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SUR UN RÉSULTAT DE TYPE GELFAND-MAZUR

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ABSTRACT. We show that any subalgebra of $\mathbf{C}(X)$ endowed with a sequentially complete m-convex topology is isomorphic to $\mathbf{C}$.

Dans [2], s'intéressant à certaines questions de topologisation dans le corps $\mathbf{C}(X)$, nous avons montré qu'aucune sous-algèbre du corps $\mathbf{C}(X)$ contenant $\mathbf{C}[X]$, (dites ordres de $\mathbf{C}(X)$), ne peut être munie d'une topologie d'algèbre localement multiplicativement convexe séquentiellement complète. C'est le caractère principal de ces ordres qui était à la base de la preuve du résultat ci-dessus. Dans cette note, nous montrons que si A est une sous-algèbre unitaire de $\mathbf{C}(X)$ munie d'une topologie d'algèbre localement multiplicativement convexe séquentiellement complète, alors elle est égale à $\mathbf{C}$. Notons qu'une sous-algèbre quelconque de $\mathbf{C}(X)$ n'est pas nécessairement principale. C'est le cas, par exemple, de l'algèbre $\mathbf{C}[X^2, X^3]$.

Toutes les algèbres considérées sont, sauf mention du contraire, complexes, commutatives et unitaires. Soit A une algèbre et B une A-algèbre. Un élément x de B est dit entier, sur A, s'il existe un polynôme P unitaire (*i.e.*, le coefficient de son monôme du plus haut degré est égal à 1), à coefficients dans A tel que $P(x) = 0$ [1, définition 1, p. 6]. On dit que B entière, sur A, si tout élément de B l'est. L'ensemble des éléments de B entiers sur A est une A-algèbre [1, corollaire 1, p. 9]. Le spectre d'un élément x de A est la partie de $\mathbf{C}$, notée $\mathrm{Sp}_A(x)$, formée des λ tel que $x-\lambda e$ n'est pas inversible dans A (e étant l'unité de A). Une algèbre localement multiplicativement convexe (a.l.m.c.) est une algèbre topologique dont la topologie est définie par une famille de semi-normes sous-multiplicatives. Une telle algèbre est dite de Fréchet si elle est metrisable et complète. Dans la suite, on note par $\mathbf{C}[X]$ (resp. $\mathbf{C}(X)$) l'algèbre des polynômes (resp. fractions rationnelles) à une indéterminée à coefficients complexes. On note également par $\mathbf{H}(\mathbf{C})$ l'algèbre des fonctions complexes holomorphes sur $\mathbf{C}$ munie de la topologie de la convergence uniforme sur les compacts de $\mathbf{C}$. Celle ci est définie par la famille de semi-normes $(q_r)_{r\in\mathbf{N}^*}$, où $q_r(\sum_n \alpha_n z^n) = \sup_{|z|\leq r} \sum_n |\alpha_n|\,|z^n|)$. L'algèbre $\mathbf{H}(\mathbf{C})$ est ainsi une a.l.m.c. de Fréchet.

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Le résultat fondamental de cette note est le suivant qui est en fait un théorème de type Gelfand-Mazur.

THÉORÈME. *Soit A une sous-algèbre de $\mathbf{C}(X)$ munie d'une topologie d'a.l.m.c. séquentiellement complète. Alors $A = \mathbf{C}$.*

Pour la preuve de ce théorème, nous aurons besoin des deux lemmes suivants dont le premier est à caractère purement algébrique.

LEMME 1. *Soit A une sous-algèbre de $\mathbf{C}(X)$ qui n'est pas un corps. Alors tout idéal maximal de A est de codimension 1.*

PREUVE. Soit M un idéal maximal de A. Supposons d'abord que A contient l'algèbre $\mathbf{C}[X]$. Comme A n'est pas un corps, M est non nul. Alors M contient nécessairement un polynôme non nul. L'algèbre A/M est donc algébrique (*i.e.*, chacun de ses éléments est racine d'un polynôme non nul à coefficients complexes). Elle est alors isomorphe à $\mathbf{C}$ vu que ce dernier est algébriquement clos. Considérons maintenant le cas où il existe $U, V \in \mathbf{C}[X]$ avec $d^\circ V < d^\circ U$ et $U/V \in A$. Alors, dans ce cas, X est entier sur A; et par suite $A[X]$ est entière sur A [1, corollaire 1, p. 9] ($A[X]$ étant la sous A-algèbre de $\mathbf{C}(X)$ engendrée par A et X). D'après [1, lemme 2, p. 32], $A[X]$ n'est pas un corps. De plus, par [1, théorème 1, p. 84], il existe un idéal premier N de $A[X]$ tel que $M = N \cap A$. Par ailleurs, on voit facilement que la A/M-algèbre B/N est entière. Donc, via [1, lemme 2, p. 32], N est maximal. D'après le premier cas, N est de codimension 1 dans $A[X]$. Il en est de même de M dans A. Considérons maintenant le cas où il existe $x = U/V \in A \setminus \mathbf{C}$ avec $d^\circ U \leq d^\circ V$. Posons $V = X^n + \alpha_{n-1}X^{n-1} + \cdots + \alpha_0$ et $U = \beta_r X^r + \beta_{r-1}X^{r-1} + \cdots + \beta_0$. En multipliant par x^{n-1} les deux membres de l'égalité $Vx = U$, nous obtenons $(xX)^n + \alpha_{n-1}(xX)^{n-1} + \cdots + \alpha_0 x^{n-1} - \beta_0 x^{n-1} - \beta_1 x^{n-2}(xX) - \cdots - \beta_r x^{n-r-1}(xX)^r = 0$. Ainsi xX est entier sur A et par suite $A[xX]$ est entière sur A. De proche en proche, on mettera en évidence un entier naturel m tel que $A[xX^m]$ soit entière sur A [1, proposition 6, p. 10] et, de plus, xX^m est de la forme examinée dans le second cas. On conclut par ce qui précède.

LEMME 2. *Soit A une a.l.m.c. séquentiellement complète. Pour tout $x \in A$ et pour tout $f = \sum_n \alpha_n z^n \in \mathbf{H}(\mathbf{C})$, la série $\sum_n \alpha_n x^n$ est convergente vers un élément, qu'on notera $f(x)$, dans A. De plus, l'application φ de $\mathbf{H}(\mathbf{C})$ vers A définie par $\varphi(f) = f(x)$ est un morphisme d'algèbres continu.*

PREUVE. Soit $(p_\lambda)_{\lambda \in \Lambda}$ une famille de semi-normes sous-multiplicatives définissant la topologie de A. Pour $\lambda \in \Lambda$ et $\sum_n \alpha_n z^n \in \mathbf{H}(\mathbf{C})$, on a $\sum_n p_\lambda(\alpha_n x^n) \leq \sum_n |\alpha_n| p_\lambda(x)^n < \infty$. D'où la convergence de la série $\sum_n \alpha_n x^n$ dans A. Par ailleurs, on montre aisément que φ est bien un morphisme d'algèbres. Quant à la continuité, nous avons pour $\lambda \in \Lambda$ et $f = \sum_n \alpha_n z^n \in \mathbf{H}(\mathbf{C})$, $p_\lambda(f(x)) = p_\lambda(\sum \alpha_n x^n) \leq \sum_n |\alpha_n| p_\lambda(x)^n \leq q_r(f)$, où r est un entier plus grand que $p_\lambda(x)$. D'où la continuité de φ.

PREUVE DU THÉORÈME. Soit $x \in A$ et φ le morphisme continu considéré dans le lemme 2. $\operatorname{Ker}\varphi$ étant un idéal fermé de $\mathbf{H}(\mathbf{C})$; donc l'algèbre $\mathbf{H}(\mathbf{C})/\operatorname{Ker}\varphi$, munie de la topologie quotient, est une a.l.m.c. de Fréchet. De plus, nous avons un isomorphisme algébrique entre $\mathbf{H}(\mathbf{C})/\operatorname{Ker}\varphi$ et $\operatorname{Im}\varphi$. Moyennant le lemme 1, tout idéal maximal de $\mathbf{H}(\mathbf{C})/\operatorname{Ker}\varphi$ est de codimension 1. Donc, d'après [3], tout élément de $\mathbf{H}(\mathbf{C})/\operatorname{Ker}\varphi$ est de spectre compact. Il en sera de même pour les éléments de $\operatorname{Im}\varphi$. Par ailleurs, $\operatorname{Ker}\varphi$ n'est pas nul vu que $\mathbf{H}(\mathbf{C})$ contient des éléments de spectre non borné. Considérons alors $f \in \mathbf{H}(\mathbf{C})$, non nul, tel que $f(x) = 0$. Soit $\lambda \in \operatorname{Sp}_{\operatorname{Im}\varphi}(x)$. Considérons $g \in \mathbf{H}(\mathbf{C})$ tel que $f(z) - f(\lambda) = (z - \lambda)g(z)$, pour tout $z \in \mathbf{C}$. On a $f(x) - f(\lambda) = -f(\lambda) = (x - \lambda)g(x)$ n'est pas inversible dans $\operatorname{Im}\varphi$. D'où $f(\lambda) = 0$. Il s'en suit que f, qui est holomorphe, s'annule sur $\operatorname{Sp}_{\operatorname{Im}\varphi}(x)$. Ce dernier est alors fini. Soient $\lambda_1, \ldots, \lambda_r$ ses éléments. Ecrivons $f(z) = \Pi_{1\leq i\leq r}(z - \lambda_i)^{\alpha_i}h(z)$, où $h \in \mathbf{H}(\mathbf{C})$ avec $h(\lambda_i) \neq 0$ pour tout i. Alors $h(x) \neq 0$, car, sinon, h s'annulerait sur $\operatorname{Sp}_{\operatorname{Im}\varphi}(x)$. Donc, vu l'intégrité de A, $\Pi_{1\leq i\leq r}(x - \lambda_i)^{\alpha_i} = 0$; et par suite x est l'un des λ_i. D'où $A = \mathbf{C}$.

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ON SOME RESULTS OF GRZAŚLEWICZ CONCERNING QUADRATIC MAPS AND ADDITIVE MAPS

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ABSTRACT. In a 1979 paper, A. Grząślewicz proved several results characterizing special quadratic and additive maps by means of functional equations and systems of functional equations for real-valued functions. One of the main results was that if $n: \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ = the field of real numbers) satisfies $[n(x+y)-n(x-y)]^2 = 16n(x)n(y)$ and $n(x) = x^4n(x^{-1})$ for $x \neq 0$, then n is either the square of a linear function or the square of a derivation on $\mathbb{R}$. In the present paper we extend those results to more general rings and fields.

RÉSUMÉ. Dans un article de 1979, A. Grząślewicz a prouvé plusieurs résultats concernant la caractérisation de certaines fonctions additives et quadratiques à l'aide d'équations fonctionnelles et de systèmes d'équations fonctionnelles pour des fonctions à valeurs réelles. L'un des principaux résultats était que, si $n: \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ = le corps des nombres réels) satisfait $[n(x+y) - n(x-y)]^2 = 16n(x)n(y)$ et $n(x) = x^4n(x^{-1})$ pour $x \neq 0$, alors n est soit le carré d'une fonction linéaire, soit le carré d'une dérivation sur $\mathbb{R}$. Dans l'article présent, nous généralisons ces résultats à des anneaux et des corps plus généraux.

