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HYPERCONTACT METRIC THREE-MANIFOLDS

Dedicated to the memory of Professor S. I. Goldberg.

DOMENICO PERRONE

Presented by Vlastimil Dlab, FRSC

RÉSUMÉ. Une variété d'hypercontact métrique, de dimension 3, est (i) une variété 3-sasakienne, localement isométrique à la sphère $S^3(1)$, ou (ii) est localement isométrique au groupe de Lie SL$(2, R)$, ayant une structure d'hypercontact métrique invariante à gauche. Notamment, une variété d'hypercontact métrique, de dimension 3, compacte et simplement connexe, est homéomorphe à la sphère S^3.

1. Introduction. J. Martinet [Ma] showed that a compact and orientable three-manifold M possesses a contact form α and hence a contact metric structure (α, g, ϕ). S. I. Goldberg [Go], in order to give credibility to the classical Poincaré conjecture, showed that if M is simply connected and the structure (α, g, ϕ) is Sasakian with Ric $+\lambda g$ positive definite for some $\lambda < 2$, then M is homeomorphic to the sphere S^3. More recently, Geiges and Thomas [GT] have proved that a compact and orientable three-manifold M possesses a hypercontact structure, namely a triple of contact forms $(\alpha_1, \alpha_2, \alpha_3)$ and an almost contact metric 3-structure $(g, \eta_i, \xi_i, \phi_i)$ such that $d\alpha_i = g(\cdot, \phi_i \cdot)$. If $\eta_i = \alpha_i$, g is a associated metric for α_i, the hypercontact structure is a contact 3-structure and hence, by a recent and remarkable result of Kashiwada [Ka], 3-Sasakian. So, we consider hypercontact structures which satisfy the weaker condition (than $\eta_i = \alpha_i$) that the Reeb vector field of α_i is dual to α_i with respect to g, and call such structures hypercontact metric structures.

Then, in this short note we show the following extension of Goldberg's result.

THEOREM 1.1. *If a compact simply connected three-manifold admits a hypercontact metric structure, then it is homeomorphic to the sphere S^3.*

This result is one consequence of the following classification theorem.

THEOREM 1.2. *A three-dimensional hypercontact metric manifold is either*

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(i) a 3-Sasakian manifold locally isometric to the sphere $S^3(1)$, *or*

(ii) locally isometric to the Lie group SL$(2, R)$ *equipped with a left invariant hypercontact metric structure (which is not a contact 3-structure).*

Another consequence of this theorem is the following.

COROLLARY 1.3 ([KA]). *A contact 3-structure on a three-manifold is a Sasakian 3-structure.*

REMARK. Kashiwada gets her theorem as a consequence of a Hitchin's lemma (*cf.* [Ka, lemma H]). Boyer and Galicki [BG, p. 2428] posed the question to give a direct proof of Kashiwada's theorem working exclusively on M. Our Corollary 1.3 answers, in the three-dimensional case, this question.

REMARK. In [GT] was given, in the non compact case, only the example of R^{4n+3} with the standard hypercontact structure. To our knowledge, (j) of Theorem 1.2 gives the first example of a non compact 3-manifold with a hypercontact structure $(\alpha_i, g, \eta_i, \xi_i, \phi_i)$ which is not 3-Sasakian, but g is a associated metric for α_i. We believe that the notion of hypercontact metric structure introduced in this note is also interesting in the search for suitable quaternionic analogues of contact manifolds. So we propose to investigate on hypercontact metric structures in dimension > 3.

2. **Preliminaries.**

In this section we recall some definitions about contact and hypercontact metric manifolds. All manifolds are supposed to be connected and smooth.

A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field or the *Reeb vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. A Riemannian metric g is said to be an associated metric if there exists a tensor field ϕ of type $(1, 1)$ such that

$$\eta = g(\xi, \cdot), \quad d\eta = g(\cdot, \phi\cdot), \quad \phi^2 = -I + \eta \otimes \xi.$$

In this case, (g, η, ξ, ϕ) is called a *contact metric structure* and M is called a *contact metric manifold.* If the almost complex structure J on $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, M is said to be *Sasakian.* If ξ is a Killing vector field, or equivalently if the tensor $L_\xi \phi$ vanishes, M is said to be *K-contact.* A Sasakian manifold is K-contact and the converse also holds for three-dimensional spaces. Moreover for an arbitrary contact metric manifold the Ricci curvature in the direction of ξ is given by

$$\mathrm{Ric}(\xi, \xi) = 2n - \mathrm{tr}\left(\frac{1}{2}L_\xi \phi\right)^2.$$

An *almost contact metric structure* is defined as a tensor field (g, η, ξ, ϕ) satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad g(\phi\cdot, \phi\cdot) = g - \eta \otimes \eta.$$

Note that these conditions imply $\phi(\xi) = 0$, $\eta\phi = 0$ and $\eta = g(\cdot, \xi)$. Of course, a contact metric structure is an almost contact metric structure. We refer to [Bl] for more information about contact (and almost contact) metric manifolds.

An *almost contact metric 3-structure* is defined as three almost contact metric structures $(g, \eta_i, \xi_i, \phi_i)$, $i = 1, 2, 3$, such that

$$\phi_i\phi_j - \xi_i \otimes \eta_j = \phi_k = -\phi_j\phi_i + \xi_j \otimes \eta_i \tag{2.1}$$

for cyclic permutation (i, j, k) of $(1, 2, 3)$. In this case M has to be of dimension $4m + 3$ for a non-negative integer m.

If $(g, \eta_i, \xi_i, \phi_i)$, $i = 1, 2, 3$, are three almost contact metric structures, then the condition (2.1) implies the conditions

$$\phi_i\xi_j = \varepsilon_{ijk}\xi_k \tag{2.2}$$

and

$$\eta_i\xi_j = 0, \quad \eta_i\phi_j = \varepsilon_{ijk}\eta_k,$$

where (i, j, k) is a permutation of $(1, 2, 3)$; moreover, in the three-dimensional case, (2.1) is equivalent to (2.2).

A *contact 3-structure* is defined as three contact metric structures $(g, \eta_i, \xi_i, \phi_i)$, satisying (2.1). If each of them is Sasakian, then the contact 3-structure is called a *3-Sasakian structure* [Ku] and the manifold is called a *3-Sasakian manifold.*

Geiges and Thomas [GT] define a *hypercontact structure* as a triple of contact forms $(\alpha_1, \alpha_2, \alpha_3)$ together with an almost contact metric 3-structure $(g, \eta_i, \xi_i, \phi_i)$ such that

$$d\alpha_i = g(\cdot, \phi_i\cdot). \tag{2.3}$$

If $\eta_i = \alpha_i$, then the hypercontact structure is a contact 3-structure and hence 3-Sasakian.

Now, we define a *hypercontact metric structure* as a hypercontact structure $(\alpha_i, g, \eta_i, \xi_i, \phi_i)$ whose Reeb vector field ζ_i of α_i is dual of α_i with respect to the metric g.

3. Proof of Theorem 1.2. Let $(\alpha_i, g, \eta_i, \xi_i, \phi_i)$ be a hypercontact metric structure on a manifold M of dimension 3 and ζ_i the Reeb vector of α_i. From (2.3) we have

$$\phi_i\zeta_i = 0$$

and hence, because $\phi_i^2 = -I + \eta_i \otimes \xi_i$,

$$\zeta_i = \eta_i(\zeta_i)\xi_i.$$

Since $\alpha_i = g(\cdot, \zeta_i)$, $1 = g(\zeta_i, \zeta_i) = \{\eta_i(\zeta_i)\}^2 g(\xi_i, \xi_i)$ implies

$$\zeta_i = \pm\xi_i, \quad \text{and} \quad \alpha_i = \pm\eta_i.$$

Consequently,

$$\phi_i^2 = -I + \eta_i \otimes \xi_i = -I + \alpha_i \otimes \zeta_i.$$

On the other hand, (2.3) holds. Therefore, each $(g, \alpha_i, \zeta_i, \phi_i)$ is a contact metric structure. In particular, the integral curves of ζ_i are geodesics, and thus

$$\nabla_{\xi_i}\xi_i = \nabla_{\zeta_i}\zeta_i = 0$$

where ∇ is the Levi-Civita connection. Then, for (i, j, k) cyclic permutation of $(1, 2, 3)$, $\nabla_{\xi_i}\xi_j$ and $\nabla_{\xi_j}\xi_i$ are parallel to ξ_k and we put

$$[\xi_i, \xi_j] = c_k\xi_k. \tag{3.1}$$

Then,

$$(d\alpha_k)(\xi_i, \xi_j) = \frac{1}{2}\{\xi_i\alpha_k(\xi_j) - \xi_j\alpha_k(\xi_i) - \alpha_k[\xi_i, \xi_j]\} = \pm\frac{1}{2}c_k$$

and

$$(d\alpha_k)(\xi_i, \xi_j) = g(\xi_i, \phi_k\xi_j) = g(\xi_i, \varepsilon_{kji}\xi_i) = -1$$

give $c_k = \pm 2$. Consequently, from (3.1), we have

$$[\zeta_2, \zeta_3] = a_1\zeta_1, \quad [\zeta_3, \zeta_1] = a_2\zeta_2, \quad [\zeta_1, \zeta_2] = a_3\zeta_3, \tag{3.2}$$

where $a_i = \pm 2$, more precisely $a_i = +2$ (resp. -2) if $\zeta_i = \xi_i$ (resp. $-\xi_i$). From (3.2), we have that $\zeta_1, \zeta_2, \zeta_3$ are global vector fields satisfying the conditions of Propositon 1.9 of [TV]. Then the universal covering $\tilde{M}$, for an arbitrary point $p \in \tilde{M}$, has a unique Lie group structure such that p is the identity and the vector fields ζ_i are left invariant. Consequently also the metric is left-invariant. In [Mi], Milnor gave a complete classification of three-dimensional Lie groups. From this classification we see that $\tilde{M}$ is SU(2) or $\tilde{\mathrm{SL}}(2, R)$. If $a_i = 2$ (or -2) for each $i = 1, 2, 3$, and hence $[\zeta_i, \zeta_j] = +2\varepsilon_{ijk}\zeta_k$ (or $-2\varepsilon_{ijk}\zeta_k$), with (i, j, k) permutation of $(1, 2, 3)$, from [Mi] we obtain $\tilde{M} = \mathrm{SU}(2)$. Moreover,

$$\begin{aligned}(L_{\zeta_i}\phi_i)(\zeta_j) &= \phi_i[\zeta_j, \zeta_i] - [\phi_i\zeta_j, \zeta_i] = \pm 2\phi_i\varepsilon_{jik}\zeta_k \mp \varepsilon_{ijk}[\zeta_k, \zeta_i] = \\ &= \pm 2\varepsilon_{jik}\varepsilon_{ikj}\zeta_j \mp 2\varepsilon_{ijk}\varepsilon_{kij}\zeta_j = 0.\end{aligned}$$

Thus, in this case, $(g, \alpha_i, \zeta_i, \phi_i)_i$ are three Sasakian structures with $\phi_i\zeta_j = \varepsilon_{ijk}\zeta_k$, therefore the hypercontact structure is 3-Sasakian. Moreover, since $\mathrm{Ric}(\zeta_i, \zeta_i) = 2$, for each $i = 1, 2, 3$, g is of constant curvature $+1$ and thus M is locally isometric to the sphere $S^3(1) \equiv \mathrm{SU}(2)$ equipped with the standard left invariant Sasakian 3-structure. If two $c_i = 2$ (resp. -2) and one $c_i = 2$ (resp. -2), from [Mi] we get $\tilde{M} = \tilde{\mathrm{SL}}(2, R)$.

