Mathematical Reports

DÉCEMBRE / DECEMBER 2001

IN THIS ISSUE / DANS CE NUMÉRO

23

No 4

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (4), 2001 pp. 97–125

REDUCTION ALGORITHM AND REPRESENTATIONS OF BOXES AND ALGEBRAS

YURIY A. DROZD

Presented by Vlastimil Dlab, FRSC

ABSTRACT. This paper is a survey of applications of the reduction algorithm for boxes to the representation theory of finite dimensional algebras. The topic seems important in two respects. First of all, the main advantage of the notion of box is just the possibility to study representations inductively, reducing the corresponding matrices step by step. Second, there are several principal facts in the representation theory that cannot be proved (at least have never been proved till now) without using representations of boxes and the reduction algorithm. I have chosen for the presentation here three main results. They are:

- tame-wild dichotomy [12], [6];
- relation between tameness and generic modules [7];
- coverings of tame boxes and algebras [14].

Since there is a certain prejudice to the notion of box and especially to the reduction algorithm, I have decided to give some technical details of the main constructions and to sketch proofs. I hope that they are not so complicated and understandable well enough, and the astonishing resemblance of these proofs is itself a good publicity for the techniques of boxes.

1. Categories and functors. In this article we consider *linear categories* (in particular, algebras, which we identify with the categories with one object) over a fixed field $\mathbf{k}$.[1] It means that the sets of morphisms $\mathsf{A}(A, B)$ between two objects of such a category A are vector spaces over $\mathbf{k}$ and the multiplication of morphisms is $\mathbf{k}$-bilinear. All functors (bifunctors) between such categories are also supposed $\mathbf{k}$-linear (bilinear). We denote by Vec (vec) the category of vector spaces (respectively finite dimensional vector spaces) over $\mathbf{k}$. A *module* over a category A is, by definition, a (linear) functor $M\colon \mathsf{A} \to \mathsf{Vec}$; an A-B-*bimodule* is, by definition, a (bilinear) functor $\mathsf{A}^{\circ} \times \mathsf{B} \to \mathsf{Vec}$, where A° denotes the *opposite* (or *dual*) category to A. We often say A-*bimodule* instead of A-A-bimodule. In particular, any A-module can be considered as an A-$\mathbf{k}$-bimodule. We write dim, Hom, $\otimes$, *etc.*, instead of $\dim_{\mathbf{k}}$, $\mathrm{Hom}_{\mathbf{k}}$, $\otimes_{\mathbf{k}}$, *etc.*, and denote by V the *dual vector space* $\mathrm{Hom}(V, \mathbf{k})$. If V is an A-B-bimodule, we write bva instead of $V(a, b)v$ for $v \in V(A, B)$, $a \in \mathsf{A}(A', A)$, $b \in \mathsf{B}(B, B')$ (it is an element from $V(A', B')$). Every

Received by the editors August 27, 2001.
Partially supported by the CRDF Grant UM2-2094.
AMS subject classification: Primary 16G10, 16G60; secondary 16G20,15A21.

[1] Further we shall mostly suppose that $\mathbf{k}$ is algebraically closed, but it is not essential for the first definitions.

category A (rather its set of morphisms) can be considered as an A-bimodule, which we call the *regular* A*-bimodule.*

An additive category A is said to be *fully additive* if every idempotent in it splits, *i.e.*, corresponds to a decomposition of the object into a direct sum (equivalently, every idempotent has a kernel). For any category A, there is a unique (up to equivalence) fully additive category $\operatorname{add}\mathsf{A}$ containing A. It can be defined either as the category of matrix idempotents over A or as the category of *finitely generated projective* A°-modules. Every functor $F\colon \mathsf{A} \to \mathsf{B}$ prolongs uniquely (up to isomorphism) to a functor $\operatorname{add}\mathsf{A} \to \operatorname{add}\mathsf{B}$, which we denote by the same letter F. In particular, the categories of A-modules and $\operatorname{add}\mathsf{A}$-modules are equivalent.