In this paper, $(X, +)$ denotes an abelian group and R is used to denote any structure which is at least a (commutative) ring with 1. Any additional assumptions about R will be stated explicitly. Any function $A: X \to R$ satisfying

$$A(x+y) = A(x) + A(y) \quad \text{for all } x, y \in X$$

is called an *additive* function. A function $q: X \to R$ is called *quadratic* if it satisfies

$$q(x+y) + q(x-y) = 2q(x) + 2q(y) \quad \text{for all } x, y \in X.$$

A function $d: R \to R$ is called a *derivation* on R if d is additive and satisfies also $d(xy) = xd(y) + yd(x)$ for all $x, y \in R$. It is known that if $\operatorname{char} R \neq 2$, then any additive function $d: R \to R$ satisfying also $d(x^2) = 2xd(x)$ for all $x \in R$ is a derivation. And if K is a field with $\operatorname{char} K \neq 2$, then any additive $d: K \to K$ satisfying also $d(x) = -x^2d(x^{-1})$ for $x \in K \setminus \{0\}$ is a derivation. (For the equivalence of these conditions, see [7] or [6].)

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The main aim of this paper is to present generalizations of some results due to A. Grząślewicz [3]. That paper contained three main results. The first is that a positive-valued function $q: X \to \mathbb{R}$ ($\mathbb{R}$ = the field of real numbers) satisfies

$$(1) \qquad [q(x+y) - q(x-y)]^2 = 16q(x)q(y).$$

if and only if q is the square of an (arbitrary) additive function from X into $\mathbb{R}$. We show in Theorem 1 below that the co-domain $\mathbb{R}$ may be replaced by any ordered unique factorization domain.

The second main result in [3] is that an additive function $A: \mathbb{R} \to \mathbb{R}$ satisfies

$$A(x)^2 = x^4 A(x^{-1})^2, \quad x \neq 0,$$

if and only if A is either linear (*i.e.*, $A(x) = A(1)x$) or a derivation. In Theorem 2 we show that $\mathbb{R}$ may be replaced by any field of characteristic different from 2, 3, 5, and 7.

Finally, combining the first two results, one sees that $q: \mathbb{R} \to \mathbb{R}$ satisfies both (1) and

$$q(x) = x^4 q(x^{-1}), \quad x \neq 0,$$

if and only if q is the square of either a linear function or a derivation. Now this is true also if $\mathbb{R}$ is replaced by any ordered field.

Also we fill a gap in the original proof of the second result.

THEOREM 1.

(a) *Let R be a unique factorization domain with* char $R \neq 2$. *Then a quadratic function $q: X \to R$ satisfies (1) if and only if there exists an additive map $A: X \to R$ and a constant $c \in R$ for which*

$$(2) \qquad q(x) = cA(x)^2.$$

(b) *If R is an ordered ring, then a map $q: X \to R$ satisfies (1) if and only if q is quadratic.*

(c) *Let R be an ordered unique factorization domain. Then $q: X \to R$ satisfies (1) if and only if there is an additive $A: X \to R$ and a constant $c \in R$ for which (2) holds.*

PROOF. (a) The map q is quadratic if and only if there is a symmetric bi-additive $B: X \times X \to R$ such that $q(x) = B(x, x)$. (See [1].) Thus q satisfies (1) if and only if B satisfies

$$16B(x,y)^2 = 16B(x,x)B(y,y).$$

By Lemma 3 in [2], since char $R \neq 2$ the last equation is equivalent to

$$B(x,y) = cA(x)A(y)$$

for arbitrary additive $A\colon X \to R$ and constant $c \in R$. Thus (2) holds. The converse is easy to verify.

(b) The proofs of Propositions 1 and 2 in [3] carry over word-for-word to establish this part. (Note: Any ordered ring is automatically an integral domain of characteristic 0. See [8].)

(c) This part follows immediately from parts (a) and (b).

In order to prove the next main result we need a couple of lemmas. We observe that the proofs of Proposition 4 and Theorem 4 in [3] actually yield the following more general result.

LEMMA 1 ([3]). *If* char $R \neq 2$ *and* $A\colon R \to R$ *is additive, then the following are equivalent.*

$$A(x^2) = 2xA(x) - x^2A(1), \tag{3}$$

$$A(xy) = xA(y) + yA(x) - xyA(1), \tag{4}$$

$$A(x) = ax + d(x), \tag{5}$$

for an arbitrary constant $a \in R$ *and derivation* d *on* R. *If in addition* R *is a field, then the condition*

$$A(x) = -x^2A(x^{-1}) + 2xA(1), \quad x \neq 0, \tag{6}$$

is also equivalent to (3), (4), and (5).

At this point we note that the proof of Proposition 5 in [3] contains a gap. The possibility that $f(x^2) = -2xf(x)$ and $f(1) = 0$ is not ruled out there; such a situation would preclude the existence of r such that inequality (21) in [3] holds. That gap can be filled with the aid of the following lemma.

LEMMA 2.

(a) *Let* R *be an integral domain, with* char $R \neq 2, 3$. *Then* $A\colon R \to R$ *is additive and satisfies*

$$A(x^2)^2 = 4x^2A(x)^2 \tag{7}$$

if and only if A *is a derivation on* R.

(b) *Let* R *be an integral domain with* char $R = 3$. *Then* $A\colon R \to R$ *is an additive map satisfying (7) if and only if* A *is either linear or a derivation.*

(c) *In parts (a) and (b), equation (7) can be replaced by*

$$A(xy)^2 = [xA(y) + yA(x)]^2. \tag{8}$$

PROOF. (a) Putting $x = 1$ in (7), we see that $A(1)^2 = 4A(1)^2$, or $3A(1)^2 = 0$. Since char $R \neq 3$, we have therefore $A(1) = 0$. Now replace x by $x + 1$ in (7) and expand using additivity to get

$$[A(x^2) + 2A(x) + A(1)]^2 = 4(x^2 + 2x + 1)[A(x) + A(1)]^2.$$

By (7) and the fact that $A(1) = 0$, this simplifies to $4A(x^2)A(x) = 8xA(x)^2$, or, since char $R \neq 2$,

$$A(x)[A(x^2) - 2xA(x)] = 0. \tag{9}$$

For any $x \in R$ such that $A(x) = 0$, (7) implies that $A(x^2) = 0$, hence it follows from (9) that

$$A(x^2) = 2xA(x), \quad x \in R,$$

which means that A is a derivation. The converse is easily verified, and this finishes the proof of part (a).

(b) The only place we used the condition char $R \neq 3$ in part (a) was in deducing that $A(1) = 0$. So the same proof shows that (7) together with $A(1) = 0$ is equivalent to A being a derivation on R in the case char $R = 3$.

Assume now that $A(1) \neq 0$. Proceeding as in the proof of part (a), we arrive at

$$\begin{aligned} A(x^2)[A(x) + 2A(1)] &= 2xA(x)^2 + (2x^2 + x + 1)A(x)A(1) \\ &\quad + (x^2 + 2x)A(1)^2 \end{aligned} \tag{10}$$

instead of (9). Replacing x by $2x$ and using the additivity of A, we obtain

$$A(x^2)[2A(x) + 2A(1)] = xA(x)^2 + (x^2 + x + 2)A(x)A(1) + (x^2 + x)A(1)^2.$$

Adding this last equation to (10) and using the fact that $A(1) \neq 0$, we find that

$$A(x^2) = 2xA(x) + 2x^2A(1), \tag{11}$$

which is equivalent to (3). We could now appeal to Lemma 1, but it is more direct to square (11) and compare the result to (7). This yields

$$2A(x) + xA(1) = 0, \quad x \neq 0,$$

hence A is linear. The converse is straightforward, using the fact that char $R = 3$.

(c) For the "only if" parts, put $y = x$ in (8) and apply parts (a) and (b). The "if" parts are easy to verify.

Now we are ready to prove the next main result.

THEOREM 2. *Let K be a field with* char $K \notin \{2, 3, 5, 7\}$. *Then $A: K \to K$ is an additive map satisfying*

$$A(x^2) = x^4A(x^{-1})^2, \quad x \neq 0, \tag{12}$$

if and only if A is either linear or a derivation on K.

PROOF. If A is either linear or a derivation, then clearly A is additive and satisfies (12). For the converse, we show first that (12) implies (3).

If $A = 0$, then (3) holds trivially. Assume $A \neq 0$, and let an arbitrary $x \in K$ be chosen. If x is an integer multiple of 1, then the fact that A is additive implies that $A(x) = A(1)x$ and $A(x^2) = A(1)x^2$. Thus (3) holds also in this case. Now assume that x is not an integer multiple of 1. Since A is additive, we have

$$2nA((x^2 - n^2)^{-1}) = A((x-n)^{-1}) - A((x+n)^{-1}) \tag{13}$$

for all integers n ($\neq \pm x$). Converting (13) by means of (12), we obtain

$$2na_n(x^2-n^2)^{-2}A(x^2-n^2) = b_n(x-n)^{-2}A(x-n) - c_n(x+n)^{-2}A(x+n)$$

for some $a_n, b_n, c_n \in \{\pm 1\}$. By additivity, this is equivalent to

$$\begin{aligned} 2nA(x^2) &= [(\alpha_n - \beta_n)x^2 + 2n(\alpha_n + \beta_n)x + n^2(\alpha_n - \beta_n)]A(x) \\ &\quad - n[(\alpha_n + \beta_n)x^2 + 2n(\alpha_n - \beta_n)x + n^2(\alpha_n + \beta_n - 2)]A(1), \end{aligned} \tag{14}$$

where $\alpha_n = b_n a_n^{-1}$ and $\beta_n = c_n a_n^{-1}$ satisfy $\alpha_n^2 = \beta_n^2 = 1$. The four possibilities $(\alpha_n, \beta_n) = (1,1), (1,-1), (-1,1), (-1,-1)$ lead respectively to:

Case (i): $A(x^2) = 2xA(x) - x^2A(1)$.

Case (ii): $nA(x^2) = (x^2+n^2)A(x) - n^2(2x-n)A(1)$.

Case (iii): $nA(x^2) = -(x^2+n^2)A(x) + n^2(2x+n)A(1)$.

Case (iv): $A(x^2) = -2xA(x) + (x^2+2n^2)A(1)$.

Since the characteristic of K is at least 11 (or 0) we have at least 11 different integers n ($\neq \pm x$) for which (14) must hold. If $A(1) \neq 0$, there are at most 8 values of n which satisfy Cases (ii), (iii) and (iv). Thus Case (i) must hold, *i.e.*, we have (3).

On the other hand, suppose $A(1) = 0$. If $A(x) = 0$, then (14) shows that $A(x^2) = 0$ too, so (3) holds again. If $A(x) \neq 0$, then the previous argument shows that Cases (ii) and (iii) may be avoided by appropriate choices of n. Hence we are left with Case (i) or Case (iv). That is,

$$A(x^2) = 2xA(x) \quad \text{or} \quad A(x^2) = -2xA(x).$$

In either case, we have (7). By Lemma 2 (a), A is a derivation. Hence $A(x^2) = 2xA(x)$, and (3) follows.