To finish the proof, we have to exihibit explicitly the left invariant hypercontact metric structure on $\mathrm{SL}(2,R)$. On a three-dimensional unimodular Lie group G with a left-invariant metric g, there exists an orthonormal basis (e_1, e_2, e_3) in its Lie algebra such that

$$[e_2, e_3] = \lambda_1 e_1 \quad [e_3, e_1] = \lambda_2 e_2 \quad [e_1, e_2] = \lambda_3 e_3,$$

where λ_k are constant (see [Mi]). If $\lambda_k \neq 0$, the dual 1-form θ^k is a contact form; in particular if $\lambda_k = 2$, then $(g, \theta^k, \xi_k = e_k, \phi_k)$ is a contact metric structure where $\phi_k(e_k) = 0$ and $\phi_k(e_i) = \varepsilon_{kij} e_j$ with (k, i, j) permutation of $(1,2,3)$ (see [Pe]).

If G is $\mathrm{SL}(2,R)$, we can consider the left-invariant metric g corresponding to the structure constant $\lambda_1 = \lambda_3 = 2$ and $\lambda_2 = -2$. Then $(g, \theta^k, \xi_k = e_k, \phi_k)$, $k = 1,2,3$, are three almost contact metric structure satisfying (2.2). Moreover, $(\alpha_1, \alpha_2, \alpha_3) = (\theta^1, -\theta^2, \theta^3)$ are three contact forms satisfying

$$d\alpha_i = g(\cdot, \phi_i \cdot) \quad \text{and} \quad \alpha_i = g(\zeta_i, \cdot),$$

where $\zeta_1 = e_1$, $\zeta_2 = -e_2$, $\zeta_3 = e_3$ are the Reeb vector fields of $\alpha_1, \alpha_2, \alpha_3$. Therefore, $(\alpha_k, g, \theta^k, \xi_k = e_k, \phi_k)$ defines a hypercontact metric structure which is not a contact 3-structure because $\alpha_2 \neq \theta^2$.

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CONTINUED FRACTIONS BEEPERS AND FIBONACCI NUMBERS

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ABSTRACT. We introduce the concept of *continued fraction beepers* and *punctuated beepers*. We use this notion to present infinite families of quadratic polynomials $\{D_k(X)\}_{k\in\mathbb{N}}$ such that the continued fraction expansions of $\sqrt{D_k(X)}$ have period length $\ell\big(\sqrt{D_k(X)}\big)$ going to infinity with k, while for fixed k, $\ell\big(\sqrt{D_k(X)}\big) = \ell\big(\sqrt{D_k(X+1)}\big)$ for all $X \in \mathbb{N}$. We are also able to explicitly determine the fundamental unit of the underlying quadratic order. Moreover, we exhibit infinite families of continued fraction beepers for which $\ell\big(\sqrt{D_k(X)}\big) = \ell\Big(\big(1+\sqrt{D_k(X)}\big)/2\Big)$; for which $\ell\big(\sqrt{D_k(X)}\big) = \ell\Big(\big(1+\sqrt{D_k(X)}\big)/2\Big)+4$; for which $\ell\big(\sqrt{D_k(X)}\big) = 5\Big(1+\ell\big(\sqrt{D_k(X)}\big)\Big)/2$, the maximum possible according to [8]; and for which $\ell\big(\sqrt{D_k(X)}\big) = \frac{1}{3}\ell\Big(\big(1+\sqrt{D_k(X)}\big)/2\Big)$, the minimum possible according to [8]. Furthermore, each of the $D_k(X)$ is expressed in terms of Fibonacci numbers and the continued fraction expansions are all beepers or punctuated beepers. This continues work in [4]–[6].

RÉSUMÉ. Nous introduisons les concepts de beepers de fractions continues et de beepers pointés. Nous utilisons ces notions pour présenter des familles infinies de polynômes quadratiques $\{D_k(X)\}_{k\in\mathbb{N}}$ telles que la longueur ℓ de la période du développement en fraction continue de $\sqrt{D_k(X)}$ tend vers l'infini en fonction de k, alors que pour k fixe, $\ell\big(\sqrt{D_k(X)}\big) = \ell\big(\sqrt{D_k(X+1)}\big)$ pour tout $X \in \mathbb{N}$. Nous sommes aussi capables de déterminer explicitement l'unité fondamentale de l'ordre quadratique sous-jacent. De plus, nous exhibons des familles infinies de beepers de fractions continues pour lesquelles $\ell\big(\sqrt{D_k(X)}\big) = \ell\Big(\big(1+\sqrt{D_k(X)}\big)/2\Big)$; pour lesquelles $\ell\big(\sqrt{D_k(X)}\big) = \ell\Big(\big(1+\sqrt{D_k(X)}\big)/2\Big)+4$; pour lesquelles $\ell\big(\sqrt{D_k(X)}\big) = 5\Big(1+\ell\big(\sqrt{D_k(X)}\big)\Big)/2$, ce qui correspond au maximum possible selon [8]; et pour lesquelles $\ell\big(\sqrt{D_k(X)}\big) = \frac{1}{3}\ell\Big(\big(1+\sqrt{D_k(X)}\big)/2\Big)$, ce qui correspond au minimum possible selon [8]. De plus, chacun des $D_k(X)$ s'exprime en

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termes de nombres de Fibonacci et les développements en fractions continues considérés sont tous des beepers ou des beepers pointés. Ce travail est la suite de [4]–[6].

In [8], the authors provide upper and lower bounds for the period length $\ell(\sqrt{D})$ of the simple continued fraction expansion of $\sqrt{D}$ for $D \equiv 1 \pmod 4$ not a perfect square, in terms of $\ell\big((1+\sqrt{D})/2\big)$ and provide infinitely many families to illustrate the extreme values (upper and lower bounds in 6 possible cases). In only one of these cases is there a determination of D in terms of Fibonacci numbers. In this article, we express three of the cases in terms of Fibonacci numbers and provide the fundamental units of the underlying quadratic orders. We also look at the interesting case, not covered in [8], where $\ell\big(\sqrt{D_k(X)}\big) = \ell\Big(\big(1+\sqrt{D_k(X)}\big)/2\Big)$ and again provide infinite families with each D determined by Fibonacci numbers. Each of the infinite families we provide satisfy the property of being either what we call *beepers* or *punctuated beepers* defined below.

DEFINITION 1. Let $c_1, c_2, \ldots, c_n$, with $c_j \in \mathbb{N}$ for $1 \le j \le n$, be a palindrome, $C \in \mathbb{N}$ nonsquare, and $\alpha = (P + \sqrt{C})/Q$ a quadratic irrational having simple continued fraction expansion $\alpha = \langle c_0\ ;\ \overline{c_1, \ldots, c_n, c_\ell}\rangle$. Furthermore, for a given nonnegative integer m let w_m denote a string of m copies of $c_1, c_2, \ldots, c_n, c_\ell$ followed by one iteration of $c_1, c_2, \ldots, c_n$ (which is the empty string if $n = 0$, and if $m = 0$, the initial string is empty, but we do not allow $m = n = 0$). Then a simple continued fraction of the form

$$\langle q_0\ ;\ \overline{w_m, q_k}\rangle = \Big\langle q_0\ ;\ \overline{\underbrace{c_1, c_2, \ldots, c_n, c_\ell, \ldots, c_1, c_2, \ldots, c_n, c_\ell}_{m \text{ copies of } c_1, c_2, \ldots, c_n, c_\ell}, c_1, c_2, \ldots, c_n, q_k}\Big\rangle,$$

for some $q_0, q_k \in \mathbb{N}$ is called an *m-beeper* for α. (Note that $k = m\ell + n + 1$. See Example 1 and Theorem 1 for explicit families of examples.)

REMARK 1. If $m = 0$, then of course α is a zero-beeper for itself. However, it is a classical result that, for example, if $\alpha = \langle c_0\ ;\ \overline{c_1, \ldots, c_n, 2c_0}\rangle$, then there exist infinitely many $D, d_0 \in \mathbb{N}$ such that $\sqrt{D} = \langle d_0\ ;\ \overline{c_1, \ldots, c_n, 2d_0}\rangle$ (see [1, Corollary 4.1, p. 1022] for a two-sentence proof of this result). Hence, there exist infinitely many zero-beepers for a given α. However, a generalization of this fact is that there are infinitely many beepers of any kind for a given $\sqrt{C}$, which follows from [1, Theorems 4.1–4.2, pp. 1009–1021]. Similarly from [4] the same can be said for $(1 + \sqrt{C})/2$ when $C \equiv 1 \pmod 4$. It follows from these comments that the result holds for any quadratic irrational α (see Remark 2 below).

We now provide a nontrivial family of illustrations of Definition 1.

EXAMPLE 1. If $C = 2$, then $\sqrt{2} = \langle 1\ ;\ \overline{2}\rangle$, so $n = 0$. Define $B_k + A_k\sqrt{2} = (3 + 2\sqrt{2})^k$ and for any $X \in \mathbb{N}$, set $D_k(X) = (B_k + 1)^2 A_k^2 X^2 + 2(B_k + 1)^2 X = 2$.

