \sum_{m|n} 1,$$

and the work of Moreno and Shahidi [3] on a non-trivial estimate of the fourth moment of coefficients. However, using the property of Euler products, it might be possible to improve the result.

(ii) An interesting observation is following: whatever the methods we use, the sum of the exponents of Q and of x is always equal to $1+\epsilon$. It will be very interesting to see how to break this bound.

(iii) Our theorem computes the conductor dependence. It will be useful in the study of average values of families of elements of the Selberg class. We hope to undertake this study in forthcoming research.

References

1. K. Chandrasekharan and R. Narasimhan, *Functional equations with multiple gamma factors and the average order of arithmetical functions.* Ann. of Math. **76**(1962), 93–136.
2. W. Kuo and M. Ram Murty, *Summatory functions of elements in Selberg's class.* To appear.
3. C. J. Moreno and F. Shahidi, *The fourth moment of Ramanujan τ-function.* Math. Ann. **266**(1983), 233–239.
4. M. Ram Murty, *On the estimation of eigenvalues of Hecke operations.* Rocky Mountain J. Math. **15**(1985), 521–533.
5. ———, *Problems in Analytic Number Theory.* Graduate Texts in Math. **206**, Springer-Verlag, 2001.
6. R. A. Rankin, *Sums of cusp form coefficients.* In: Automorphic forms and analytic number theory (Montreal, PQ, 1989), Université de Montréal, 115–121.
7. A. Selberg, *Old and new conjectures and results about a class of Dirichlet series.* In: Collected Papers, Vol. **II**, Springer-Verlag, Berlin, 1991, 47–63.

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