Now we have shown that (12) implies (3). In the case $A(1) = 0$, (3) shows that $A(x^2) = 2xA(x)$, which means that A is a derivation (as in fact seen in the previous paragraph).

In the case $A(1) \neq 0$, Lemma 1 yields the validity of (6). Squaring (6) and comparing the result with (12), we find that

$$A(x^{-1}) = x^{-1}A(1), \quad x \neq 0.$$

Since $A(0) = 0$, this means that A is linear, and that completes the proof of the theorem.

REMARK. The exclusion of some fields of small characteristic in Theorem 2 is necessary. For example, let K be the field of 9 elements. Consider K as the vector space over $\mathbb{Z}_3$ generated by $\{1, \alpha\}$, where α satisfies $\alpha^2 + 1 = 0$. Define a map $A: K \to K$ on the basis $\{1, \alpha\}$ by $A(1) = 1$, $A(\alpha) = 2\alpha$, and extend A to an additive map on K. This A satisfies (12), but not (3). In particular, A is neither linear nor a derivation on K.

We conclude with the following.

THEOREM 3. *Let K be an ordered field. Then a map $q: K \to K$ satisfies (1) and*

$$q(x) = x^4 q(x^{-1}), \quad x \neq 0, \tag{15}$$

if and only if either

$$q(x) = q(1)x^2,$$

or there exist a nonzero constant $c \in K$ and a derivation d on K for which

$$q(x) = c[d(x)]^2.$$

PROOF. If q satisfies (1), then by Theorem 1 there exist a constant c ($\neq 0$) and an additive map A such that (2) holds. Now (15) and (2) imply that A satisfies (12). By Theorem 2, A must be either linear or a derivation, and then (2) gives the asserted forms of q. The converse is easily checked.

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EXACT SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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ABSTRACT. Infinite-dimensional families of exact solutions are derived for classical Navier-Stokes equations with constant kinematic viscosity.

RÉSUMÉ. Des familles à dimensions infinies de solutions exactes sont dérivées pour les équations classiques de Navier-Stokes avec viscosités kinématiques constantes.

The Navier-Stokes equations (NSE) for incompressible fluid have the form

$$\rho\frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\cdot\operatorname{grad})\mathbf{V} = \mathbf{f} - \operatorname{grad} P + \mu\Delta\mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0, \tag{1}$$

where $\mathbf{V}$ is the fluid velocity vector field, $\mathbf{f}$ is the field of external forces and P is the pressure. We assume that the density ρ and viscosity μ are constant; the kinematic viscosity $\nu = \mu/\rho > 0$. There are many papers on the NSE, starting with the works of Leray [4] and Kolmogorov [3], see the reviews [2], [6]; only highly symmetric exact solutions are known [6]. In this paper, we solve the Cauchy problem for the NSE with the special initial conditions

$$\mathbf{V}(0,\mathbf{x}) = \mathbf{V}_0(\mathbf{x}), \quad \mathbf{x} = (x,y,z) \in \mathbb{R}^3 \tag{2}$$

and construct the infinite-dimensional spaces S_α of exact solutions to the NSE that are non-symmetric and depend on all four variables t, x, y, z.

THEOREM 1. *Let the initial vector field* $\mathbf{V}_0(\mathbf{x})$ *satisfy the Beltrami equation*

$$\operatorname{curl}\mathbf{B}(\mathbf{x}) = \alpha\mathbf{B}(\mathbf{x}), \quad \alpha = \text{const}, \tag{3}$$

and $\mathbf{f} = -\rho\operatorname{grad}\Phi$, $\Phi(t,\mathbf{x})$ *is an external gravitational potential. Then the Cauchy problem (1)-(2) has the exact solution*

$$\mathbf{V}(t,\mathbf{x}) = \exp(-\nu\alpha^2 t)\mathbf{V}_0(\mathbf{x}), \quad P = \rho\big(C - \mathbf{V}_0^2(\mathbf{x})/2 - \Phi(t,\mathbf{x})\big). \tag{4}$$

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PROOF. Applying operator curl to the equation (3), we get

$$\operatorname{curl}\operatorname{curl}\mathbf{B}(\mathbf{x}) = \alpha^2\mathbf{B}(\mathbf{x}). \tag{5}$$

The known identity $\operatorname{curl}(\operatorname{curl}\mathbf{B}) = \operatorname{grad}(\operatorname{div}\mathbf{B}) - \Delta\mathbf{B}$ implies for any divergence-free vector field $\mathbf{B}$: $\Delta\mathbf{B} = -\operatorname{curl}(\operatorname{curl}\mathbf{B})$. Thus equation (5) yields $\Delta\mathbf{V}_0(\mathbf{x}) = -\alpha^2\mathbf{V}_0(\mathbf{x})$. Hence the vector field $\mathbf{V}(t,\mathbf{x})$ (4) satisfies the equations

$$\frac{\partial\mathbf{V}}{\partial t} = -\nu\alpha^2\mathbf{V}, \quad \Delta\mathbf{V} = -\alpha^2\mathbf{V}, \quad \operatorname{curl}\mathbf{V} = \alpha\mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0. \tag{6}$$

As is known, the NSE (1) for $\rho = \text{const}$ take the form

$$\frac{\partial\mathbf{V}}{\partial t} = \mathbf{V}\times\operatorname{curl}\mathbf{V} - \operatorname{grad}(\Phi + \mathbf{V}^2/2 + P/\rho) + \nu\Delta\mathbf{V}.$$

Substituting here formulae (6), we obtain that the vector field $\mathbf{V}(t,\mathbf{x})$ (4) is an exact solution to the Cauchy problem (1)–(2). ∎

REMARK 1. For each $\alpha \neq 0$, the vector fields $\mathbf{V}(t,\mathbf{x})$ (4) form an infinite-dimentional linear space S_α of exact solutions to the NSE and $S_\alpha \cap S_\beta = 0$ for $\alpha \neq \beta$. Using the Fourier method, we proved that any smooth solution to the Beltrami equation (3) has the form

$$\mathbf{V}_0(\mathbf{x}) = \int\!\!\int_{S^2} \big(\sin(\alpha\mathbf{k}\cdot\mathbf{x})\mathbf{T}(\mathbf{k}) + \cos(\alpha\mathbf{k}\cdot\mathbf{x})\mathbf{k}\times\mathbf{T}(\mathbf{k})\big)\, d\sigma, \tag{7}$$

where the integral is taken with respect to an arbitrary measure $d\sigma$ on the 2-dimensional unit sphere S^2: $\mathbf{k}\cdot\mathbf{k} = 1$ and $\mathbf{T}(\mathbf{k})$ is an arbitrary smooth vector field tangent to the unit sphere, $\mathbf{T}(\mathbf{k})\cdot\mathbf{k} = 0$. Formula (7) gives the explicit form of the exact solutions (4). The vector fields $\mathbf{V}_0(\mathbf{x})$ (7) have the form $\mathbf{V}_0(\mathbf{x}) = \operatorname{curl}\mathbf{V}_1(\mathbf{x}) + \alpha^{-1}\operatorname{curl}\operatorname{curl}\mathbf{V}_1(\mathbf{x})$ where

$$\mathbf{V}_1(\mathbf{x}) = \int\!\!\int_{S^2} \cos(\alpha\mathbf{k}\cdot\mathbf{x})\mathbf{S}(\mathbf{k})\, d\sigma, \tag{8}$$

$\mathbf{T}(\mathbf{k}) = -\alpha\mathbf{k}\times\mathbf{S}(\mathbf{k})$ and $\mathbf{S}(\mathbf{k})$ is an arbitrary vector field.

REMARK 2. Since the Beltrami equation (3) is invariant with respect to the arbitrary directional derivations, translations and rotations, the linear spaces S_α of the exact solutions (4) also are invariant with respect to the transforms

$$\mathbf{V}(t,\mathbf{x}) \longrightarrow (\mathbf{a}\cdot\nabla)\mathbf{V}(t,\mathbf{x}) = a_1\frac{\partial\mathbf{V}}{\partial x_1} + a_2\frac{\partial\mathbf{V}}{\partial x_2} + a_3\frac{\partial\mathbf{V}}{\partial x_3}, \tag{9}$$

$$\mathbf{V}(t,\mathbf{x}) \longrightarrow \mathbf{V}(t,\mathbf{x}-\mathbf{b}), \quad \mathbf{V}(t,\mathbf{x}) \longrightarrow Q\Big(\mathbf{V}\big(t, Q^{-1}(\mathbf{x})\big)\Big), \tag{10}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and Q is an orthogonal matrix.

SOLITON-LIKE SOLUTIONS TO THE NAVIER-STOKES EQUATIONS. Let $d\sigma$ in the integral (8) be the standard Euclidean measure on the unit sphere S^2 and the vector field $\mathbf{S}(\mathbf{k})$ be constant: $\mathbf{S}(\mathbf{k}) = \mathbf{A}/(4\pi)$. It is easy to verify that

$$\int\int_{S^2} \cos(\alpha \mathbf{k}\cdot\mathbf{x})\, d\sigma = \frac{4\pi \sin(\alpha|\mathbf{x}|)}{\alpha|\mathbf{x}|}.$$

Therefore the vector field $\mathbf{V}_1(\mathbf{x})$ (8) takes the form

$$\mathbf{V}_{1F}(\mathbf{x},\mathbf{A}) = F(\mathbf{x})\mathbf{A}, \quad F(\mathbf{x}) = \frac{\sin(\alpha|\mathbf{x}|)}{\alpha|\mathbf{x}|}. \tag{11}$$

The function $F(\mathbf{x}) = F(|\mathbf{x}|)$ is uniquely defined as a smooth spherically symmetric solution to the Helmholtz equation $\Delta F(\mathbf{x}) = -\alpha^2 F(\mathbf{x})$, $F(0) = 1$. We have $\operatorname{curl} \mathbf{V}_{1F}(\mathbf{x},\mathbf{A}) = \operatorname{grad} F(\mathbf{x}) \times \mathbf{A} = h(\mathbf{x})\mathbf{x} \times \mathbf{A}$ where

$$h(\mathbf{x}) = h(|\mathbf{x}|) = F'(|\mathbf{x}|)|\mathbf{x}|^{-1} = \cos(\alpha|\mathbf{x}|)|\mathbf{x}|^{-2} - \sin(\alpha|\mathbf{x}|)\alpha^{-1}|\mathbf{x}|^{-3} \tag{12}$$

is a smooth function. Hence the corresponding exact solution (4), (7) is

$$\begin{aligned}\mathbf{V}_F(t,\mathbf{x},\mathbf{A}) = \exp(-\nu\alpha^2 t)\Big((\alpha F(\mathbf{x}) + \alpha^{-1}h(\mathbf{x}))\mathbf{A} + h(\mathbf{x})\mathbf{x} \\ \times \mathbf{A} - (\mathbf{x}\cdot\mathbf{A})|\mathbf{x}|^{-2}(\alpha F(\mathbf{x}) + 3\alpha^{-1}h(\mathbf{x}))\mathbf{x}\Big).\end{aligned} \tag{13}$$

For $|\mathbf{x}| > 1$, the functions $F(\mathbf{x})$ and $h(\mathbf{x})$ satisfy the inequalities $|F(\mathbf{x})| \leq |\alpha|^{-1}|\mathbf{x}|^{-1}$, $|h(\mathbf{x})| \leq 2|\mathbf{x}|^{-2}$ that yield the inequality

$$|\mathbf{V}_F(t,\mathbf{x},\mathbf{A})| < \exp(-\nu\alpha^2 t)\left(3 + \frac{8}{|\alpha \mathbf{x}|}\right)\frac{|\mathbf{A}|}{|\mathbf{x}|}. \tag{14}$$

Thus $|\mathbf{V}_F(t,\mathbf{x},\mathbf{A})|$ tends to zero as $C\exp(-\nu\alpha^2 t)|\mathbf{x}|^{-1}$ when $|\mathbf{x}| \longrightarrow \infty$.