By [1, Theorem 4.1 (b), p. 1021], for any $j \in \mathbb{N}$, we have that if $k = 2j + 1$:

$$\begin{aligned}\sqrt{D_k(X)} &= \langle (B_k - 1)A_kX + 1\,;\,\overline{w_{k-1}, 2[(B_k - 1)A_kX + 1]}\rangle \\ &= \Big\langle (B_k - 1)A_kX + 1\,;\,\overline{\underbrace{2, 2, \ldots, 2}_{2j \text{ copies}}, 2[(B_k - 1)A_kX + 1]}\Big\rangle,\end{aligned}$$

which is a $2j$-beeper for $\sqrt{2}$.

Similarly, if $C = 3$, then $\sqrt{3} = \langle 1\,;\,\overline{1, 2}\rangle$, so $n = 1$. Define $B_k + A_k\sqrt{3} = (2 + \sqrt{3})^k$, then by [1, Theorem 4.2 (c)-(iii), p. 1021], if $k = 2j$ for any $j \in \mathbb{N}$:

$$\begin{aligned}\sqrt{D_k(X)} &= \langle (B_k + 1)A_kX + 1\,;\,\overline{w_{k/2-1}, 2[(B_k + 1)A_kX + 1]}\rangle \\ &= \Big\langle (B_k + 1)A_kX + 1\,;\,\overline{\underbrace{1, 2, 1, 2, \ldots, 1, 2}_{j-1 \text{ copies of } 1,2}, 1, 2[(B_k + 1)A_kX + 1]}\Big\rangle,\end{aligned}$$

which is a $(j - 1)$-beeper for $\sqrt{3}$.

There is another notion that we wish to introduce and illustrate as follows.

DEFINITION 2. Let α be as given in Definition 1. Also, let v_m denote m copies of $c_1, \ldots, c_n, c_\ell$. Then for any $m \in \mathbb{N}$, if $v_{m_j} = r_1, r_2, \ldots, r_{m_j}$ ($m_j \in \mathbb{N}$) is a string of natural numbers, called *punctures*, for $j = 1, \ldots, s \in \mathbb{N}$, and v is one of v_m or w_{m-1}, then

$$\Big\langle q_0\,;\,\overline{v_m, v_{m_1}, v_{m-1}, v_{m_2}, \ldots, v_{m-1}, v_s, v, q_k}\Big\rangle,$$

is called an *s-times punctuated m-beeper* for α. (Note that when $v = v_m$, we have $k = 2\ell m + \ell(m - 1)(s - 1) + 1 + \sum_{j=1}^{s} m_j$, and if $v = w_{m-1}$, then $k = 2\ell m + n - \ell + \ell(m - 1)(s - 1) + 1 + \sum_{j=1}^{s} m_j$.)

EXAMPLE 2. If $\alpha = \sqrt{5} = \langle 2\,;\,\overline{4}\rangle$, then

$$\sqrt{1957} = \langle 44\,;\,\overline{4, 4, 1, 21, 3, 4, 3, 21, 1, 4, 4, 88}\rangle = \langle q_0\,;\,\overline{v_2, v_{m_1}, v_1, v_{m_2}, v_2, 2q_0}\rangle$$

is a 2-times punctuated 2-beeper for α, where $q_0 = 44$, $v_2 = 4, 4$; $v_{m_1} = 1, 21, 3$; $v_1 = 4$; and $v_{m_2} = 3, 21, 1$. For a generalization of this example, see Example 5 below.

EXAMPLE 3. Let $\alpha = \sqrt{245} = \langle 15\,;\,\overline{1, 1, 1, 7, 6, 7, 1, 1, 1, 30}\rangle$, and set

$$q_0 = 32264490539,$$

$$v_{m_1} = 1, 1, 1, q_0 - 1, 6, 7, 1, 1, 1, 30, \quad v_{m_2} = 1, 1, 1, 7, 6, q_0 - 1, 1, 1, 1, 30,$$

and v_m is m copies of $1, 1, 1, 7, 6, 7, 1, 1, 1, 30$. Then

$$\sqrt{4163989398920031282517} = \langle q_0\,;\,\overline{v_2, v_{m_1}, v_1, v_{m_2}, w_1, 2q_0}\rangle$$

is a 2-times punctuated 2-beeper for α. We know of no example of a punctuated beeper ending in w_{m-1} for which the punctures are *not* of the form $c_1, \ldots, c_\ell$ with *exactly one* of the c_j replaced by another value. For a generalization of this example, see Example 4 (d) below.

REMARK 2. Several heretofore undefined notions related to continued fractions have been in the folklore for some time, with nobody exercising the formality of a definition to date. For instance, my colleagues/co-authors, Hugh Williams and Alf van der Poorten, have been looking at several concepts: *creepers*, *sleepers*, and *beepers* in conjunction with simple continued fraction expansions. This largely arose out of correspondence with Irving Kaplansky, who originated the study, but did not formally define such objects. (See [2] for some related results emanating from that correspondence.) In this paper, we have sufficient results related to the notion of beepers to define and illustrate the concepts with some applications.

We need the following, which was proved in [4]. In the first result, the following notation holds. Let $C, X \in \mathbb{N}$ with $C \equiv 5 \pmod 8$ not a perfect square and $(x, y) = (B, A)$ the smallest positive solution of $x^2 - Cy^2 = 4$ with $\gcd(x, y) = 1$. Set $A = 2y$, $B = 2x$ and for each $k \in \mathbb{N}$,

$$B_k + A_k\sqrt{C} = (B + A\sqrt{C})^k/4^{k-1} = (x + y\sqrt{C})^k/2^{k-2}.$$

Also, set

$$\frac{1+\sqrt{C}}{2} = \langle c_0\ ; \overline{c_1, \ldots, c_n, 2c_0 - 1}\rangle.$$

We proved the following in [4]. Below, for a positive discriminant Δ, ε_Δ denotes the fundamental unit of the order $\mathbb{Z}[(\sigma - 1 + \sqrt{\Delta})/\sigma]$, where $\sigma = 1$ if $\Delta \not\equiv 1 \pmod 4$ and $\sigma = 2$ otherwise.

THEOREM 1. *Let* $D_k(X) = A_k^2X^2 + 2B_kX + C$. *Then*

$$\varepsilon_{D_k(X)} = \frac{A_k^2X + B_k + A_k\sqrt{D_k(X)}}{4},$$

and if $q_0 = A_kX/2 + c_0$, *then:*

(a) If $n \geq 0$ *is even, then* $\left(1 + \sqrt{D_k(X)}\right)/2 = \langle q_0\ ; \overline{w_{2k-1}, 2q_0 - 1}\rangle$ *with* $\ell\Big(\left(1 + \sqrt{D_k(X)}\right)/2\Big) = 2k(n+1)$,

(b) If n *is odd, then* $\left(1 + \sqrt{D_k(X)}\right)/2 = \langle q_0\ ; \overline{w_{k-1}, 2q_0 - 1}\rangle$ *where*

$$\ell\Big(\left(1 + \sqrt{D_k(X)}\right)/2\Big) = k(n+1).$$

REMARK 3. We see that for a fixed C (and so a fixed $n \in \mathbb{N}$), we may let $k \to \infty$ in which case $\ell\Big(\left(1 + \sqrt{D_k(X)}\right)/2\Big) = \ell\Big(\left(1 + \sqrt{D_k(1)}\right)/2\Big) \to \infty$ for all $X \in \mathbb{N}$. We also see that we have infinitely many distinct radicands $D_k(X)$ for a fixed $k \in \mathbb{N}$ with $\ell\Big(\left(1 + \sqrt{D_k(X)}\right)/2\Big) = \ell\Big(\left(1 + \sqrt{D_k(X+1)}\right)/2\Big)$ for all $X \in \mathbb{N}$.

In what follows F_n, and L_n for $n \in \mathbb{N}$, denote the n-th Fibonacci number and Lucas number, respectively.

EXAMPLE 4. In Theorem 1, let $C = 245$, for which $A_1 = 6$, $B_1 = 94$, and let $k = 7 \cdot 2^{2s}$ ($s \geq 0$), then:

(a) $\ell(\sqrt{D_k(X)}) = 5\ell(\frac{1+\sqrt{D_k(X)}}{2}) = 5 \cdot 7 \cdot 2^{2s+1}$,

(b) $D_k(X) = D_{7\cdot 2^{2s}}(X) = (\frac{2F_{7\cdot 2^{3+2s}}}{7})^2 X^2 + 4(\frac{F_{7\cdot 2^{4+2s}}}{F_{7\cdot 2^{3+2s}}})X + 245 =$ $(\frac{2F_{8k}}{7})^2 X^2 + 4(\frac{F_{16k}}{F_{8k}})X + 245 = (\frac{2F_{8k}}{7})^2 X^2 + 4L_{8k}X + 245$,

(c) $(1+\sqrt{D_k(X)})/2 = \langle q_0\, ;\, \overline{w_{k-1}, 2q_0 - 1}\rangle$, where w_{k-1} is given in Definition 2 with $\alpha = (1 + \sqrt{245})/2 = \langle 8\, ;\, \overline{3, 15}\rangle = \langle c_0\, ;\, \overline{c_n, c_\ell}\rangle$ for $n = 1$, $\ell = 2$; and $q_0 = A_k X/2 + 8$. In other words, $(1 + \sqrt{D_k(X)})/2$ is a $(k-1)$-beeper for α.

(d) $\sqrt{D_k(X)} = \langle 2q_0 - 1\, ;\, \overline{v_t, 1, 1, 1, q_0 - 1, 6, 7, 1, 1, 1, 30, v_{t-1}, 1, 1, 1, 7, 6, q_0 - 1, 1, 1, 1, 30, w_{t-1}, 2(2q_0 - 1)}\rangle$, where v_t, for $t = \lfloor 7 \cdot 2^{2s}/3 \rfloor$, is given in Definition 2, with

$$\alpha = \sqrt{245} = \langle 15\, ;\, \overline{1, 1, 1, 7, 6, 7, 1, 1, 1, 30}\rangle = \langle c_0\, ;\, \overline{c_1, c_2, \ldots, c_\ell}\rangle$$

for $n = 9$, $\ell = 10$; and q_0 is given in part (c). In other words, $\sqrt{D_k(X)}$ is a 2-times punctuated t-beeper for α.

(e) $\varepsilon_{D_k(X)} = (A_k^2 X + B_k + A_k\sqrt{D_k(X)})/4$, and

$$\varepsilon_{4D_k(X)} = \left((A_k^2 X + B_k + A_k\sqrt{D_k(X)})/4\right)^3.$$

The facts about $(1 + \sqrt{D_k(X)})/2$ follow from Theorem 1, and the results on $\sqrt{D_k(X)}$ follow in a similar fashion. The same comment applies to the results below.