Just as for usual bimodules over rings, one can define operations such as Hom or $\otimes$. Formally, if M is an A-B-bimodule and N is an C-A-bimodule, we define their tensor product $M\otimes_{\mathsf{A}}N$ as the C-B-bimodule such that $(M\otimes_{\mathsf{A}}N)(C,B)$ is the factor space of the direct sum $\bigoplus_{A\in\operatorname{Ob}\mathsf{A}} M(A,B)\otimes N(C,A)$ modulo the subspace generated by all differences $ua\otimes v - u\otimes av$, where $u\in M(A,B)$, $v\in N(C,A')$ and $a\in\mathsf{A}(A',A)$ for some objects $A,A'\in\operatorname{Ob}\mathsf{A}$. On the other hand, for an A-B-bimodule M and an A-C-bimodule N, the B-C-bimodule $\operatorname{Hom}_{\mathsf{A}}(M,N)$ has the values $\big(\operatorname{Hom}_{\mathsf{A}}(M,N)\big)(B,C) = \operatorname{Hom}_{\mathsf{A}}\big(M(\_,B),N(\_,C)\big)$, the right side being the space of morphisms of functors $\mathsf{A}\to\mathsf{Vec}$. One can easily check that the usual identities (*cf.* [5, Chapter IX, Section 2]) for $\otimes$ and Hom hold, especially:

$$L\otimes_{\mathsf{B}}(M\otimes_{\mathsf{A}}N)\simeq(L\otimes_{\mathsf{B}}M)\otimes_{\mathsf{A}}N,\quad \text{where } {}_{\mathsf{D}}L_{\mathsf{B}},\ {}_{\mathsf{B}}M_{\mathsf{A}},\ {}_{\mathsf{A}}N_{\mathsf{C}};$$
$$\operatorname{Hom}_{\mathsf{B}}(M\otimes_{\mathsf{A}}N,L)\simeq\Big(\operatorname{Hom}_{\mathsf{A}}\big(M,\operatorname{Hom}_{\mathsf{B}}(N,L)\big)\Big),\quad \text{where } {}_{\mathsf{D}}L_{\mathsf{B}},\ {}_{\mathsf{C}}M_{\mathsf{A}},\ {}_{\mathsf{A}}N_{\mathsf{B}}$$

(both are isomorphisms of C-D-bimodules). We shall freely use these isomorphisms as well as the analogous ones established for bimodules over rings in [5], [19].

If $F\colon\mathsf{A}\to\mathsf{B}$ is a functor and V is a B-C-bimodule (or a C-B-bimodule), one can define the A-C-bimodule V^F such that $V^F(A,C)=V(FA,C)$ (respectively the C-A-bimodule ${}^FV$ such that ${}^FV(C,A)=V(C,FA)$). We often omit the superscript ^{F} if the sense of the notation is quite clear. Especially one can consider the A-B-bimodule B^F, or the B-A-bimodule ${}^F\mathsf{B}$, or the A-bimodule ${}^F\mathsf{B}^F$. Certainly, if $M\colon\mathsf{B}\to\mathsf{Vec}$ is a B-module, the A-module ${}^FM$ is just the composition MF. It is easy to see that $V^F\simeq\operatorname{Hom}_{\mathsf{B}}({}^F\mathsf{B},V)\simeq V\otimes_{\mathsf{B}}\mathsf{B}^F$ for every B-C-bimodule V (respectively ${}^FV\simeq\operatorname{Hom}_{\mathsf{B}}(\mathsf{B}^F,V)\simeq{}^F\mathsf{B}\otimes_{\mathsf{B}}V$ for every C-B-bimodule V). Therefore, in particular,