Applying the transforms (9)–(10) to the flow (13), we obtain the exact solutions

$$(\mathbf{a}\cdot\nabla)\mathbf{V}_F(t,\mathbf{x}-\mathbf{b},\mathbf{A}), \quad \mathbf{V}_F(t,\mathbf{x}-\mathbf{c},\mathbf{B}). \tag{15}$$

Theorem 1 and Remark 1 imply that the interaction of the $N+M$ solutions (15) (with the same constant α) is described by their sum

$$\mathbf{V}_{N+M}(t,\mathbf{x}) = \sum_{i=1}^{N}(\mathbf{a}_i\cdot\nabla)\mathbf{V}_F(t,\mathbf{x}-\mathbf{b}_i,\mathbf{A}_i) + \sum_{j=1}^{M}\mathbf{V}_F(t,\mathbf{x}-\mathbf{c}_j,\mathbf{B}_j) \tag{16}$$

that also is an exact solution to the NSE. Hence the solutions (15) actually do not interact with each other and thus manifest a soliton-like behaviour. The inequality (14) yields that the vector fields (15)–(16) decrease as $C\exp(-\nu\alpha^2 t)|\mathbf{x}|^{-1}$ when $|\mathbf{x}| \longrightarrow \infty$. Therefore we refer to the exact solutions (15) as the soliton-like solutions to the viscous NSE, or the viscons.

The viscon $(\mathbf{a}\cdot\nabla)\mathbf{V}_F(t,\mathbf{x},\mathbf{A})$ (15) is non-symmetric if the vectors $\mathbf{a}$ and $\mathbf{A}$ are non-collinear. The corresponding fluid streamlines are dense in certain 3-dimensional domains. For the generic vectors $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_j$, $\mathbf{A}_i$ and $\mathbf{B}_j$, the flows $\mathbf{V}_{N+M}(t,\mathbf{x})$ (16) are non-symmetric and have dense streamlines.

The viscon $\mathbf{V}_F(t,\mathbf{x},\mathbf{A})$ (13) is invariant under rotations around the axis $\mathbf{A}$. The corresponding streamlines have first integral $\psi_1(\mathbf{x}) = |\mathbf{x}\times\mathbf{A}|^2 h(\mathbf{x})$ and hence possess invariant surfaces $\psi_1(\mathbf{x}) = \text{const}$. The condition $\psi_1(\mathbf{x}) = 0$ defines a line $\mathbf{x} = \lambda\mathbf{A}$ and an infinite family of invariant spheres $|\mathbf{x}| = R_k$ where radii R_k are roots of the equation $h(R_k) = 0$: $\alpha R_k = \tan(\alpha R_k)$. Inside the ball $0 \le |\mathbf{x}| < R_1$ and in the domains $R_k < |\mathbf{x}| < R_{k+1}$, the streamlines enjoy a quasi-periodic winding on the $\mathbf{A}$-axially symmetric tori $\psi_1(\mathbf{x}) = \text{const} \neq 0$.

For $N = 0$ and $\mathbf{c}_j = \mu_j\mathbf{B}$ and $\mathbf{B}_j = \lambda_j\mathbf{B}$, the flows (16) take the form

$$\mathbf{V}_{M\mathbf{B}}(t,\mathbf{x}) = \sum_{j=1}^{M} \lambda_j \mathbf{V}_F(t,\mathbf{x}-\mu_j\mathbf{B},\mathbf{B}) \tag{17}$$

and are $\mathbf{B}$-axially symmetric. The flow (17) has first integral $\psi_{M\mathbf{B}}(\mathbf{x}) = |\mathbf{x}\times\mathbf{B}|^2(\lambda_1 h(\mathbf{x}-\mu_1\mathbf{B})+\cdots+\lambda_M h(\mathbf{x}-\mu_M\mathbf{B}))$. The invariant surfaces $\psi_{M\mathbf{B}}(\mathbf{x}) = 0$ generalize the spherical surfaces $\psi_1(\mathbf{x}) = 0$ for $M = 1$ and are not spherical. In the nonviscous limit $\nu \longrightarrow 0$, the formulae (13), (15)–(17) describe steady flows of an ideal fluid. In view of (14), all these exact solutions decrease as $C|\mathbf{x}|^{-1}$ when $|\mathbf{x}| \longrightarrow \infty$.

SPATIALLY PERIODIC SOLUTIONS. These solutions satisfy the periodicity conditions

$$\mathbf{V}(t,\mathbf{x}+L_j\mathbf{e}_j) = \mathbf{V}(t,\mathbf{x}), \quad P(t,\mathbf{x}+L_j\mathbf{e}_j) = P(t,\mathbf{x}), \tag{18}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the coordinate unit orts and periods $L_j > 0$ are arbitrary constants. The formulae (4), (7) for $d\sigma = \delta(\mathbf{k}_1) + \cdots + \delta(\mathbf{k}_N)$ give the exact solutions

$$\mathbf{V}(t,\mathbf{x}) = e^{-\nu\alpha^2 t}\sum_{i=1}^{N}(\sin(\alpha\mathbf{k}_i\cdot\mathbf{x})\mathbf{T}(\mathbf{k}_i) + \cos(\alpha\mathbf{k}_i\cdot\mathbf{x})\mathbf{k}_i\times\mathbf{T}(\mathbf{k}_i)), \tag{19}$$

where $\mathbf{k}_i\cdot\mathbf{k}_i = 1$, $T(\mathbf{k}_i)\cdot\mathbf{k}_i = 0$, and $\mathbf{k}_i = k_{i1}\mathbf{e}_1 + k_{i2}\mathbf{e}_2 + k_{i3}\mathbf{e}_3$. The vector fields (19) satisfy the periodicity conditions (18) if $\alpha k_{ij}L_j = 2\pi n_{ij}$ where n_{ij} are some integers. The condition $\mathbf{k}_i\cdot\mathbf{k}_i = 1$ yields

$$(n_{i1}/L_1)^2 + (n_{i2}/L_2)^2 + (n_{i3}/L_3)^2 = (\alpha/2\pi)^2. \tag{20}$$

For any given periods L_1, L_2, L_3, equation (20) defines a countable set of parameters α for which formula (19) gives the exact periodic solutions to the NSE; they form a finite-dimensional linear space $S_{\alpha p}$. The dimension of the $S_{\alpha p}$ depends on the number of integer solutions to equation (20). For the generic case there are only solutions $(\pm n_{i1}, \pm n_{i2}, \pm n_{i3})$. The space of vectors $\mathbf{T}(\mathbf{k}_i)$ is 2-dimensional. Hence the generic space $S_{\alpha p}$ is 8-dimensional, $S_{\alpha p} = \mathbb{R}^8$. For $L_1 = L_2 = L_3$, any permutation of $(\pm n_{i1}, \pm n_{i2}, \pm n_{i3})$ is a solution. Hence the space $S_{\alpha p}$ is 48-dimensional if the three integers $|n_{ij}|$ are different, $S_{\alpha p} = \mathbb{R}^{48}$. If equation (20) has two different triples of solutions, as for $2^2+3^2+7^2 = 1^2+5^2+6^2 = 62$, then the space $S_{\alpha p} = \mathbb{R}^{96}$, *etc.*

ATTRACTORS. Let us suppose that the vector field of the external forces $\mathbf{f}(t,\mathbf{x})$ for all t satisfies the Beltrami equation (3): $\operatorname{curl}\mathbf{f}(t,\mathbf{x}) = \alpha\mathbf{f}(t,\mathbf{x})$. The $\mathbf{f}(t,\mathbf{x})$ can be either periodic (18) or non-periodic. For the Beltrami initial data (2), the Cauchy problem for the NSE (1) reduces to the equations

$$\frac{\partial \mathbf{V}(t,\mathbf{x})}{\partial t} = -\nu\alpha^2\mathbf{V}(t,\mathbf{x}) + \mathbf{f}(t,\mathbf{x}), \quad \mathbf{V}(0,\mathbf{x}) = \mathbf{V}_0(\mathbf{x})$$

in the space of the Beltrami vector fields $\mathbf{V}(t,\mathbf{x})$. The complete solution to the Cauchy problem is

$$\mathbf{V}(t,\mathbf{x}) = \exp(-\nu\alpha^2 t)\left(\mathbf{V}_0(\mathbf{x}) + \int_0^t \exp(\nu\alpha^2\tau)\mathbf{f}(\tau,\mathbf{x})\,d\tau\right). \tag{21}$$

For the time-independent forces $\mathbf{f}(t,\mathbf{x}) = \mathbf{f}(\mathbf{x})$, the solution has the form

$$\mathbf{V}(t,\mathbf{x}) = \exp(-\nu\alpha^2 t)\big(\mathbf{V}_0(\mathbf{x}) - (\nu\alpha^2)^{-1}\mathbf{f}(\mathbf{x})\big) + (\nu\alpha^2)^{-1}\mathbf{f}(\mathbf{x}). \tag{22}$$

For this case, the Navier-Stokes dynamics in the space of the Beltrami vector fields has the point attractor $\mathbf{V}_a(\mathbf{x}) = (\nu\alpha^2)^{-1}\mathbf{f}(\mathbf{x})$.

For any domain $\Omega \subset \mathbb{R}^3$, the time averaged energy dissipation rate (per unit volume) is defined [3] as

$$\epsilon = \frac{\nu}{|\Omega|} \lim_{T\longrightarrow\infty} \sup \frac{1}{T}\int_0^T \int_\Omega |\nabla\mathbf{V}(t,\mathbf{x})|^2\,dx\,dy\,dz\,dt.$$

For the solutions (22), this quantity is $\epsilon = (\nu\alpha^4)^{-1}\int_\Omega |\nabla\mathbf{f}(\mathbf{x})|^2\,dx\,dy\,dz/|\Omega|$. Hence $\epsilon = C\nu^{-1} \longrightarrow \infty$ as the Reynolds number $Re \approx \nu^{-1} \longrightarrow \infty$.