We see that $\lim_{s\to\infty} \ell(\sqrt{D_k(X)}) = \infty$, and for fixed $s \geq 0$, $\ell(\sqrt{D_k(X)}) = 5\ell\left((1 + \sqrt{D_k(X)})/2\right) = 35 \cdot 2^{2s+1}$ for all $X \in \mathbb{N}$.

EXAMPLE 5. Let $C = 5$ in Theorem 1. Then $A_1 = 2$, $B_1 = 6$, $(1+\sqrt{5})/2 = \langle\overline{1}\rangle$, $D_k(X) = 4F_{2k}^2 X^2 + (20F_k^2 + 8(-1)^k)X + 5$,

$$\frac{1 + \sqrt{D_k(X)}}{2} = \langle q_0, \overline{w_{2k-1}, 2q_0 - 1}\rangle = \left\langle A_k X/2 + 1\, ;\, \overline{\underbrace{1, 1, \ldots, 1}_{2k-1 \text{ copies}}, A_k X + 1)}\right\rangle,$$

which is a $(2k-1)$-beeper for $(1 + \sqrt{5})/2$, and $\ell\left((1 + \sqrt{D_k(X)})/2\right) = 2k$. On the other hand, if $k = 3t$ for $t \in \mathbb{N}$, then $\sqrt{D_k(X)} = \langle 2q_0\, ;\, \overline{w_{2t-1}, 4q_0}\rangle$, which is a $(2t-1)$-beeper for $\sqrt{5} = \langle 2\, ;\, \overline{4}\rangle$, consisting of $2t - 1$ copies of 4. Moreover,

$$\ell(\sqrt{D_k(X)}) = \frac{1}{3}\ell\left((1 + \sqrt{D_k(X)})/2\right) = 2t.$$

If $k = 3t + 1$, then a 2-times punctuated $2t$-beeper for $\sqrt{5}$ is:

$$\sqrt{D_k(X)} = \langle 2q_0\, ;\, \overline{v_{2t}, 1, q_0 - 1, 3, v_{2t-1}, 3, q_0 - 1, 1, v_{2t}, 4q_0}\rangle$$

$$= \left\langle 2q_0\,;\overline{\underbrace{4,4,\ldots,4}_{2t\text{ copies}},1,q_0-1,3,\underbrace{4,4,\ldots,4}_{2t-1\text{ copies}},3,q_0-1,1,\underbrace{4,4,\ldots,4}_{2t\text{ copies}},4q_0}\right\rangle,$$

and

$$\ell(\sqrt{D_k(X)}) = \ell\Big((1+\sqrt{D_k(X)})/2\Big) + 4.$$

We conclude with an infinite family of beepers related to Fibonacci numbers for which the period lengths of the two continued fraction expansions under consideration are equal.

EXAMPLE 6. Let $C = 45$, for which $A_1 = 2$, $B_1 = 14$, $(1+\sqrt{45})/2 = \langle 3\,;\overline{1,5}\rangle$, $\sqrt{45} = \langle 6\,;\overline{1,2,2,2,1,12}\rangle$. If $k = 3^t$ for $t \in \mathbb{N}$, then

$$D_k(X) = \left(\frac{2F_{4k}}{3}\right)^2 X^2 + 4\frac{F_{8k}}{F_{4k}}X + 45,$$

and for $q_0 = A_kX/2 + 3$:

$$\begin{aligned}\sqrt{D_k(X)} &= \langle 2q_0\,;\overline{w_{3^{t-1}-1},4q_0}\rangle\\ &= \left\langle 2q_0\,;\overline{\underbrace{1,2,2,2,1,12,\ldots,1,2,2,2,1,12}_{3^{t-1}-1\text{ copies of }1,2,2,2,1,12},1,2,2,2,1,4q_0}\right\rangle,\end{aligned}$$

which is a $(3^{t-1}-1)$-beeper for $\sqrt{45}$, and

$$\ell(\sqrt{D_k(X)}) = \ell\Big((1+\sqrt{D_k(X)})/2\Big) = 2\cdot 3^t.$$

REMARK 4. It is worthy of note that van der Poorten [7] showed that any palindrome appears as the symmetric part of some quadratic irrational, and he applied this to Fibonacci numbers. However, in [5], we demonstrated that this quadratic irrational can *always* be chosen as one of $\sqrt{D}$ or $(1+\sqrt{D})/2$ for a suitably chosen nonsquare $D \in \mathbb{N}$, a result known to Perron. Moreover, in [5] we vastly simplified the approach in [7] by using Perron's result. What we have done herein is to introduce new ideas for the generation of infinite families of continued fraction expansions related to Fibonacci numbers, which also extends what was sought in [7].

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REMARKS ON THE VISIBILITY PROBLEM IN THE FUNCTION FIELD CASE

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ABSTRACT. We extend results of [1], [2], [3] on the visibility problem for lattice points in $\mathbb{Z}^d$ to the case of function fields over finite fields which are related to important questions regarding the corresponding q-Jacobsthal function.

RÉSUMÉ. Nous étendons les résultats de [1], [2], [3] sur le problème de la visibilité des points du réseau $\mathbb{Z}^d$ au cas des corps de fonctions sur un corps fini, en rapport avec la fonction de q-Jacobsthal.

1. Introduction. Denote by $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in the fixed finite field $\mathbb{F}_q$. Furthermore for $n \in \mathbb{N}$ set

$$\Delta_n = \Delta_n(q) = \{(f, g) \in \mathbb{F}_q[x]^2, \text{ such that } \deg f \leq n \text{ and } \deg g \leq n\}.$$

Clearly $|\Delta_n| = q^{2(n+1)}$. Given distinct $P_1 = (f_1, g_1), P_2 = (f_2, g_2) \in \Delta_n$, as in the classical case, we say that P_1 is *visible* from P_2 if $(f_1 - f_2, g_1 - g_2) = 1$. This is equivalent to say that there are no elements of Δ_n in the line connecting P_1 and P_2. Similarly, if $S \subseteq \Delta_n$, we say that Δ_n is visible from S if for any $P \in \Delta_n$, there is $Q \in S$ such that P is visible from Q. We are interested in the following function:

$$\mathcal{F}_q(n) = \min\{|S|, S \subseteq \Delta_n, \Delta_n \text{ is visible from } S\}. \tag{1}$$

We will prove the following result which is analogous to [2, Theorem 1]:

THEOREM 1. *Let q be fixed and let $\beta_q > 4q^2/(1-\alpha_q)^2$ (where $\alpha_q = \alpha_q^3$ is defined in part 2 of Lemma 1) be any number. Then for all n large enough one can explicitly construct a subset $X_n(q)$ of Δ_n such that Δ_n is visible from $X_n(q)$ and*

$$|X_n(q)| \leq \beta_q \frac{n \log\log n}{\log_q n}.$$

Therefore, in particular $\mathcal{F}_q(n) \leq \beta_q \frac{n \log\log n}{\log_q n}$, for all n large enough.

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It is natural to generalize the concept of visibility to the d-dimensional space. If we write $\Delta_n^d = \{(f_1, \ldots, f_d) \in (\mathbb{F}_q[x])^d, \deg f_i \leq n\}$, then $|\Delta_n^d| = q^{d(n+1)}$. It is obvious what one means by saying that two points of Δ_n^d are visible from each other.

We will prove, as in [3, Theorem 3], that Theorem 1 can be improved in the higher dimensional case:

THEOREM 2. *Let q be fixed, $d \geq 3$ and let $\gamma_q > q/(1-\alpha_q^d)$ be any number. Then for all n large enough one can explicitly construct a subset $X_n^d(q)$ of Δ_n^d such that Δ_n^d is visible from $X_n^d(q)$ and $|X_n^d(q)| \leq \gamma_q \frac{n}{\log_q n}$. Therefore, if we define $\mathcal{F}_q^d(n)$ as the minimum number of elements in a subset of Δ_n^d, from which Δ_n^d is visible, we have for n large enough,*

$$\mathcal{F}_q^d(n) \leq \gamma_q \frac{n}{\log_q n}.$$

Further, let $\delta_q < \frac{1}{q}$ be any positive number. Then for all n large enough

$$\mathcal{F}_q^d(n) \geq \delta_q \frac{n}{\log_q n}.$$

We will need the following facts about distribution of polynomials in finite fields. The proofs can be found in the book of Lidl and Nieddereiter [10]. See also the book of Shparlinski [12]. The last statement can be found in [6]:

LEMMA 1. *Let q be a fixed power of a fixed prime and denote by $\mathcal{I}(q)$ the set of monic irreducible polynomials in $\mathbb{F}_q[x]$, by $\mathcal{I}_k(q)$ the set of irreducible monic polynomials of degree k and by $I_k(q)$ the order $|\mathcal{I}_k(q)|$. Then*

1. $I_k(q) = \frac{1}{k}(q^k + O(q^{k/2}))$;
2. *If $d \geq 3$, the series $\alpha_q^d = \sum_{k=1}^{\infty} \frac{I_k(q)}{q^{(d-1)k}}$ converges to a number less than 1;*
3. $\sum_{k \leq m} \frac{I_k(q)}{q^k} = (1 + o(1)) \log m$;
4. $\sum_{k \leq m} k I_k(q) = \frac{q}{q-1} q^m + O(q^{m/2})$.
5. *Let $m \in \mathbb{F}_q[x]$, and denote by $\omega_q(m)$ the number of distinct monic irreducible polynomials which divide m. If the degree of m is at most n, then if n is large enough, we have*

$$\omega_q(m) \leq \frac{n}{\log_q n - 3}. \qquad \blacksquare$$

LEMMA 2. *Given $a, b \in \mathbb{F}_q[x]$, the number of polynomials with degree up to s which are congruent to a modulo b is at most $q^{s+1-\deg b} + 1$.* $\blacksquare$

2. Proof of the lower bound in Theorem 2. We follow the proof of Abbott [1]. Suppose $S \subset \Delta_n^d$ is visible from every point of Δ_n^d, assume that