$$\operatorname{Hom}_{\mathsf{A}\text{-}\mathsf{C}}(W_1,V^F)\simeq\operatorname{Hom}_{\mathsf{B}\text{-}\mathsf{C}}(W_1\otimes_{\mathsf{A}}{}^F\mathsf{B},V),$$
$$\operatorname{Hom}_{\mathsf{C}\text{-}\mathsf{A}}(W_2,{}^FV)\simeq\operatorname{Hom}_{\mathsf{C}\text{-}\mathsf{B}}(\mathsf{B}^F\otimes_{\mathsf{A}}W_2,V),$$
$$\operatorname{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(W,{}^FV^F)\simeq\operatorname{Hom}_{\mathsf{B}\text{-}\mathsf{B}}(\mathsf{B}^F\otimes_{\mathsf{A}}W\otimes_{\mathsf{A}}{}^F\mathsf{B},V),$$

where W_1 (respectively W_2 and W) is a A-C-bimodule (respectively C-A-bimodule and A-bimodule).

Let Γ be an oriented graph (or a *quiver*), perhaps with multiple arrows and loops. Remind that the *(linear) category* $\mathbf{k}\Gamma$ *freely generated* by Γ is defined as follows:

- The objects of $\mathbf{k}\Gamma$ are the vertices of the graph Γ.
- The vector space of morphisms from a vertex A to another vertex B has a basis consisting of all *paths* starting from A and ending at B, that is words $p = a_n \cdots a_2 a_1$, where a_i are arrows of the graph Γ, the source of a_{i+1} coincide with the target of a_i for $i = 1, \dots, n-1$, the source of a_1 is A and the target of a_n is B. We write $p: A \to B$. If $A = B$, we allow an "empty" path ι_A (*i.e.*, with $n = 0$) starting and ending at A.
- The product of two paths $p = a_n \cdots a_2 a_1 : A \to B$ and $q = b_m \cdots b_2 b_1 : C \to A$ is defined as their concatenation $pq = a_n \cdots a_1 b_m \cdots b_1$. Certainly, if $q = \iota_A$ (or $p = \iota_A$), one has $pq = p$ (respectively $pq = q$). The products of any morphisms are defined by linearity.

A category A is called *free* if it is isomorphic (not simply equivalent!) to a category of the form $\mathbf{k}\Gamma$ for some graph Γ. The images of the arrows of Γ under an isomorphism $\mathbf{k}\Gamma \to$ A are called *a set of free generators* of the category A. Just as for free algebras, one can check that $\mathbf{k}\Gamma \simeq \mathbf{k}\Gamma'$ if and only if $\Gamma \simeq \Gamma'$, hence, there is a one-to-one correspondence between isomorphism classes of graphs and of free categories. On the other hand, there can be plenty of sets of free generators in the same free category (it is always the case if there are oriented cycles in Γ or there is an arrow $a: A \to B$ and a path $p: A \to B$ such that $p \neq a$). We denote by add Γ the fully additive category add $\mathbf{k}\Gamma$.

Especially, if the graph Γ is *trivial*, *i.e.*, has no arrows, the category $\mathbf{k}\Gamma$ is the *trivial category* whose set of objects equals the set of vertices of the graph Γ. It means that there are no morphisms between different objects and the endomorphism ring of every object coincides with $\mathbf{k}$.

We shall also use *semi-free* categories defined as follows. Let Γ be an oriented graph, $\mathfrak{S}$ be the set of loops from Γ and $g: a \mapsto g_a$ be a mapping $\mathfrak{S} \to \mathbf{k}[t]$ such that neither of g_a is zero. The category $\mathsf{A} = \mathbf{k}\Gamma[g_a(a)^{-1} \mid a \in \mathfrak{S}]$ is called a *semi-free category*, the arrows of Γ are called a set of *semi-free generators* of A. Evidently, we can (and shall always) suppose that all polynomial g_a are *unital* (with the leading coefficient 1). The polynomial g_a is called the *marking polynomial* of the loop a. The set of arrows of Γ is called a set of *semi-free generators* of A. If $g_a \neq 1$ the loop a is called a *marked loop* of A. Especially, if Γ only contain loops and there is at most one loop $a: A \to A$ for every object A, the corresponding semifree category is called a *minimal category*.