NON-NEWTONIAN FLUIDS. There are generalizations of the NSE with some hyperviscosities ν_k [6], [5]:

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\operatorname{grad})\mathbf{V} = \mathbf{f} - \operatorname{grad} P - \sum_{k=1}^N \nu_k(-\Delta)^k\mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0.$$

These equations have exact solutions $\mathbf{V}(t,\mathbf{x}) = \exp\big(-(\sum_{k=1}^N \nu_k\alpha^{2k})t\big)\mathbf{V}_0(\mathbf{x})$ for $\mathbf{f} = 0$ and the solutions analogous to (21), (22) where $\mathbf{V}_0(\mathbf{x})$ and $\mathbf{f}(t,\mathbf{x})$ are arbitrary Beltrami fields (7). The results of this paper are generalized in [1] for the viscous magnetohydrodynamics equations.

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AN ELEMENTARY PROOF OF A THEOREM OF DELANGE

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ABSTRACT. We will give a short elementary proof of the following theorem: There exists a constant Θ (given in (8) below), such that for all x we have

$$\sum_{n\leq x}\big(\omega(n)-\log\log x\big)^2 = x\log\log x+\Theta x+O\left(\frac{x\log\log x}{\log x}\right).$$

RÉSUMÉ. Nous démontrons, d'une façon élémentaire, le théorème suivant: Il existe une constante Θ (définie en (8)) pour laquelle pour tout x,

$$\sum_{n\leq x}\big(\omega(n)-\log\log x\big)^2 = x\log\log x+\Theta x+O\left(\frac{x\log\log x}{\log x}\right).$$

1. Theorem of Hardy and Ramanujan.

1.1. Let $n\in\mathbb{N}$, and let p denote a prime. Define $\pi(x)$ to be the number of primes $\leq x$, $\pi_k(x)$ to be the number of positive integers $\leq x$ having exactly k prime factors, and recall the definitions of arithmetic functions $d(n)$, $\omega(n)$ and $\Omega(n)$:

$$d(n)\stackrel{\text{def}}{=}\sum_{d|n}1,\quad \omega(n)\stackrel{\text{def}}{=}\sum_{p|n}1\quad\text{and}\quad \Omega(n)\stackrel{\text{def}}{=}\sum_{p^\alpha\|n}\alpha.$$

Also, let $[x]$ be the integer part of x, and let $\{x\}=x-[x]$. An arithmetic function $f(n)$ is said to have a normal order $F(n)$, if for any $\epsilon>0$, for almost all $n\leq x$,

$$(1)\qquad (1-\epsilon)F(n)<f(n)<(1+\epsilon)F(n).$$

In 1916 Hardy and Ramanujan [9] showed that both functions $\omega(n)$ and $\Omega(n)$ have normal order $\log\log n$, which follows from the estimate[1] [9, p. 92]:

$$2^{\log\log n-f\sqrt{\log\log n}}<d(n)<2^{\log\log n+f\sqrt{\log\log n}},$$

[1] Earlier, in 1907, S. Wigert [21] used the prime number theorem to prove that the maximum order $D(n)$ of $d(n)$ satisfies: $2^{(1-\epsilon)\log n/\log\log n}<D(n)<2^{(1+\epsilon)\log n/\log\log n}$, for $\forall\epsilon>0$. While studying 'highly composite numbers', in 1915, Ramanujan [16] gave an elementary proof of this.

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for almost all $n \le x$, for every diverging f. The key idea in the proof was to show

$$(2) \qquad k\pi_{k+1}(x) \le \sum_{p^2 \le x} \pi_k\left(\frac{x}{p}\right) \Longrightarrow \pi_{k+1}(x) < \frac{Ax}{\log x}\frac{(\log\log x + B)^k}{k!}, \quad \forall x \ge 2,$$

for absolute $A, B \in \mathbb{R}$; *i.e.*, an upper bound version of a theorem of Landau [11].

2. Theorem of Turán.

2.1. In 1934, P. Turán [20] found a completely different proof of the Hardy-Ramanujan theorem. The new method he devised turned out to be analogous to Chebyshev's proof of the law of large numbers (*cf.* [4]). It yielded the following

$$(3) \qquad \mathfrak{S}(x) \stackrel{\text{def}}{=} \sum_{n \le x} \big(\omega(n) - \log\log x\big)^2 \ll x \log\log x.$$

At the end of [20] he also states that $\mathfrak{S}(x) = x\log\log x + o(x\log\log x)$ could be obtained. Turán's theorem (3) implies that the normal order of $\omega(n)$ is $\log\log n$.

2.2. Problems of sharpening of estimates of $\mathfrak{S}(x)$ and questions concerning $\pi_k(x)$ are clearly related. In a series of papers [18] L. G. Sathe proved[2] that:

$$(4) \qquad \pi_k(x) = \big(1 + o(1)\big) f(c) \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!},$$

$$\text{where } f(c) \stackrel{\text{def}}{=} \frac{1}{\Gamma(c+1)}\left(\prod_p \left(1 - \frac{1}{p}\right)e^{1/p}\right)^c \prod_p \left(1 + \frac{c}{p}e^{-c/p}\right),$$

with $k = c\log\log x$. His complicated (over 100 pages long) proofs were simplified by A. Selberg [19] in 1954, and some of the ideas behind Selberg's analytic method were refined by Delange [1], [2] to obtain sharper versions of these results. One of his theorems states: Let $f(n)$ be a non-negative additive arithmetical function, so that $f(p) = 1$, for all primes p. Then for every $s \ge 0$ there exist functions $A_0(z), \dots, A_m(z)$ analytic on $|z| \le 1$, such that $A_0(0) = \cdots = A_m(0) = 0$, and

$$(5) \qquad \sum_{n \le x} z^{f(n)} = x(\log x)^{z-1}\left(\sum_{i=0}^{m} \frac{A_i(z)}{(\log x)^i} + O\left(\frac{1}{(\log x)^{m+1}}\right)\right),$$

[2] This extended some earlier work of P. Erdős [6] of 1948, who showed that, for any $c \in \mathbb{R}$,

$$\log\log x - c\sqrt{\log\log x} < k < \log\log x + c\sqrt{\log\log x} \Longrightarrow \pi_k(x) = \big(1 + o(1)\big)\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Already during the 1930s Erdős [5] generalized Turán's theorem to $\omega\big(f(p)\big)$, with $f(p) = \phi(p)$.

where the O-constant is uniform for $|z| \leq 1$. As a special case of (5) one[3] obtains:

$$\sum_{n\leq x} \omega(n) = x\log\log x + Bx + \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m} G^{(m)}(1) \frac{x}{(\log x)^m} + O\left(\frac{x}{(\log x)^{n+1}}\right), \tag{6}$$

for all n, where B is a constant (defined in (8) below), and $G(s) = (s-1)\zeta(s)/s$.

2.3. Finally, one should also note that some weaker results in the direction of the asymptotic version of (3) could be deduced from the important Erdős-Kac theorem [7] of 1939 (see [4]): For constants $\alpha, \beta \in \mathbb{R}$, with $\alpha \leq \beta$, we have

$$S(x;\alpha,\beta) \stackrel{\text{def}}{=} \#\left\{n \leq x : \alpha \leq \frac{\omega(n) - \log\log n}{(\log\log n)^{1/2}} \leq \beta\right\} \Longrightarrow$$
$$\lim_{x\to\infty} \frac{S(x;\alpha,\beta)}{x} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2}\,dt \stackrel{\text{def}}{=} \Phi(\alpha,\beta), \tag{7}$$

where $\Phi(\alpha,\beta)$ is the Gaussian normal distribution. However, (7) doesn't imply (6).

3. Main theorem.

3.1. In this short note we show how one can improve the upper bound in (3). In fact, we will modify Turán's original method to obtain the asymptotic estimate:

THEOREM 1. *There is a constant Θ, such that[4] for all $x > 2$ we have:*

$$\sum_{n\leq x} \left(\omega(n) - \log\log x\right)^2 = x\log\log x + \Theta x + O\left(\frac{x\log\log x}{\log x}\right). \tag{9}$$

Our proof of (9) is more elementary and straight-forward than the one due to Delange, and it uses virtually nothing beyond the Abel's summation and a couple of estimates related to Mertens' theorem. To be more precise, we'll only need:

[3] For higher moments Delange (see [2, p. 136]) just states the existence (implied by the general formula (5)), but he does not give any explicit functions $A_i(z)$, like those exhibited in (6).

[4] Θ can be given explicitly as: $\Theta = \zeta(2) - A + B + B^2$, where $\zeta(2) = \sum_n \frac{1}{n^2} = \pi^2/6$, and A and B are constants defined as (the sums are extended over all primes, γ is Euler's constant):

$$A = \sum_p \frac{1}{p^2} \quad \text{and} \quad B = \gamma - \sum_p \left\{\log\left(\frac{1}{1-\frac{1}{p}}\right) - \frac{1}{p}\right\} \approx 0.26149\cdots. \tag{8}$$

LEMMA 1 (MERTENS [12], HARDY [8]). *For the constant B, defined in (8),*

$$\mathfrak{M}(x) \stackrel{\text{def}}{=} \sum_{p\le x} \frac{1}{p} = \log\log x + B + O\left(\frac{1}{\log x}\right). \tag{10}$$

LEMMA 2 (MURTY [13, Problem 9.4.4]). *For all $x > 0$ we have:*

$$\sum_{p\le \frac{x}{2}} \frac{1}{p\log(\frac{x}{p})} \ll \frac{\log\log x}{\log x}. \tag{11}$$

3.2. In order to prove Theorem 1, we shall also require one new lemma:

LEMMA 3. *Let B be the constant defined in (8), and let $\Psi = \zeta(2) + B^2$, then*

$$\sum_{pq\le x} \frac{1}{pq} = (\log\log x)^2 + 2B(\log\log x) + \Psi + O\left(\frac{\log\log x}{\log x}\right). \tag{12}$$

PROOF. For p, q, with $pq \le x$, we have $p \le \frac{x}{2}$, since $q \ge 2$. And by Lemma 1,

$$\begin{aligned}
\sum_{pq\le x} \frac{1}{pq} &= \sum_{p\le\frac{x}{2}} \frac{1}{p} \sum_{q\le\frac{x}{p}} \frac{1}{q} = \sum_{p\le\frac{x}{2}} \frac{1}{p}\left\{\log\log\left(\frac{x}{p}\right) + B + O\left(\frac{1}{\log(\frac{x}{p})}\right)\right\} \\
&= \sum_{p\le\frac{x}{2}} \frac{1}{p}\log\log\left(\frac{x}{p}\right) + B\left\{\log\log\left(\frac{x}{2}\right) + B + O\left(\frac{1}{\log x}\right)\right\} \\
&\qquad + O\left(\sum_{p\le\frac{x}{2}} \frac{1}{p\log(\frac{x}{p})}\right) \\
&= \sum_{p\le\frac{x}{2}} \frac{1}{p}\log\log\left(\frac{x}{p}\right) + B\log\log x + B^2 + O\left(\frac{\log\log x}{\log x}\right),
\end{aligned}$$

by Lemma 2. From the definition of $\mathfrak{M}(x)$, and the partial summation, we get

$$\begin{aligned}
\sum_{p\le\frac{x}{2}} \frac{1}{p}\log\log\left(\frac{x}{p}\right) &= \int_1^{\frac{x}{2}} \log\log\left(\frac{x}{t}\right) d\mathfrak{M}(t) \\
&= \mathfrak{M}\left(\frac{x}{2}\right)\log\log 2 - \int_1^{\frac{x}{2}} \mathfrak{M}(t)\, d\log\log\left(\frac{x}{t}\right).
\end{aligned}$$