$|S| = r$ and $S = \{\underline{f}_1, \dots, \underline{f}_r\}$ where we write $\underline{f}_i = (f_{i1}, \dots, f_{id})$ $(i = 1, \dots, d)$. Let m be the least integer defined by the property that

$$\sum_{k \leq m} I_k(q) \geq r \tag{2}$$

and let $p_1, \dots, p_r$ be monic irreducible polynomials with degree less or equal than m. Next consider polynomials $f_{01}, \dots, f_{0d}$ which are respectively the solutions of the system of equations

$$\begin{cases} X \equiv f_{i1} \bmod p_i \\ i = 1, \dots, r \end{cases} \quad \cdots \quad \text{and} \quad \begin{cases} X \equiv f_{id} \bmod p_i \\ i = 1, \dots, r \end{cases}$$

with the property that $\underline{f}_0 = (f_{01}, \dots, f_{0d}) \notin S$. Indeed, by the Chinese remainder theorem one can find such a solution with $\deg f_{0j} \leq ([\log_q r] + 1) + \sum_{i \leq r} \deg p_i$, $j = 1, \dots, d$. In fact if $\underline{\tilde{f}}_0 = (\tilde{f}_{01}, \dots, \tilde{f}_{0d})$ is a fundamental solutions and $P = p_1 \cdots p_r$, then the set of solutions $\{(\tilde{f}_{01} + hP, \dots, \tilde{f}_{0d} + hP) \mid \deg(h) \leq [\log_q r] + 1\}$ contains more then r elements therefore it contains one at least outside S. Now from part 4 of Lemma 1 and from the inequality (2) above we deduce

$$\sum_{i \leq r} \deg p_i \leq \sum_{k \leq m} k I_k(q) = \big(1 + o(1)\big) \frac{q}{q-1} q^m.$$

Furthermore $r \geq \sum_{k \leq m-1} I_k(q) \geq \frac{1}{m-1} \sum_{k \leq m-1} k I_k(q) = \big(1 + o(1)\big) \frac{q^{m+1}}{q(q-1)(m-1)}$ implies that $([\log_q r] + 1) + \sum_{i \leq r} \deg p_i \leq \big(q + o(1)\big) r \log_q r$. Therefore all $\deg f_{01}$, $\dots$, $\deg f_{0d}$ are less than or equal to $\big(q + o(1)\big) r \log_q r$, which is smaller than n for $r \leq \big(\frac{1}{q} + o(1)\big) \frac{n}{\log_q n}$.

Finally if $r < \delta_q \frac{n}{\log_q n}$ and n is large enough, $\underline{f}_0 \in \Delta_n^d$. Therefore $r \geq \delta_q \frac{n}{\log_q n}$ and this completes the proof. ■

3. **Proof of Theorem 1.** We will need the following:

LEMMA 3. *Suppose that n is large enough, let $\beta > 0$ be any fixed number and let t be the least integer such that $q^{t+1} \geq \beta \log \log n$. Then for every given $f \in \Delta_n$ there exists $g \in \mathbb{F}_q[x]$ with $\deg g \leq t$ such that*

$$\sum_{\substack{p \in \mathcal{I}(q) \\ p \mid f - g}} \frac{1}{q^{\deg p}} < \alpha_q + \frac{1}{\beta} + o(1). \tag{3}$$

PROOF OF LEMMA 3. Consider the sum

$$\sum_{\substack{\deg g \leq t \\ g \neq f}} \sum_{\substack{p \in \mathcal{I}(q) \\ p \mid f - g}} \frac{1}{q^{\deg p}}. \tag{4}$$

We split the sum in three sums Σ_1, Σ_2 and Σ_3 where Σ_1 counts the irreducibles p with $\deg p \le t$, the second counts those with $t < \deg p \le (\log n) \log\log n$ and the third counts those with $(\log n)\log\log n < \deg p \le n$.

$$\text{Now } \Sigma_1 \le \sum_{\substack{p\in\mathcal{I}(q)\\ \deg p\le t}} \sum_{\substack{\deg g\le t\\ g\ne f, p\mid f-g}} \frac{1}{q^{\deg p}}$$

$$\le \sum_{\substack{p\in\mathcal{I}(q)\\ \deg p\le t}} \frac{1}{q^{\deg p}}\left(\frac{q^{t+1}}{q^{\deg p}}+1\right) = \sum_{k\le t}\left(q^{t+1}\frac{I_k(q)}{q^{2k}}+\frac{I_k(q)}{q^k}\right)$$

by Lemma 2 and from Lemma 1 we obtain

$$\Sigma_1 \le q^{t+1}(\alpha_q + o(1)) + (1+o(1))\log t = q^{t+1}(\alpha_q+o(1)). \tag{5}$$

As for Σ_2, note that there are no irreducible dividing $f-g'$ and $f-g''$ with degree larger then t. Therefore, from part 3 of Lemma 1,

$$\Sigma_2 \le \sum_{\substack{p\in\mathcal{I}(q)\\ \deg p\le \log n\log\log n}} \frac{1}{q^{\deg p}} = (1+o(1))\log\log n. \tag{6}$$

Furthermore

$$\Sigma_3 \le \sum_{\substack{\deg g\le t\\ g\ne f}} \frac{1}{q^{\log n\log\log n}} \sum_{\substack{p\in\mathcal{I}(q)\\ p\mid f-g}} 1 \ll \frac{q^{t+1}}{q^{\log n\log\log n}}\frac{n}{\log n} = o(1). \tag{7}$$

Finally by (5), (6) and (7) we deduce that the sum in (4) is

$$\le q^{t+1}(\alpha_q+o(1)) + (\beta+o(1))\log\log n + o(1) \le q^{t+1}\left(\alpha_q + \frac{1}{\beta} + o(1)\right).$$

Hence, for some $g\in\mathbb{F}_q[x]$ with $\deg g < t$, (3) is satisfied. ∎

We define the *q-Jacobsthal function* of $m\in\mathbb{F}_q[x]$ as follows:

$$\mathcal{J}_q(m) = \min\{t \mid \forall a\in\mathbb{F}_q[x], \exists h\in\mathbb{F}_q[x], \deg h < t, \gcd(a+h,m)=1\}. \tag{8}$$

It is immediate to see that $\mathcal{J}_q(m)$ is well defined and that $\mathcal{J}_q(m) < \deg m$. Indeed, for any $a\in\mathbb{F}_q[x]$, if r is the remainder of the division of $1-a$ by m, then it clear that $\deg r < \deg m$ and $\gcd(a+r,m)=1$. We will need the following:

LEMMA 4. *Suppose $m\in\mathbb{F}_q[x]$ and that $\gamma = \sum_{\substack{p\in\mathcal{I}(q)\\ p\mid m}} \frac{1}{q^{\deg p}} < 1$. Then for n large enough, $q^{\mathcal{J}_q(m)+1} \le (1-\gamma)^{-1}\omega_q(m)$.*

Proof of Lemma 4. For any $a \in \mathbb{F}_q[x]$, consider the set $S = \{a + h \mid h \in \mathbb{F}_q[x], \deg h \leq k\}$. Then $|S| = q^{k+1}$. We want to estimate the size of the set

$$S_m = \{y \in S \mid \gcd(y, m) \neq 1\}.$$

Note that by Lemma 2

$$\begin{aligned} \#S_m &\leq \sum_{\substack{p \in \mathcal{I}(q) \\ p|m}} \#\{h \in \mathbb{F}_q[x] \mid \deg h < k, p|h + k\} \\ &\leq \sum_{\substack{p \in \mathcal{I}(q) \\ p|m}} (q^{k+1-\deg p} + 1) \leq q^{k+1}\gamma + \omega(m) \end{aligned}$$

which is smaller than q^{k+1} if $q^{k+1} > (1-\gamma)^{-1}\omega(m)$. Finally, there is an element of S not in S_m if k satisfies the above, so that

$$q^{\mathcal{J}_q(m)+1} \leq (1-\gamma)^{-1}\omega(m). \qquad \blacksquare$$

We are now ready to prove Theorem 1. Consider the set

$$X_n(q) = \{(f, g) \in \Delta_n, \deg f \leq t, \deg g \leq s\}$$

where t is the least integer such that $q^{t+1} > \frac{2}{1-\alpha_q} \log\log n$ and s is the least integer such that $q^{s+1} > (\frac{1-\alpha_q}{2} + \epsilon)\frac{n}{\log_q n - 3}$ where $\epsilon > 0$ is small and will be chosen later.

Then (if ϵ is small enough)

$$|X_n(q)| = q^{s+1}q^{t+1} \leq \beta_q \frac{n \log\log n}{\log_q n}.$$

We need to show that Δ_n is visible from X_n for n large enough. Indeed, for $(a, b) \in \Delta_n$, from Lemma 3 we know that there exists $g \in \mathbb{F}_q[x]$ with $\deg g \leq t$ such that $\sum_{\substack{p \in \mathcal{I}(q) \\ p|a-g}} \frac{1}{q^{\deg g}} \leq (\alpha_q + 1)/2 + o(1)$. Furthermore Lemma 4 implies that $q^{\mathcal{J}_q(a-g)+1} \leq ((1-\alpha_q)/2 + o(1))\omega_q(a-g)$. Note that from the fifth part of Lemma 1, for n large enough

$$\left(\frac{1-\alpha_q}{2} + o(1)\right)\omega_q(a-g) \leq \left(\frac{1-\alpha_q}{2} + \epsilon\right)\frac{n}{\log_q n} \leq q^{s+1}.$$

Therefore $\mathcal{J}_q(a-g) \leq s$ and this implies that there exists $h \in \mathbb{F}_q[x]$ with $\deg h \leq s$ such that $\gcd(a-g, b-h) = 1$. So, (a, b) and (f, h) are visible from each other and this concludes that proof. $\blacksquare$

4. Proof of the upper bound in Theorem 2. In this section we follow the method of [3] to investigate the concept of visibility in higher dimensional space. For $d \geq 3$, consider the set

$$X_n^d = \{(g_1, \dots, g_{d-1}, g_d) \in (\mathbb{F}_q[x])^d, \deg g_i \leq s \text{ for } i < d \text{ and } \deg g_d = 0\}.$$

Clearly $|X_n^d| = q^{(d-1)*(s+1)+1}$.