A *free module* over a category A is, by definition, a module isomorphic to a direct sum of *representable* (or *principal*) modules, *i.e.*, those of the form $\mathsf{A}(A, \_)$. If $M \simeq \bigoplus_i \mathsf{A}(A_i, \_)$, the images in M of the identity morphisms 1_{A_i} are called a *set of free generators* of M. Just in the same way, a *free* A-B-*bimodule* is a bimodule V isomorphic to a direct sum $\bigoplus_i \mathsf{B}(B_i, \_) \otimes \mathsf{A}(\_, A_i)$ and the images in V of the elements $1_{B_i} \otimes 1_{A_i}$ are called a *set of free generators* of V.

A category A is said to be *skeletal* if:

- there are no nontrivial idempotents in $\mathsf{A}(A, A)$ for each $A \in \operatorname{Ob}\mathsf{A}$;
- each object from $\operatorname{add}\mathsf{A}$ decomposes into a direct sum of objects from A;
- if $\bigoplus_{i=1}^n A_i \simeq \bigoplus_{j=1}^m B_j$ in $\operatorname{add}\mathsf{A}$, where $A_i, B_j \in \operatorname{Ob}\mathsf{A}$, then $n = m$ and there is a permutation σ such that $A_i \simeq B_{\sigma i}$ for all $i = 1, \dots, n$.

For instance, if the category A is *local*, *i.e.*, all algebras $\mathsf{A}(A, A)$ are local, and has no isomorphic objects, it is skeletal (*cf.* [1, Theorem 3.6]). Each semi-free category is skeletal too (in a bit different setting it is proved in [20]).

2. **Boxes and their representations.** A *coalgebra over a category* A is defined as an A-bimodule V together with homomorphisms of A-bimodules $\Delta: \mathsf{V} \to \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V}$ (*comultiplication*) and $\varepsilon: \mathsf{V} \to \mathsf{A}$ (*counit*) such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{\Delta} & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \\ \Delta \downarrow & & \downarrow \Delta \otimes 1V \\ \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} & \xrightarrow[1 \otimes \Delta]{} & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \end{array}$$

$$\begin{array}{ccc} \mathsf{V} & \xrightarrow{\sim} & \mathsf{A} \otimes_{\mathsf{A}} \mathsf{V} \\ \Delta \downarrow & & \downarrow 1 \\ \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} & \xrightarrow[\varepsilon \otimes 1]{} & \mathsf{A} \otimes_{\mathsf{A}} \mathsf{V} \end{array} \qquad \begin{array}{ccc} \mathsf{V} & \xrightarrow{\sim} & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{A} \\ \Delta \downarrow & & \downarrow 1 \\ \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} & \xrightarrow[1 \otimes \varepsilon]{} & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{A} \end{array}$$

(the first rows of the last two diagrams are the natural isomorphisms).

A *box* is defined as a pair $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$, where A is a category and V is an A-coalgebra. The kernel $\overline{\mathsf{V}} = \operatorname{Ker}\varepsilon$ of the counit is called the *kernel* of the box $\mathfrak{A}$. If $v \in \mathsf{V}(A, B)$, we often write $v: A \cdots\!> B$. If C is a category, we define the *category of representations* $\operatorname{Rep}(\mathfrak{A}, \mathsf{C})$ of the box $\mathfrak{A}$ in the category C in the following way:

- The *objects* of this category are functors $M: \mathsf{A} \to \mathsf{C}$.
- A *morphism* from M to N in $\operatorname{Rep}(\mathfrak{A}, \mathsf{C})$ is a homomorphism of A-bimodules $f: \mathsf{V} \to \operatorname{Hom}_{\mathsf{C}}({}^M\mathsf{C}, {}^N\mathsf{C})$. We denote the set of all morphisms from M to N by $\operatorname{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(M, N)$.
- The *product* of two morphisms, $f \in \operatorname{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(M, N)$ and $g \in \operatorname{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(L, M)$, is defined as the composition