Now, realizing that $\left(\log\log(\frac{x}{t})\right)' = \frac{-1}{\log(\frac{x}{t})}\frac{dt}{t}$, we can apply (10) again and write:

$$\begin{aligned}
\sum_{p\le\frac{x}{2}} \frac{1}{p}\log\log\left(\frac{x}{p}\right) &= \left\{\log\log\left(\frac{x}{2}\right) + B + O\left(\frac{1}{\log x}\right)\right\}(\log\log 2) \\
&\qquad + \int_1^{\frac{x}{2}} \frac{\mathfrak{M}(t)}{\log(\frac{x}{t})}\frac{dt}{t}
\end{aligned}$$

$$= (\log\log 2)\log\log\left(\frac{x}{2}\right) + B\log\log 2 + O\left(\frac{1}{\log x}\right)$$
$$+ \int_1^{\frac{x}{2}} \frac{\log\log t + B + O(\frac{1}{\log t})}{\log x - \log t}\frac{dt}{t}.$$

The last integral $I(x)$ can be broken into three parts, and estimated as follows:

$$I(x) = \int_1^{\frac{x}{2}} \frac{\log\log t + B + O(\frac{1}{\log t})}{\log x - \log t}\frac{dt}{t} = \int_1 + \int_2 + \int_3$$
$$\stackrel{\text{def}}{=} \int_1^{\frac{x}{2}} \frac{\log\log t}{\log x - \log t}\frac{dt}{t} + \int_1^{\frac{x}{2}} \frac{B}{\log x - \log t}\frac{dt}{t} + \int_1^{\frac{x}{2}} \frac{O(1)dt}{t\log t(\log x - \log t)}.$$

3.3. It is now easy to see that if we let $v = \log x - \log t$, then $dv = -\frac{dt}{t}$, and

$$\int_1 \stackrel{\text{def}}{=} \int_1^{\frac{x}{2}} \frac{\log\log t}{\log x - \log t}\frac{dt}{t} = -\int_{\log x}^{\log 2} \frac{\log(\log x - v)}{v}dv$$
$$= \int_{\log 2}^{\log x} \frac{\log(\log x - v)}{v}dv = \int_{\log 2}^{\log x} \frac{\log\{\log x \cdot (1 - \frac{v}{\log x})\}}{v}dv$$
$$= \log\log x \int_{\log 2}^{\log x} \frac{dv}{v} + \int_{\log 2}^{\log x} \frac{\log(1 - \frac{v}{\log x})}{v}dv$$
$$= (\log\log x)\{\log\log x - \log\log 2\} + \int_{\log 2}^{\log x} \frac{\log(1 - \frac{v}{\log x})}{v}dv,$$

and here, if we take $s = \frac{v}{\log x}$, then $ds = \frac{dv}{\log x}$, and the integral $\int_1$ becomes

$$\int_1 = (\log\log x)\{\log\log x - \log\log 2\} + \int_{\frac{\log 2}{\log x}}^1 \frac{\log(1-s)}{s}ds. \tag{13}$$

But, as we already noted, $\log(1-s) = O(s)$, for $0 < s < r < 1$. Also notice that

$$\int_0^1 \frac{\log(1-t)}{t}dt = \left[\sum_{n=1}^{\infty}\frac{t^n}{n^2}\right]_0^1 = \zeta(2) = \frac{\pi^2}{6} \Longrightarrow$$
$$\int_{\frac{\log 2}{\log x}}^1 \frac{\log(1-s)}{s}ds = \frac{\pi^2}{6} - \int_0^{\frac{\log 2}{\log x}} \frac{\log(1-s)}{s}ds = \frac{\pi^2}{6} + O\left(\frac{1}{\log x}\right).$$

3.4. Making the same substitution for the integrals $\int_2$ and $\int_3$, we get

$$\int_2 \stackrel{\text{def}}{=} \int_1^{\frac{x}{2}} \frac{B}{\log x - \log t}\frac{dt}{t} = B\int_1^{\frac{x}{2}} \frac{1}{\log x - \log t}\frac{dt}{t} = -B\int_{\log x}^{\log 2}\frac{dv}{v}$$
$$= B(\log\log x) - B\log\log 2,$$

and

$$\int_3 \stackrel{\text{def}}{=} \int_1^{\frac{x}{2}} \frac{O(1)dt}{t\log t(\log x - \log t)} \ll \frac{\log\log x}{\log x}.$$

Combining estimates of these three integrals proves the above Lemma 3. ∎

4. Method of moments.

4.1. Let us now estimate the first moment of the function $\omega(n)$:

$$\begin{aligned}\sum_{n\le x}\omega(n) &= \sum_{n\le x}\sum_{p\mid n} 1 = \sum_{n\le x}\sum_{n\equiv 0(\mathrm{mod}\, p)} 1 = \sum_{p\le x}\left[\frac{x}{p}\right] = \sum_{p\le x}\left(\frac{x}{p}-\left\{\frac{x}{p}\right\}\right)\\ &= x\sum_{p\le x}\frac{1}{p} - \sum_{p\le x}\left\{\frac{x}{p}\right\} = x\sum_{p\le x}\frac{1}{p} + O(\pi(x))\\ &= x\log\log x + Bx + O\big(\pi(x)\big),\end{aligned}$$

by the estimate (10) in Lemma 1, with the same constant $B \approx 0.26149\cdots$. ■

4.2. In order to obtain the second moment of $\omega(n)$, we similarly write

$$\sum_{n\le x}\omega^2(n) = \sum_{\substack{p\neq q\\ pq\le x}}\sum_{\substack{n\le x\\ pq\mid n}} 1 + \sum_{n\le x}\omega(n) = \sum_{\substack{p\neq q\\ pq\le x}}\left[\frac{x}{pq}\right] + \sum_{n\le x}\omega(n),$$

and instead of crudely estimating $\{x\}$ in $[x] = x - \{x\}$, note that by Lemma 2

$$\sum_{pq\le x} 1 \le \sum_{p\le\frac{x}{2}}\sum_{q\le\frac{x}{p}} 1 \ll \sum_{p\le\frac{x}{2}}\frac{\frac{x}{p}}{\log(\frac{x}{p})} \ll \frac{x\log\log x}{\log x} \Longrightarrow$$

$$(14)\qquad \sum_{\substack{p\neq q\\ pq\le x}}\left[\frac{x}{pq}\right] = \sum_{\substack{p\neq q\\ pq\le x}}\frac{x}{pq} + O\Big(\sum_{pq\le x} 1\Big) = \sum_{\substack{p\neq q\\ pq\le x}}\frac{x}{pq} + O\left(\frac{x\log\log x}{\log x}\right).$$

4.3. Thus, with the help of (14), and our new Lemma 3, we can deduce

$$\begin{aligned}\sum_{n\le x}\omega(n)^2 &= \sum_{\substack{p\neq q\\ pq\le x}}\frac{x}{pq} + O\left(\frac{x\log\log x}{\log x}\right) + x\log\log x + Bx + O\left(\frac{x}{\log x}\right)\\ &= \sum_{pq\le x}\frac{x}{pq} - \sum_{p\le\sqrt{x}}\frac{x}{p^2} + x\log\log x + Bx + O\left(\frac{x\log\log x}{\log x}\right)\\ &= x\sum_{pq\le x}\frac{1}{pq} + x\log\log x + (B-A)x + O\left(\frac{x\log\log x}{\log x}\right)\\ &= x(\log\log x)^2 + (2B+1)x(\log\log x) + \Theta x + O\left(\frac{x\log\log x}{\log x}\right),\end{aligned}$$

where $\Theta = B - A + \Psi$. ■

Combining this with the first moment estimate gives

$$\begin{aligned}\mathfrak{S}(x) &= \sum_{n\le x}\big(\omega(n) - \log\log x\big)^2\\ &= \sum_{n\le x}\omega^2(n) - 2\log\log x\sum_{n\le x}\omega(n) + (\log\log x)^2\sum_{n\le x} 1\end{aligned}$$

$$= 2x(\log\log x)^2 + (2B+1)x\log\log x + \Theta x + O\left(\frac{x\log\log x}{\log x}\right)$$

$$- 2\log\log x\left\{x\log\log x + Bx + O\left(\frac{x}{\log x}\right)\right\}$$

$$= x\log\log x + \Theta x + O\left(\frac{x\log\log x}{\log x}\right). \quad \blacksquare$$

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ON A CLASS OF LÉVY PROCESSES USED TO MODEL STOCK PRICE MOVEMENTS WITH POSSIBLE DOWNWARD JUMPS

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Presented by Donald A. Dawson, FRSC

RÉSUMÉ. Afin de modéliser le cours des actions en bourse, on introduit un nouveau modèle stochastique qui généralise de façon naturelle le mouvement brownien géométrique au cas d'évènements discontinus. Le modèle est en accord avec le comportement de sousmartingale des fluctuations du cours des actions en bourse et est fondé sur une transformation exponentielle d'une classe d'un processus de Lévy avec des sauts négatifs possibles. On détermine une mesure martingale à risque neutre et on l'applique à la détermination du prix des calls européens pour les modèles considérés. Dans le cas spécifique d'un mouvement brownien géométrique, la formule utilisée correspond à la formule de Black-Scholes. Les résultats obtenus impliquent que le mouvement brownien géométrique et la formule de Black-Scholes peuvent être utilisés comme approximations si l'amplitude des sauts est petite. On définit les distributions des premiers temps de passage pour les processus de Lévy sous-jacents qui généralisent au cas discontinu les résultats classiques obtenus par Schrödinger [7] et Smoluchowsky [9]. On révèle la présence d'un point critique dans la formation de grands écarts qui sont en relation avec le comportement à long terme du modèle présenté. Enfin, le modèle présenté est comparé avec d'autres modèles qui prennent en considération les discontinuités dans les fluctuations du cours des actions en bourse.

1. Introduction. In this paper, we consider the class of Hougaard stochastic processes, which are particular univariate Lévy processes, *i.e.*, stochastic processes with stationary independent increments which start from the origin. A subclass of Hougaard processes was considered by Lee and Whitmore [3] in connection to a problem of financial mathematics. Hereinafter, we denote Hougaard processes by $X_{p,\mu,\lambda}(t)$, where $t \geq 0$. These processes are described in Section 2. In this paper, we assume that parameter $p \in (-\infty, 0]$, in which case it characterizes the magnitude of downward jumps (see below).