We want to show that for a suitable choice of s, Δ_n^d is visible from X_n^d. Clearly all the elements of Δ_n^d which have a degree 0 polynomial in the last coordinate are visible from X_n^d. Therefore fix $(f_1, \dots, f_d) \in \Delta_n^d$ such that $\deg f_d \geq 1$. We want to estimate the size of the set

$$\mathcal{A} = \{(g_1, \dots, g_{d-1}, g_d) \in X_n^d, \deg((f_1 - g_1, f_2 - g_2, \dots, f_d - g_d)) \geq 1\}.$$

First of all, we observe that

$$\begin{aligned} |\mathcal{A}| &\leq \sum_{\substack{g_1,\dots,g_{d-1} \\ \deg g_i \leq s, g_d \in \mathbf{F}_q}} \sum_{\substack{p \in \mathcal{I}(q) \\ p \mid \gcd(f_1-g_1, f_2-g_2, \dots, f_d-g_d)}} 1 \\ &= \sum_{g_d \in \mathbf{F}_q} \sum_{\substack{p \in \mathcal{I}(q) \\ p \mid f_d - g_d}} \sum_{\substack{g_1,\dots,g_{d-1} \\ \deg g_i \leq s, p \mid (f_i - g_i)}} 1 \\ &= \sum_{g_d \in \mathbb{F}_q} \sum_{\substack{p \in \mathcal{I}(q) \\ p \mid f_d - g_d}} \prod_{i=1}^{d-1} \Big(\sum_{\deg g_i \leq s, p \mid (f_i - g_i)} 1 \Big). \end{aligned}$$

From Lemma 2 we deduce that

$$|\mathcal{A}| \leq \sum_{g_d \in \mathbf{F}_q} \sum_{\substack{p \in \mathcal{I}(q) \\ p \mid f_d - g_d}} \left(1 + \frac{q^{s+1}}{q^{\deg p}}\right)^{d-1}.$$

Now we have

$$\begin{aligned} |\mathcal{A}| &\leq \sum_{g_d \in \mathbf{F}_q} \sum_{\substack{p \in \mathcal{I}(q) \\ p \mid f_d - g_d}} \sum_{j=0}^{d-1} \binom{d-1}{j} \left(\frac{q^{s+1}}{q^{\deg p}}\right)^j \\ &\leq \sum_{g_d \in \mathbb{F}_q} \sum_{\substack{p \in \mathcal{I}(q) \\ p \mid f_d - g_d}} 1 + \sum_{\substack{p \in \mathcal{I}(q) \\ \deg(p) \leq n}} \sum_{j=1}^{d-2} \binom{d-1}{j} \left(\frac{q^{s+1}}{q^{\deg p}}\right)^j \\ &\quad + |X_n^d| \sum_{p \in \mathcal{I}(q)} \frac{1}{q^{(d-1)\deg p}}. \end{aligned}$$

We evaluate each of the three terms separately. For the last one, we have to use part 2 of Lemma 1. For the middle one just uses part 3 of Lemma 1 observing that

$$\begin{aligned}\sum_{\substack{p\in\mathcal{I}(q)\\ \deg(p)\leq n}}\sum_{j=1}^{d-2}\binom{d-1}{j}\left(\frac{q^{s+1}}{q^{\deg p}}\right)^j &\leq 2^{d-1}q^{(s+1)(d-2)}\sum_{\substack{p\in\mathcal{I}(q)\\ \deg(p)\leq n}}\frac{1}{q^{\deg p}}\\ &\leq 2^{d-1}q^{(s+1)(d-2)}\sum_{j\leq n}\frac{I_j(q)}{q^j}\\ &\leq (1+o(1))2^{d-1}\left(\frac{|X_n^d|}{q}\right)^{(d-2)/(d-1)}\log n,\end{aligned}$$

and for the first sum we use the fifth part of Lemma 1. Putting all these together we obtain:

$$|\mathcal{A}|\leq q^{\frac{n}{\log_q n-3}}+\alpha_q^d|X_n^d|+\big(1+o(1)\big)\log n\left(\frac{|X_n^d|}{q}\right)^{(d-2)/(d-1)}.$$

Finally, in order to have $|\mathcal{A}|<|X_n^d|$ for n large enough, it is enough to choose s in such a way that $(1-\alpha_q^d)|X_n^d|>q^{\frac{n}{\log_q n-3}}$, and this gives the claim. ∎

5. Final remarks. The order of the q-Jacobsthal function.

The classical Jacobsthal function has been investigated in [4], [7], [8], [9], [13], [14]. We have already defined in (8) the natural analogue of the Jacobsthal function for $\mathbb{F}_q[x]$. If we set

$$Y_n=\{(0,h)\in\Delta_n,\deg h\leq\max_{g\in\mathbb{F}_q[x],\deg g\leq n}\mathcal{J}_q(g)\},$$

then clearly Δ_n is visible from Y_n as for every $(f,g)\in\Delta_n$ there is an $h\in Y_n$ (also $-h\in Y_n$) and $\gcd(f,g-h)=1$ so that (f,g) is visible from $(0,h)$.

It is conjectured (see [11]) that for any $m\in\mathbb{F}_q[x]$, $\mathcal{J}_q(m)\leq\log_q\deg m$. This would imply that $\mathcal{F}_q(n)\leq n$, which is weaker than the upper bound in Theorem 1.

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COMPLEX OSCILLATION THEORY AND SPECIAL FUNCTIONS

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Presented by R. S. C. Wong, FRSC

RÉSUMÉ. Ceci est une annonce de nôtre resultat [10] dans lequel nous montrons que les représentations des solutions non-oscillatoires de certaines équations differentielles ordinaires proviennent des fonctions hypergéométriques confluents avec un nombre fini de zéros et aussi des polynômes orthogonaux.

Nous caractérisons les distributions nulles et les représentations des solutions de deux classes des équations differentielles ordinaires et indiquons que la solution du problème dans les autres cas est relié à un problème de Heine.

1. **Introduction.** Let $f(z)$ be an entire function and denote its order by $\sigma(f)$. The exponent of convergence of f is $\lambda(f) = \limsup_{r\to+\infty} \log n(r,f)/\log r$, where $n(r,f)$ denotes the number of the zeros of f in $|z| < r$. It is an easy consequence of Weierstrass' theorem that $\lambda(f) \le \sigma(f)$ (see [15]).

The *complex oscillation problem* (see [18]) we are interested in is to study the quantity $\lambda(f)$ in relation to $\sigma(f)$ for entire functions f satisfying

$$f''(z) + A(z)f(z) = 0, \tag{1.1}$$

and $A(z)$ is an entire transcendental function of finite order. One of the main problems is to seek sufficient conditions on $A(z)$ that guarantee that each solution f of (1.1) to satisfy

$$\lambda(f) = +\infty. \tag{1.2}$$

Some basic oscillation problems are considered in [5], followed by [6] which deals with periodic coefficients, and [7] which deals with meromorphic solutions. We refer to [18, Chapter 5] and the references therein for subsequent works.

The equation (1.1) considered by Bank and Laine in [6] has periodic coefficient

$$A(z) = B(e^z), \quad B(\zeta) = K_k\zeta^{-k} + \cdots + K_0 + \cdots + K_\ell\zeta^\ell, \quad \ell > 0, k \ge 0, \tag{1.3}$$

where $K_\ell K_k \neq 0$. They established the following theorem when (1.2) is violated.

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THEOREM 1.1. *Let $A(z)$ be an entire function given by (1.3). If (1.1) admits a non-trivial solution f with $\lambda(f) < +\infty$, then there exist constants d, d_j, and a polynomial $\psi(\zeta)$ with only simple roots, such that if ℓ is odd in (1.3), then $k = 0$, and*

$$f(z) = \psi(e^{z/2}) \exp\Big(\sum_{j=0}^{\ell} d_j e^{jz/2} + dz\Big), \tag{1.4}$$

where $d_j = 0$ for even j. If ℓ is even, then k is also even, and

$$f(z) = \psi(e^{z}) \exp\Big(\sum_{j=-k/2}^{\ell/2} d_j e^{jz} + dz\Big). \tag{1.5}$$

It appears to be a difficult problem to characterize the choices of $B(\zeta)$ in (1.3) so that equation (1.1) would admit a solution f with $\lambda(f) < +\infty$, and to write down such a solution explicitly [4], [9], [12]. The major difficulty is to verify the existence of the polynomial $\psi(\zeta)$ that appears in (1.4) and (1.5).

This announcement summarizes our findings in [10] where we show that the above mentioned problem is closely related to special functions and in some cases to orthogonal polynomials. We have shown that for the well-studied classes of (1.3), equation (1.1) can be transformed to equations satisfied by two classes of special functions—the Bessel functions and Coulomb wave functions. We solve these two subclasses of (1.1) completely—general and explicit solutions $f(z)$ to (1.1) are found regardless of whether $\lambda(f) < +\infty$ or $= +\infty$. Moreover, we reproduce the polynomial component in the solution $f(z)$ via the confluent hypergeometric functions. This extends our understanding of the polynomial components found by Bank, Laine and Langley [6], [8]. This establishes a connection between the finding of $\psi(\zeta)$ for (1.4)–(1.5) for (1.1) in the above question and to a class of orthogonal polynomials—Bessel polynomials. The key is the following transformation due to Lommel [21, p. 97]

$$x = \alpha t^{\beta}, \quad y(x) = t^{\gamma} u(t),$$

where α, β and γ are constants. We have identified the transformations needed to transform (1.1) to the above-mentioned equations. We then establish another connection to (1.1) with the remaining (1.3) not covered in the previous consideration of the *Heine problem* for differential equations with polynomial coefficients [20]. The study here is based on the solutions (1.4) and (1.5). In the case when $\ell > 0$, $k = 0$ and ℓ is odd, we shall give a characterization of the coefficients $\{K_j\}$ in (1.3) based on the Heine problem. This extends the results obtained in [9] where only zero-free solutions were studied.

We also study the closely related *subnormal solution* problem first studied by Frei [11], then by Wittich [22] and then by Bank and Laine [6].

The subnormal solution problem can be transformed to be an oscillation problem of (1.1) with specific (1.3) in one of the two above-mentioned subclasses of (1.3). Hence the corresponding subnormal solution problem can be solved completely to the same effect as the oscillation problem.

We follow notations in Watson [21] for Bessel functions and in Grosswald [13] for the Bessel polynomials. The notations of the Coulomb wave functions are as in [1].

We first review what was obtained by Bank, Laine and Langley in [8].

THEOREM 1.2. *Let K be a complex number. The equation*

$$f'' + (e^z - K)f = 0 \tag{1.6}$$

admits a non-trivial solution f with $\lambda(f) < +\infty$ if and only if

$$K = (2n+1)^2/16, \tag{1.7}$$

where n is a non-negative integer, and

$$f(z) = \psi(e^{z/2})\exp[de^{z/2} - (2n+1)z/4], \tag{1.8}$$

where ψ is a polynomial of degree n, and $d^2+4=0$. Conversely, when K is given in (1.7), then f of (1.8) is a solution to (1.6) with $\lambda(f) < +\infty$.