  $$\mathsf{V} \xrightarrow{\Delta} \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \xrightarrow{f \otimes g} \operatorname{Hom}_{\mathsf{C}}({}^M\mathsf{C}, {}^N\mathsf{C}) \otimes_{\mathsf{A}} \operatorname{Hom}_{\mathsf{C}}({}^L\mathsf{C}, {}^M\mathsf{C}) \xrightarrow{\text{mult}} \operatorname{Hom}_{\mathsf{C}}({}^L\mathsf{C}, {}^N\mathsf{C}),$$

  where mult denotes the multiplication of morphisms of functors.
- The *identity morphism* of a representation M is defined as the composition $\mathsf{V} \xrightarrow{\varepsilon} \mathsf{A} \to \operatorname{Hom}_{\mathsf{C}}({}^M\mathsf{C}, {}^M\mathsf{C})$, where the second homomorphism maps $a: A \to B$ to $\mathsf{C}(\_, M(a)): \mathsf{C}(\_, MA) \to \mathsf{C}(\_, NB)$.

One can easily check that in this way we obtain indeed a category. If $\mathsf{C} = \mathsf{Proj}\text{-}\mathbf{R}$, the category of right projective modules over an algebra $\mathbf{R}$, we write $\mathsf{Rep}(\mathfrak{A}, \mathbf{R})$ instead of $\mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ and $\mathrm{Hom}_{\mathbf{R}\text{-}\mathfrak{A}}(M, N)$ instead of $\mathrm{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(M, N)$. If $\mathbf{R} = \mathbf{k}$, we omit all and write $\mathsf{Rep}(\mathfrak{A})$ and $\mathrm{Hom}_{\mathfrak{A}}(M, N)$.

Sometimes it is convenient to identify $\mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(\mathsf{V}, \mathrm{Hom}_{\mathsf{C}}({}^{M}\mathsf{C}, {}^{N}\mathsf{C}))$ with $\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{A}}(\mathsf{V} \otimes_{\mathsf{A}} {}^{M}\mathsf{C}, {}^{N}\mathsf{C})$ and we shall do it freely.

A *principal box* is one of the form $\mathfrak{A} = (\mathsf{A}, \mathsf{A})$, where the coalgebra is the regular bimodule with identity comultiplication and counit. It is easy to see that the category of representations of this box coincide with that of the category A; in particular, $\mathsf{Rep}(\mathfrak{A}) = \mathsf{A}\text{-}\mathsf{Mod}$. We always identify a principal box with the corresponding category; it allows to consider (formally) the representation theory of algebras as a partial case of that of boxes.

A *morphism of boxes* $\Phi: \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ and $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$, is a pair (Φ_0, Φ_1), where $\Phi_0: \mathsf{A} \to \mathsf{B}$ is a functor and $\Phi_1: \mathsf{V} \to {}^{\Phi_0}\mathsf{W}^{\Phi_0}$ is a morphism of A-bimodules compatible in the evident sense with comultiplication and counit. We usually omit indices and write $\Phi(a)$ both for $a \in \mathsf{A}$ and for $a \in \mathsf{V}$. Such a morphism induces the *inverse image* functor $\Phi^*: \mathsf{Rep}(\mathfrak{B}, \mathsf{C}) \to \mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ for each category C: it maps a representation M to the composition $M\Phi_0$ and a morphism $f \in \mathrm{Hom}_{\mathsf{C}\text{-}\mathfrak{B}}(M, N)$, *i.e.*, a homomorphism of bimodules $\mathsf{W} \to \mathrm{Hom}_{\mathsf{C}}({}^{M}\mathsf{C}, {}^{N}\mathsf{C})$, to the morphism $M\Phi_0 \to N\Phi_0$, *i.e.*, the homomorphism $\Phi^* f: \mathsf{V} \to \mathrm{Hom}_{\mathsf{C}}({}^{M\Phi_0}\mathsf{C}, {}^{N\Phi_0}\mathsf{C})$, such that $\Phi^* f(v) = f(\Phi_1 v)$.