Now, consider an exponential transformation of Hougaard processes. Our interest in this transformation is due to the fact that we propose its use for modelling stock price movements. These exponentially transformed Hougaard processes are

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hereinafter referred to as *geometric Hougaard processes* and denoted by

$$S_{p,\mu,\lambda}(t) := S(0) \cdot \exp\{X_{p,\mu,\lambda}(t) - \zeta_{p,0,\lambda}(1) \cdot t\} \tag{1}$$

with $p \in (-\infty, 0]$. Here, $S(0) > 0$ is the (non-random) initial price of the stock, $S_{p,\mu,\lambda}(t)$ is the price of the stock at time t, and $\zeta_{p,0,\lambda}(1)$ is given by formula (4). Our use of the word *geometric* is motivated by the fact that process $S_{p,\mu,\lambda}(\cdot)$ with $p \in (-\infty, 0]$ generalizes the geometric Brownian motion, which is frequently employed as a stock price process. The geometric Brownian motion corresponds to the case when $p = 0$ and is hereinafter denoted by

$$\begin{aligned} S_{0,\mu,\lambda}(t) &:= S(0) \cdot \exp\{X_{0,\mu,\lambda}(t) - \zeta_{0,0,\lambda}(1) \cdot t\} \\ &\stackrel{\text{d}}{=} S(0) \cdot \exp\{(\mu - 1/(2 \cdot \lambda)) \cdot t + \lambda^{-1/2} \cdot B(t)\}. \end{aligned} \tag{2}$$

Here, $B(t)$ stands for the standard Brownian motion, and the sign $\stackrel{\text{d}}{=}$ is understood in the sense that the distributions of stochastic processes coincide. Also, in the case when $p = 0$, it is more common to denote the reciprocal $1/\lambda$ of scaling parameter λ by σ^2. Note that σ $(= \lambda^{-1/2})$ is usually referred to as the *volatility* of a stock (*cf.*, *e.g.*, [8, p. 238]).

It should be pointed out that Hougaard processes $X_{p,\mu,\lambda}(\cdot)$ with *negative* values of p have *discontinuous* trajectories and are spectrally negative. Moreover, Hougaard processes with $p \in (-\infty, 0] \cup [2, +\infty)$ have no fixed jump points. On the other hand, recall (see formula (2)) that process $X_{0,\mu,\lambda}(t) \stackrel{\text{d}}{=} \mu \cdot t + \lambda^{-1/2} \cdot B(t)$, and hence has continuous trajectories with probability 1. At the same time, Theorem 3.2.i stipulates that $S_{p,\mu,\lambda}(\cdot)$ converges in a certain sense to $S_{0,\mu,\lambda}(\cdot)$ as $p \uparrow 0$, *i.e.*, when the (downward) jumps become negligible. Various implications of the convergence results of such character are considered in Section 3. In particular, we present the rational pricing formula for a European call option, which naturally generalizes the Black-Scholes formula to the discontinuous setting (see Theorem 3.1 and Corollary 3.1). In addition, it follows from formulas (1), (3) and the definition of Hougaard processes that for each $p \in (-\infty, 0]$ and $\mu \geq 0$, process $S_{p,\mu,\lambda}(t)$ is a positive *submartingale*. This is consistent with the fact that when investing in a risky asset, there should be an incentive in terms of higher expected return.

It is now well understood that sudden price downfalls such as the 1987 stock market crash or the stock market behavior during the week of September 17, 2001 do not support an assumption that the stock market is governed by the Gaussian law (*cf.*, *e.g.*, Madan [4, Subsection 2.5]). In contrast, geometric Hougaard processes with $p < 0$ take into consideration the phenomenon when *the bottom falls out of the market* faster than trades can be executed. However, *continuous* geometric Brownian motion can be derived as a limit of exponential transformations of these processes as $p \uparrow 0$. This is due to the fact that the jumps of Hougaard process $X_{p,\mu,\lambda}(\cdot)$ become smaller when p becomes closer to zero.

It follows from Vinogradov [11, Theorem 5.3.1] that each Hougaard process with $p \in (-\infty, 0)$ has a critical point. This phenomenon pertains to the long-time behavior of these processes and to a qualitative change in the mechanism of formation of catastrophic events (*i.e.*, large deviations). In this respect, observe that there is a growing understanding that the *critical state theory could be applied for studying instability of the world's financial markets* (*cf.* Manning [5]). Note that the change in the mechanism of formation of catastrophic events for Hougaard processes with $p \in (-\infty, 0)$ is consistent with Manning's remarks. In addition, in the concluding Section 5, we compare geometric Hougaard processes with $p \in (-\infty, 0]$ with the other models which also take into consideration *discontinuities* in stock price movements.

In Section 4, we present the results pertaining to the fluctuation theory of spectrally negative Hougaard process $X_{p,\mu,\lambda}(t)$ with $p \in (-\infty, 0]$ (see Theorem 4.1 and Corollary 4.1). In particular, Corollary 4.1 generalizes the celebrated result due to Schrödinger [7] and Smoluchowsky [9] on the distribution of the first passage time for the Brownian motion with a non-negative drift (see Remark 4.1 for more detail). Hence, the results of Section 4 also stipulate that spectrally negative Hougaard processes may be regarded as a natural generalization of a (scaled) Brownian motion with non-negative drift to the discontinuous setting.

2. Hougaard processes and power-variance family of distributions. First, it is known that under mild restrictions, a Lévy process has a modification whose sample paths belong to càdlàg space $\mathbf{D}[0, \infty)$ of functions on time interval $[0, \infty)$ which are right continuous and possess left limits. This property holds for all Hougaard processes. Therefore, assume that their sample paths belong to $\mathbf{D}[0, \infty)$. Note that Theorem 3.2 employs the restriction of $\mathbf{D}[0, \infty)$ to a finite time interval $[0, T]$, which is denoted by $\mathbf{D}[0, T]$.

Next, Hougaard processes can be constructed starting from the power-variance family of probability distributions. These distributions are often referred to as *Tweedie laws* and hereinafter denoted by $\mathrm{Tw}_p(\cdot, \cdot)$. The interested reader is referred to Jørgensen [1, Chapter 4] or Vinogradov [12] for more detail on the power-variance family. Each Tweedie law is characterized by specific values of particular three parameters, which are *power* parameter $p \in \mathbf{R}^1 \setminus (0, 1)$, *scaling* parameter $\lambda > 0$, and *location* (or mean) parameter μ. The latter parameter is such that $\mu \in [0, +\infty)$ if $p \in (-\infty, 0)$; $\mu \in \mathbf{R}^1$ if $p = 0$; $\mu \in (0, +\infty)$ if $p \in [1, 2]$, and $\mu \in (0, +\infty]$ if $p \in (2, +\infty)$. Hereinafter, we refer to the values of parameters specified above as their *admissible values.* It is known that for an arbitrary fixed real $c > 0$, the following scaling property holds: $c \cdot \mathrm{Tw}_p(\mu, \lambda) \stackrel{\mathrm{d}}{=} \mathrm{Tw}_p(c \cdot \mu, c^{p-2} \cdot \lambda)$ (*cf.*, *e.g.*, Jørgensen [1, formula (4.7)]). Here, the sign $\stackrel{\mathrm{d}}{=}$ is understood in the sense that the distributions of both r.v.'s coincide.

Now, we generate the three-parametric family of real-valued Hougaard processes $\{X_{p,\mu,\lambda}(t)\}_{t \geq 0}$ with stationary independent increments starting from the power-variance family. Define $X_{p,\mu,\lambda}(1) - X_{p,\mu,\lambda}(0) \stackrel{d}{=} \mathrm{Tw}_p(\mu, \lambda)$ and recall that $\mathbf{P}\{X_{p,\mu,\lambda}(0) = 0\} = 1$.

Let us describe the Tweedie family in more detail, since Hougaard processes inherit many of their properties from the power-variance family. Each r.v. $W \stackrel{d}{=} \mathrm{Tw}_p(\mu, \lambda)$ can be characterized by virtue of its cumulant-generating function $\zeta_{p,\mu,\lambda}(s) := \log \mathbf{E} \exp\{s \cdot \mathrm{Tw}_p(\mu, \lambda)\}$, which is given by formula (3) in the special case when $p \in (-\infty, 0]$. Also, Tweedie laws can be characterized by the variance-to-mean relationship of power type (*cf.*, *e.g.*, Jørgensen [1, formulas (4.1)–(4.2)]).

The power-variance family was introduced by Tweedie [10]. It includes normal ($p = 0$), scaled Poisson ($p = 1$), non-central chi-square with zero degrees of freedom ($p = 3/2$), gamma ($p = 2$) and inverse Gaussian distributions ($p = 3$). In addition, Tweedie distributions with $p \in (2, +\infty)$ are derived by an exponential tilting of positive stable laws with index $\alpha = \alpha(p) = (p-2)/(p-1) \in (0, 1)$. Positive stable distributions also belong to the family of Tweedie laws corresponding to $p = p(\alpha) = (2 - \alpha)/(1 - \alpha) \in (2, +\infty)$ and $\mu = +\infty$. In contrast, Tweedie distributions with $p \in (-\infty, 0)$ are derived by an exponential tilting of extreme stable laws with index $\alpha = \alpha(p) = (2-p)/(1-p) \in (1, 2)$ and skewness parameter $\beta = -1$. Similarly, extreme stable distributions belong to the family of Tweedie laws corresponding to $p = p(\alpha) = (\alpha - 2)/(\alpha - 1) \in (-\infty, 0)$ and $\mu = 0$. Observe that the tails of the probability density of r.v. $\mathrm{Tw}_p(\mu, \lambda)$ with $p < 0$ are heavily asymmetric. See Vinogradov [12] for more detail.

PROPOSITION 2.1 (SEE VINOGRADOV [12, PROPOSITION 4.1]). *Fix $p \in (-\infty, 0]$ and $\lambda > 0$. Let $\mu \in [0, +\infty)$ if $p \in (-\infty, 0)$ and $\mu \in \mathbf{R}^1$ if $p = 0$. Let $s \geq -\frac{1}{1-p} \cdot \lambda \cdot \mu^{1-p}$ if $p \in (-\infty, 0)$ and $s \in \mathbf{R}^1$ if $p = 0$. Then*

$$(3) \qquad \zeta_{p,\mu,\lambda}(s) = \frac{1}{2-p} \cdot \lambda \cdot \mu^{2-p} \cdot \left(\left(1 + \frac{s}{\frac{1}{1-p} \cdot \lambda \cdot \mu^{1-p}} \right)^{\frac{2-p}{1-p}} - 1 \right).$$

COROLLARY 2.1. *Formula (3) implies that for each $p \in (-\infty, 0]$,*

$$(4) \qquad \zeta_{p,0,\lambda}(1) = \frac{(1-p)^{\frac{2-p}{1-p}}}{2-p} \cdot \lambda^{-1/(1-p)};$$

$$(5) \qquad \zeta_{p,\mu,\lambda}(s) = \zeta_{p,0,\lambda}(1) \cdot \left\{ \left(\theta(p, \mu, \lambda) + s \right)^{\frac{2-p}{1-p}} - \theta(p, \mu, \lambda)^{\frac{2-p}{1-p}} \right\}.$$

Here,

$$(6) \qquad \theta(p, \mu, \lambda) := \frac{1}{1-p} \cdot \lambda \cdot \mu^{1-p}$$

is the index of exponential tilting which should be employed for the derivation of a specific Tweedie distribution from the corresponding extreme stable law (see Vinogradov [12, formula (4.1′)] for more detail). Also, it should be mentioned that the procedure of exponential tilting is frequently referred to as Esscher transform (cf. Section 3 or [8, p. 683, formula (2)]).