Our findings show that the polynomial component $\psi(\zeta)$ in (1.8) and more generally in (1.4) and (1.5) are very special polynomials for two special classes of (1.3). One of these is essentially given in equation (1.6). Moreover, we shall show that the general solution of this equation is related to Bessel functions. The other class that has this property turns out to be the equation of the form

$$f'' + (K_{-2}e^{-2z} + K_{-1}e^{-z} + K_0)f = 0, \tag{1.9}$$

to be discussed in the next section. The equations (1.6) (and its generalization (1.10)) and (1.9) seem to be the only two subclasses among the general case of (1.1) with coefficient (1.3) that are related to classical confluent hypergeometric functions and hence can be solved completely. We now state our first main result.

THEOREM 1.3. *Let K_0 and K_1 be complex constants. Then any two linearly independent solutions of the equation*

$$f'' + (K_1e^z - K_0)f = 0 \tag{1.10}$$

are given by

$$y_\pm(z) = A_\pm J_{2\sqrt{K_0}}(\pm 2\sqrt{K_1}e^{z/2}) + B_\pm Y_{2\sqrt{K_0}}(\pm 2\sqrt{K_1}e^{z/2}). \tag{1.11}$$

The solutions (1.11) have a finite exponent of convergence if and only if

$$2\sqrt{K_0} = n + 1/2,$$

where n is an integer, and, if $n \geq 0$, then

$$y_{\pm}(z) = \theta_n(\pm 2i\sqrt{K_1}e^{z/2}) \exp(\mp 2i\sqrt{K_1}e^{z/2} + dz), \tag{1.12}$$

where $\theta_n(x)$ is the reverse Bessel polynomial of degree n and $d = (-2n-1)/4$. If, however, $n = -m$, $m \geq 1$, then we replace all the n in (1.12) by $m-1$.

We note that (1.6) was already considered by Lommel in 1871 [19].

2. **Subnormal solutions.** The oscillation problem of the second class of equation that we have found exact solutions are the ones given in the form (1.9). The following special case

$$f'' + [K + (2e^{-z} - e^{-2z})/4]f = 0 \tag{2.1}$$

is considered by Bank and Laine in [6] in connection with subnormal solutions of the differential equation (2.4) below first considered by Frei.

Let $P(\zeta)$ and $Q(\zeta)$ be two polynomials at least one of which is non-constant. Then it is known that the growth of an entire solution $g(z)$ of the equation

$$g''(z) + P(e^z)g'(z) + Q(e^z)g(z) = 0 \tag{2.2}$$

can reach infinite order. We say that $g(z)$ is *subnormal* if it satisfies

$$\limsup_{r\to+\infty} \frac{\log\log M(r,g)}{r} = 0. \tag{2.3}$$

That is, the subnormal solutions of (2.2) are those solutions that grow slower than the maximum speed allows (*i.e.*, infinite order). Wittich [22] showed that any subnormal solution can be written in the form $g(z) = e^{dz}S(e^z)$ for some polynomial $S(\zeta)$. However, Frei [11] considered the special case

$$g''(z) + e^{-z}g'(z) + Kg(z) = 0, \tag{2.4}$$

and she showed that equation (2.4) will have a subnormal solution if and only if $K = -n^2$ for some integer n. Moreover, the polynomial $S(\zeta)$ mentioned above has exact degree n.

By applying the transformation $f(z) = g(z)\exp(-e^{-z}/2)$, Bank and Laine [6] showed that equation (2.4) can be transformed to (2.1). It follows from this transformation that the solution $f(z)$ of (2.1) that satisfies $\lambda(f) < +\infty$ if and only if $g(z)$ is a subnormal solution to (2.4). Indeed, they proved:

THEOREM 2.1. *Let K be a non-zero complex number. Suppose equation (2.1) admits a non-trivial solution f that satisfies $\lambda(f) < +\infty$, then there exists a positive integer n with $K = -n^2$ and f must be equal to either*

$$f_1(z) = \Big(\sum_{j=0}^{-(n-1)} B_j e^{jz}\Big) \exp(nz + e^{-z}/2)$$

or

$$f_2(z) = \Big(\sum_{j=0}^{-n} b_j e^{jz}\Big) \exp(nz - e^{-z}/2).$$

Conversely, for each positive integer n, equation (2.1) with $K = -n^2$ possesses two linearly independent solutions of the above forms.

The equations (2.2) and (2.4) and their generalizations were also considered in [2], [14]. In particular, we mention the works of Gundersen, Langley and Ozawa (see [18] and the references therein).

We show that equation (1.9) is closely related to the *Coulomb Wave equation*

$$y'' + [1 - 2\eta x^{-1} - L(L+1)x^{-2}]y = 0, \tag{2.5}$$

where L, η are complex constants. Two linearly independent solutions of (2.5) are given by $F_L(\eta, x)$ and $G_L(\eta, x)$ known as the *regular* and *irregular Coulomb Wave functions* respectively [1]. We have:

THEOREM 2.2. *Let K_{-2}, K_{-1} and K_0 be complex numbers such that $K_{-2}K_{-1} \neq 0$. Then two linearly independent solutions of (1.9) are given by*

$$f_\pm(z) = A_\pm F_L(\eta_\pm, \alpha_\pm e^{-z}) + B_\pm G_L(\eta_\pm, \alpha_\pm e^{-z}), \tag{2.6}$$

where

$$\alpha_\pm^2 = K_{-2} \qquad -2\eta_\pm\alpha_\pm = K_{-1}, \quad L = \tfrac{1}{2}(-1 + 2i\sqrt{K_0}).$$

The solutions (2.6) have $\lambda(f_\pm) < +\infty$ if and only if there are non-negative integers n_+ and n_-, $n_+ > n_- \geq 0$ such that

$$i(2\sqrt{K_0} \pm K_{-1}/\sqrt{K_2}) = 2n_\pm + 1,$$

or equivalently

$$iK_{-1}/\sqrt{K_{-2}} = n_+ - n_- \quad \text{and} \quad 2i\sqrt{K_0} = n_+ + n_- + 1,$$

and, by writing $n = n_+$, $\hat{n} = n_+ - n_-$, we have

$$f_+(z) = e^{z/2}e^{-(1-a_+/2)z}y_n(e^z\,; a_+, b_+)\exp(-b_+e^{-z}/2), \tag{2.7}$$
$$f_-(z) = e^{z/2}e^{-(1-a_-/2)z}y_{(n-\hat{n})}(e^z\,; a_-, b_-)\exp(-b_-e^{-z}/2), \tag{2.8}$$

where $a_\pm = 2(1 + i\eta_\pm)$ and $b_\pm = -2i\alpha_\pm$, and the y_j in (2.7) and (2.8) are generalized Bessel polynomials.

We deduce immediately, as a simple corollary, the complete solution of the oscillation problem of equation (2.1).

THEOREM 2.3. *Let K be a complex constant. Then any two linearly independent solutions of equation (2.1) are given by*

$$f_\pm(z) = A_\pm F_L\left(\pm\tfrac{i}{2}, \pm\tfrac{i}{2}e^{-z}\right) + B_\pm G_L\left(\pm\tfrac{i}{2}, \pm\tfrac{i}{2}e^{-z}\right),$$

where $L = \frac{1}{2}(-1 + 2i\sqrt{K})$. Moreover, we have $\lambda(f_\pm) < +\infty$ if and only if $L = n + \frac{1}{2}$ or equivalently $K = -(L + \frac{1}{2})^2 = -n^2$ for some positive integer n, and

$$f_+(z) = y_n(e^z\ ; 1, 1)\exp(-e^{-z}/2), \tag{2.9}$$

$$f_-(z) = e^z y_{n-1}(e^z\ ; 3, -1)\exp(e^{-z}/2). \tag{2.10}$$

Here the y_j in (2.9) and (2.10) are generalized Bessel polynomials.

3. Heine problems. We next consider the remaining classes of equation (1.1) not covered by the previous discussion. We recall that the solution f to (1.1) must satisfy (1.2) if both the integers ℓ and k in (1.3) are not equal to zero [3], [12]. Thus no representation of the forms (1.4) or (1.5) are possible. Hence it suffices to consider the case where $\ell > 0$ and $k = 0$. We choose to consider ℓ being odd below.

Here we shall take an approach to the problem yet again different from that of Bank [4]. We shall suppose that (1.1) has a solution $f(z)$ with $\lambda(f) < +\infty$. Hence $f(z)$ must admit either the representation (1.4) or (1.5) depending on whether ℓ is odd or even respectively [9, Prop. 1]. Write $f(z)$ in the standard form

$$f(z) = \psi(e^{z/q})\exp\left(P(e^{z/q}) + dz\right) \tag{3.11}$$

where

$$P(\zeta) = \sum_{j=1}^{\ell/(3-q)} d_j\zeta^j,$$

$q = 1$ if ℓ is even or $q = 2$ when ℓ is odd [9, Prop. 1], d is a constant and $\psi(\zeta)$ is a polynomial of degree n. We may assume that $\psi(0) \neq 0$. One can show that the $\psi(\zeta)$ satisfies a second order differential equation with polynomial coefficients.

We can then apply a theorem of Heine (see [20]) and prove the following.

THEOREM 3.1. *Let ℓ, $\ell \geq 3$, be odd and the coefficients $K_\ell, \dots, K_{\frac{\ell+1}{2}}$ be given in (1.3). Then there are at most*

$$2\binom{n + (\ell - 1)/2}{n}$$

choices for the remaining coefficients $K_{\frac{\ell-1}{2}}, \dots, K_1$ so that the differential equation (1.1) can admit a solution $f(z)$ given by (3.11) where $\psi(\zeta)$ is a polynomial of degree n. The coefficient K_0 is determined by ℓ and n.

We note that the Heine theorem is a fundamental result in electrostatic models [16] and it also has applications in quantized physical models (see *e.g.* [17]).

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A COCYCLE PROOF THAT REVERSIBLE FLEMING-VIOT PROCESSES HAVE UNIFORM MUTATION

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ABSTRACT. Why must the mutation operator associated with a reversible Fleming-Viot process be uniform? Our explanation is based on Handa's recent result that reversible distributions must be quasi-invariant under a certain flow, forcing the mutation operator to satisfy a cocycle identity.

RÉSUMÉ. Pourquoi l'opérateur de mutation lié à un processus réversible de Fleming-Viot doit-il être uniforme? Notre explication est baseé sur le résultat récent de Handa que les distributions réversibles doivent être quasi-invariables sous un certain écoulement, forçant l'opérateur de mutation à satisfaire une identité de cocycle.

1. Introduction. The Fleming-Viot process models the evolution of the genetic profile of a population. Each individual in the population has a genetic type belonging to the type space E, and X_t denotes the empirical distribution of types at time t. The process X_t lives on the space $M_1(E)$ of probability measures on E. The changes to the genetic makeup of this population come from two opposing sources; *genetic drift* which encourages conformity by favoring the offspring of individuals with dominant type and *mutation* which continually adds fresh variation. Other mechanisms such as *selection* and *recombination* can be added to the model for more realism. As with all Markov processes, it is of great interest to find an initial distribution Π which makes the process reversible in time, so that the different genetic forces are in perfect balance.

It has long been known ([2, Chapter 10, Exercise 14(a)], [1, Theorem 2.3], [8, p. 276]) that if the mutation is uniform, then the Fleming-Viot process has a reversible distribution. In 1999, Li, Shiga, and Yao proved [5, Theorem 1.1] the "folklore" result (suggested in [3], and proved for finite E in [6]) that the converse is true. They considered a reversible Fleming-Viot process with mutation and selection, but not recombination. First reducing the problem to the inselective case, they used Dirichlet forms to argue that the mutation operator A is uniform.[1]

In a very interesting recent paper [4], Handa shows that a probability measure on $\mathcal{M}_1(E)$ is reversible for the Fleming-Viot process (with mutation, selection,

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[1] Provided the mutation operator generates an irreducible Markov semigroup on E.

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and recombination) if and only if it is quasi-invariant with respect to a certain flow on $\mathcal{M}_1(E)$. This leads to a cocycle identity (see (3) below) that Handa uses to explore reversible Fleming-Viot processes. For example, in [4, Lemma 3.5] he uses cocycles to reprove Li, Shiga, and Yao's result that a reversible distribution for the selective case is the reversible distribution for the inselective case multiplied by a simple density function. In [4, Proposition 3.1] Handa also uses cocycles to show that if the mutation operator generates an irreducible semigroup, then Π-almost every μ has full support on E. His main result [4, Theorem 2.2] extends Li, Shiga, and Yao's result to the case with recombination. Handa shows that[2] the Fleming-Viot process has a reversible distribution if and only if the recombination kernel is of a certain form, and the mutation operator A plus the diagonal part of the recombination kernel is uniform. His analysis mainly uses cocycles, but for the mutation he simply quotes [5]—though in a later remark [4, Section 4, Remark (i)] he outlines a direct proof using cocycles.

In this note we analyze an operator via the cocycle identity and show how the property of reversibility imposes uniformity.

2. **The analytical result.** Let E be a compact metric space with Borel σ-algebra $\mathcal{B}(E)$, $\mathcal{M}_1(E)$ the space of probability measures on $\mathcal{B}(E)$, and $C(E)$ the space of continuous functions on E. For $f \in C(E)$ and $\mu \in \mathcal{M}_1(E)$ we set $\langle f, \mu\rangle := \int f(x)\,\mu(dx)$.

For every $f \in C(E)$ and $\mu \in \mathcal{M}_1(E)$, we define a "shifted" probability measure $S_f\mu$ by

$$S_f\mu(dx) := \frac{e^{f(x)}\mu(dx)}{\langle e^f, \mu\rangle}. \tag{1}$$

Note that $(S_f)_{f\in C(E)}$ forms a transformation group on $\mathcal{M}_1(E)$ since $S_{f+g} = S_f(S_g)$. This group is used in [9], [7] to study Fleming-Viot operators and it is core of Handa's work in [4]. One particularly nice property of this flow can be obtained by direct calculation:

$$\frac{d}{dt}\langle f, S_{tg}\mu\rangle = \mathrm{cov}_{S_{tg}\mu}(f, g).$$

Let $(B, D(B))$ be a densely defined linear operator on $C(E)$, and define for $f \in D(B)$

$$\Lambda(f, \mu) := \int_0^1 \langle Bf, S_{uf}\mu\rangle\, du. \tag{2}$$

We assume that the *cocycle identity* holds for all $f, g \in D(B)$ and $\mu \in \mathcal{M}_1(E)$:

$$\Lambda(f + g, \mu) = \Lambda(f, S_g\mu) + \Lambda(g, \mu). \tag{3}$$

LEMMA 1. *If* $\mathrm{Var}_\mu(f) = 0$, *then* $\mathrm{Var}_\mu(Bf) = 0$.

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[2] Provided the mutation operator generates an irreducible Markov semigroup on E.

PROOF. Choose $f \in D(B)$ and $\mu \in \mathcal{M}_1(E)$ so that f is a constant μ-almost everywhere. Then $S_{uf}\mu = \mu$ for all $0 \le u \le 1$ so $\Lambda(f, S_h\mu) = \langle Bf, S_h\mu\rangle$ for any $h \in C(E)$. Applying this at $h \equiv 0$ gives $\Lambda(f,\mu) = \langle Bf, \mu\rangle$. The cocycle identity implies

$$\Lambda(f, S_g\mu) + \Lambda(g,\mu) = \Lambda(g, S_f\mu) + \Lambda(f,\mu),$$

or

$$\Lambda(f, S_g\mu) = \Lambda(f,\mu).$$

Therefore $\langle Bf, S_g\mu\rangle = \langle Bf, \mu\rangle$ is independent of g, so setting $g = t(Bf)$ and differentiating gives $0 = \frac{d}{dt}\big|_{t=0}\langle Bf, S_{t(Bf)}\mu\rangle = \operatorname{cov}_\mu(Bf, Bf)$. ■

PROPOSITION 1. *If $\big(B, D(B)\big)$ is a closed operator satisfying the cocycle identity (3), then B must be of the form*

$$Bf(x) = \alpha f(x) + \langle f, \nu\rangle, \quad f \in C(E), \tag{4}$$

for some $\alpha \in \mathbb{R}$ and some finite signed measure ν.

PROOF. Let $x \neq y \in E$ and $f, g \in D(B)$, and define the function

$$F = [g(x) - g(y)]f + [f(y) - f(x)]g.$$

Since $F(x) = F(y)$, we can apply Lemma 1 at $\mu = (\delta_x + \delta_y)/2$ and conclude that $BF(x) = BF(y)$. This can be rearranged to read

$$[g(x) - g(y)][Bf(x) - Bf(y)] = [f(x) - f(y)][Bg(x) - Bg(y)].$$

Since $D(B)$ is dense in $C(E)$ we may choose $g \in D(B)$ with $g(x) \neq g(y)$ and define $\alpha_{xy} = [Bg(x) - Bg(y)]/[g(x) - g(y)]$, so that

$$Bf(x) - Bf(y) = \alpha_{xy}[f(x) - f(y)]$$

for all $f \in D(B)$. Now take three distinct points $x, y, z \in E$ and $f \in D(B)$ with $f(z) \neq f(x)$. Then

$$\begin{aligned}\alpha_{zx} = \frac{Bf(z) - Bf(x)}{f(z) - f(x)} &= \frac{Bf(z) - Bf(y)}{f(z) - f(x)} + \frac{Bf(y) - Bf(x)}{f(z) - f(x)}\\ &= \alpha_{zy}\frac{f(z) - f(y)}{f(z) - f(x)} + \alpha_{yx}\left(1 - \frac{f(z) - f(y)}{f(z) - f(x)}\right).\end{aligned}$$

Once again, since $D(B)$ is dense we may choose $g \in D(B)$ with $g(z) \neq g(x)$, and so that $f(z) - f(y)/f(z) - f(x) \neq g(z) - g(y)/g(z) - g(x)$. Applying the equation for α_{zx} to f and g then taking the difference, we obtain

$$0 = \left(\frac{f(z) - f(y)}{f(z) - f(x)} - \frac{g(z) - g(y)}{g(z) - g(x)}\right)[\alpha_{zy} - \alpha_{yx}],$$

and conclude that $\alpha_{zy} = \alpha_{yx}$. Thus, all these α's are the same, and we can denote the common value as α. We have

$$Bf(x) - \alpha f(x) = Bf(y) - \alpha f(y)$$

for all $x, y \in E$ and $f \in D(B)$. This means that the operator $B - \alpha I$ takes $D(B)$ into the space of constant functions.

Since $\big(B, D(B)\big)$ is closed, it follows that $\big(B - \alpha I, D(B - \alpha I)\big)$ is also closed, where $D(B - \alpha I) := D(B)$. Then its null space $N(B - \alpha I)$ is closed and the quotient space $D(B)/N(B - \alpha I)$ is a vector space with norm

$$\|[f]\| = \inf\{\|f - g\| : g \in N(B - \alpha I)\}.$$

The canonical projection $\pi \colon D(B) \to D(B)/N(B - \alpha I)$ defined by $\pi(f) = [f] = f + N(B - \alpha I)$ is continuous.

Fix any $x \in E$ and define $\phi \colon D(B)/N(B - \alpha I) \to \mathbb{R}$ by $\phi([f]) = (B - \alpha I)f(x)$. This is a one-to-one linear map so that $D(B)/N(B - \alpha I)$ must be either zero or one dimensional, and hence ϕ must be continuous. Thus $f \to (B - \alpha I)f(x) = \phi\big(\pi(f)\big)$ is a continuous linear functional on $D(B)$ and can be extended continuously to $C(E)$ where it is represented by a finite signed measure ν. For $f \in D(B)$ we have

$$(B - \alpha I)f(x) = \langle f, \nu \rangle,$$

and since the right hand side is a continuous operator, and since B is closed we conclude that $D(B) = C(E)$ and

$$Bf(x) = \alpha f(x) + \langle f, \nu \rangle, \quad f \in C(E). \qquad \blacksquare$$

3. Application to the Fleming-Viot process.

Handa [4, Step 2, Proof of Theorem 2.2] proved that if Π is reversible for the Fleming-Viot process, then Π is quasi-invariant with respect to the shifts S_f, and that the cocycle identity (3) holds for the operator

$$Bf(x) := Af(x) + \rho \left(\int_E f(z) \eta(x, x; dz) - f(x) \right),$$

where A is the mutation operator and $\eta(x, y; dz)$ the recombination kernel.

Unfortunately, this gives the cocycle identity only for Π-almost every μ. However, if A generates an irreducible Markov semigroup on E, Handa's result [4, Proposition 3.1] says that Π-almost every μ has full support on E. From here it is an easy exercise to show that Π has full support on $\mathcal{M}_1(E)$, so by continuity the cocycle identity extends to all of $\mathcal{M}_1(E)$.

In this way, Proposition 1 can be fitted in to a purely cocycle based proof of Handa's main result [4, Theorem 2.2].

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