3. New rational pricing formula for European call options. In this section, we generalize the Black-Scholes formula for the rational price of European call options to the discontinuous setting. The validity of Black-Scholes formula (see (9)) relies on the hypothesis that the stock price movements follow a continuous geometric Brownian motion $S_{0,\mu,\lambda}(\cdot)$ (see formula (2)). In contrast, we take into consideration the possibility of downward jumps by assuming that the price process is a certain geometric Hougaard process $S_{p,\mu,\lambda}(\cdot)$ with $p \in (-\infty, 0]$ (see formula (1)).

Next, we introduce some additional notation. Let T be the time of the option to maturity, and $\mathbf{C}_{T,p}$ denote the rational price of the European call option that corresponds to price process $S_{p,\mu,\lambda}(t)$. This option may be exercised at time T. Assume that $K > 0$ is the exercise price of this option and that the short-term interest rate $r \geq 0$ remains constant over the time span $[0, T]$.

We employ an approach based on the construction of *risk-neutral martingale* measure and the use of *Esscher transform* (*cf.*, *e.g.*, [8, Subsections VII.3c and VIII.1b] for more detail). Namely, given $r \geq 0$ and $\lambda > 0$, one can show that price process $S_{p,\mu,\lambda}(t)$ is a martingale with respect to a particular probability measure constructed by virtue of the Esscher transform of the original measure. Moreover, the choice of such risk-neutral martingale measure does not depend on the value of μ, which is well known in the special case characterized by the value of $p = 0$ (*i.e.*, Black-Scholes formula (9)). Subsequently, by analogy to [8, p. 689, formula (43)], one should solve the following equation for θ in order to determine the index of exponential tilting θ (≥ 0) to be used for construction of the risk-neutral martingale measure (compare to formula (5)):

$$
\begin{aligned}
F(r, \lambda, p, \theta) &:= \zeta_{p,0,\lambda}(1) \cdot \{(\theta + 1)^{(2-p)/(1-p)} - (\theta^{(2-p)/(1-p)} + 1)\} - r \\
&= \log \mathbf{E}\big(S_{p,\mu(\theta),\lambda}(1)\big) - r = 0. \qquad (7)
\end{aligned}
$$

Here, the value of $\mu(\theta) := \big((1-p)\cdot\theta/\lambda\big)^{1/(1-p)}$ is obtained by solving equation (6) for θ. Next, it is not difficult to demonstrate that there exists a unique solution $\theta = \theta_{r,\lambda}(p) \in [0, +\infty)$ to equation (7). It is obvious that $\theta = 0$ if and only if $r = 0$. Subsequently, set $\mu_{r,\lambda}(p) := \mu\big(\theta_{r,\lambda}(p)\big) = \big((1-p)\cdot\theta_{r,\lambda}(p)/\lambda\big)^{1/(1-p)}$ (see the above notation). Observe that $\theta_{r,\lambda}(0) = r \cdot \lambda$, $\mu_{r,\lambda}(0) = r$ and that $F'_\theta(r, \lambda, p, \theta) = \big((1-p)/\lambda\big)^{1/(1-p)} \cdot \{(\theta+1)^{1/(1-p)} - \theta^{1/(1-p)}\} > 0$, where F'_θ denotes the first partial derivation of function F with respect to θ. A combination of the above inequality with the implicit function theorem yields that solution $\theta_{r,\lambda}(p)$ to equation (7) is *continuous at zero*. In turn, this justifies the validity of Theorem 3.2.ii below.

Next, we formulate the main results. Set $m_1 := \big((1-p)/\lambda\big)^{1/(1-p)} \cdot T^{1/(2-p)} \cdot \big(1 + \theta_{r,\lambda}(p)\big)^{1/(1-p)}$, $m_2 := \mu_{r,\lambda}(p) \cdot T^{1/(2-p)}$ and $L := L(K, S(0), T, \lambda, p) = \log\big(K/S(0)\big) \cdot T^{-(1-p)/(2-p)} + \zeta_{p,0,\lambda}(1) \cdot T^{1/(2-p)}$.

THEOREM 3.1. *Assume that the price process for a certain risky asset is given by process $S_{p,\mu,\lambda}(t)$ with $p \in (-\infty, 0]$, $\mu \geq 0$ and $\lambda > 0$, which is described by formula (1). Consider a European call option that corresponds to such price process, which is characterized by time to maturity T and exercise price K. Assume that the short-term interest rate $r \geq 0$ is constant. Then the rational price $\mathbf{C}_{T,p}$ of this option is as follows:*

$$\mathbf{C}_{T,p} = S(0) \cdot \mathbf{P}\{\mathrm{Tw}_p(m_1, \lambda) > L\} - K \cdot e^{-r \cdot T} \cdot \mathbf{P}\{\mathrm{Tw}_p(m_2, \lambda) > L\}. \tag{8}$$

THEOREM 3.2. *Fix arbitrary admissible values of μ and λ (which are specified in Section 2), and $T > 0$. Then*

(i) $S_{p,\mu,\lambda}(\cdot) \overset{\mathrm{D}[0,T]}{\longrightarrow} S_{0,\mu,\lambda}(\cdot)$ *as* $p \uparrow 0$.

(ii) Convergence with respect to the risk-neutral measures holds:
$S_{p,\mu_{r,\lambda}(p),\lambda}(\cdot) \overset{\mathrm{D}[0,T]}{\longrightarrow} S_{0,r,\lambda}(\cdot)$ *as* $p \uparrow 0$.

Here, $\overset{\mathrm{D}[0,T]}{\longrightarrow}$ *denotes convergence in* $\mathbf{D}[0,T]$.

COROLLARY 3.1. *In the case when $p = 0$, formula (8) becomes Black-Scholes formula (cf., e.g., [8, p. 740]):*

$$\begin{aligned} \mathbf{C}_{T,0} &= S(0) \cdot \Phi\left(\frac{\log(S(0)/K) + T \cdot (r + \sigma^2/2)}{\sigma \cdot \sqrt{T}}\right) \\ &\quad - K \cdot e^{-r \cdot T} \cdot \Phi\left(\frac{\log(S(0)/K) + T \cdot (r - \sigma^2/2)}{\sigma \cdot \sqrt{T}}\right), \end{aligned} \tag{9}$$

where $\Phi(\cdot)$ is the Laplace function, and $\sigma = \lambda^{-1/2}$ is the volatility.

COROLLARY 3.2. *Theorem 3.2.ii implies that for an arbitrary fixed $T > 0$,* $\mathbf{C}_{T,p} \to \mathbf{C}_{T,0}$ *as* $p \uparrow 0$.

REMARK 3.1.

(i) Theorem 3.2.i and Corollary 3.2 suggest that in *good times*, both geometric Brownian motion and Black-Scholes formula can be used as reasonably accurate approximations for the prices of stocks and options, respectively, which is due to a low magnitude of jumps of the underlying Hougaard processes.

(ii) An importance of establishing weak convergence of the families of price processes with respect to the risk-neutral martingale measures in a different context was pointed out in [8, p. 577, formula (4)].

(iii) Accurate approximations for the probabilities which emerge in option pricing formula (8) can be derived by employing Vinogradov [12, formulas (3.8)–(3.9) and (4.1′)].

4. Fluctuation property of spectrally negative Hougaard processes. First, consider a right-continuous *first passage time process* τ_u of level $u > 0$ which corresponds to spectrally negative Hougaard process $X_{p,\mu,\lambda}(\cdot)$ with $p \in (-\infty, 0]$. To this end, set $\tau_u := \inf\{t : X_{p,\mu,\lambda}(t) > u\}$, where u is understood as the time argument of stochastic process $\{\tau_u, u \geq 0\}$. Assume that $\tau_0 = 0$.

THEOREM 4.1. *Consider Hougaard process* $\{X_{p,\mu,\lambda}(t)\}_{t\geq 0}$ *with* $p \in (-\infty, 0]$. *Then its first-passage time process* $\{\tau_s, s \geq 0\}$ *has the same distribution as Hougaard process* $\{X_{3-p,1/\mu,\lambda}(s)\}_{s\geq 0}$.

COROLLARY 4.1. *Let* $\{X_{p,\mu,\lambda}(t)\}_{t\geq 0}$ *be Hougaard process with* $p \in (-\infty, 0]$. *Then for an arbitrary fixed* $b > 0$, *r.v.* $\tau_b \stackrel{\mathrm{d}}{=} \mathrm{Tw}_{3-p}(b/\mu, b^{2-p} \cdot \lambda)$.

REMARK 4.1. In the case when $p = 0$, Corollary 4.1 coincides with the result by Schrödinger [7] and Smoluchowsky [9] which stipulates that for a scaled Brownian motion $X_{0,\mu,\lambda}(t) = \mu \cdot t + \sigma \cdot B(t)$ with non-negative drift μ, the distribution of the first passage time of level b follows the inverse Gaussian law $\mathrm{Tw}_3(b/\mu, b^2/\sigma^2)$. Here, the mean and scaling parameters attain the values b/μ and b^2/σ^2, respectively.

5. Concluding remarks. The author believes that an applicability of processes $S_{p,\mu,\lambda}(\cdot)$ to model stock price movements is justified by the scaling and fluctuation properties of the underlying Hougaard processes with $p \in (-\infty, 0)$. Also, it is supported by the compliance of such Hougaard processes with the following empirical properties that are usually imposed on such models:

(i) long tailedness relative to the normal for the daily returns;
(ii) finite moments for at least the lower powers of returns;
(iii) consistency with an underlying, continuous-time process with independent stationary increments, and with the distribution of any increment belonging to the same simple family of distributions irrespective of the length of time to which the increment corresponds (*cf.*, *e.g.*, Madan [4, Section 2 and references therein]).

Also, the density of Lévy spectral measure of r.v. $X_{p,\mu,\lambda}(1)$ given in Vinogradov [12, formulas (4.6)–(4.7)] is consistent with the formula given in Madan [4, p. 114]. However, our family is *three-parametric* and hence simpler than the *four-parametric VG-process-model* which was considered by Madan [4, p. 109] and is used by Morgan Stanley Dean Witter. The other reason in support of applicability of geometric Hougaard processes for modelling stock price movements is based on the fact that Tweedie laws and Hougaard processes can be used for approximation of wide classes of distributions and stochastic processes, respectively (*cf.*, *i.e.*, Jørgensen and Vinogradov [2] for more detail).

In addition, our model incorporates the *log-stable model* for stock price movements suggested by McCulloch [6] as a special case which corresponds to zero value of index θ of exponential tilting. An empirical comparative analysis of the model given by price process $S_{p,\mu,\lambda}(\cdot)$ with the other models is required.

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