Suppose given a box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ and a functor $F: \mathsf{A} \to \mathsf{B}$. We define the new box $\mathfrak{A}^F = (\mathsf{B}, \mathsf{W})$ in the following way:

- $\mathsf{W} = \mathsf{B}^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^{F}\mathsf{B}$.
- The comultiplication $\mathsf{W} \to \mathsf{W} \otimes_{\mathsf{B}} \mathsf{W} \simeq \mathsf{B}^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^{F}\mathsf{B}^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^{F}\mathsf{B}$ maps $a \otimes v \otimes b$ to $\sum_i a \otimes v_i^{(1)} \otimes 1 \otimes v_i^{(2)} \otimes b$, where $\Delta(v) = \sum_i v_i^{(1)} \otimes v_i^{(2)}$.
- The counit $\mathsf{W} \to \mathsf{B}$ maps $a \otimes v \otimes b$ to $aF(\varepsilon(v))b$.

The functor F can be prolonged to the morphism $\mathfrak{A} \to \mathfrak{A}^F$, which we denote by F too, setting, for $v \in \mathsf{V}(A, B)$, $F(v) = F(1_B) \otimes v \otimes F(1_A) \in \mathsf{W}(FA, FB)$.

THEOREM 2.1. *Let* $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ *be a box,* $F: \mathsf{A} \to \mathsf{B}$ *be a functor and* $\mathfrak{A}^F$ *be the above defined box. The inverse image functor* F^* *corresponding to the morphism of boxes* $F: \mathfrak{A} \to \mathfrak{A}^F$ *induces an equivalence of the category* $\mathsf{Rep}(\mathfrak{A}^F, \mathsf{C})$ *onto the full subcategory* $\mathsf{Rep}(\mathfrak{A}, \mathsf{C} \mid F) \subseteq \mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ *consisting of all representations that are isomorphic to the composition* MF *for some functor* $M: \mathsf{B} \to \mathsf{C}$. *In particular, if every representation is isomorphic to such a composition, the functor* F^* *establishes an equivalence between* $\mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ *and* $\mathsf{Rep}(\mathfrak{A}^F, \mathsf{C})$.

PROOF. It follows immediately from the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(\mathsf{V}, \mathrm{Hom}_{\mathsf{C}}({}^{MF}\mathsf{C}, {}^{NF}\mathsf{C})) &\simeq \mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(\mathsf{V}, {}^{F}\mathrm{Hom}_{\mathsf{C}}({}^{M}\mathsf{C}, {}^{N}\mathsf{C})^{F}) \\ &\simeq \mathrm{Hom}_{\mathsf{B}\text{-}\mathsf{B}}(\mathsf{B}^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^{F}\mathsf{B}, \mathrm{Hom}_{\mathsf{C}}({}^{M}\mathsf{C}, {}^{N}\mathsf{C})). \blacksquare\end{aligned}$$

We shall often use the following corollary of this theorem.

COROLLARY 2.2. *Let* $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ *be a box,* A' *be a subcategory of* A *and* $F'\colon \mathsf{A}' \to \mathsf{B}'$ *be a functor. Denote by* $\mathsf{B} = \mathsf{A} \coprod^{\mathsf{A}'} \mathsf{B}'$ *the amalgamation (or pullback) of* A *and* B' *under* A' *and by* $F\colon \mathsf{A} \to \mathsf{B}$ *the natural functor. Then* F^* *induces an equivalence between* $\mathsf{Rep}(\mathfrak{A}^F, \mathsf{C})$ *and the full subcategory* $\mathsf{Rep}(\mathfrak{A}, \mathsf{C} \mid \mathsf{A}', F') \subseteq \mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ *consisting of all representations* M *such that the restriction of* M *onto* A' *can be factored through* F'.

For every box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ we denote by $\mathsf{add}\,\mathfrak{A}$ the box $(\mathsf{add}\,\mathsf{A}, \mathsf{V})$ (we denote by the same letter V the prolongation of V onto $\mathsf{add}\,\mathsf{A}$). For every fully additive category C (*e.g.* for Vec) there is an equivalence $\mathsf{Rep}(\mathfrak{A}, \mathsf{C}) \simeq \mathsf{Rep}(\mathsf{add}\,\mathfrak{A}, \mathsf{C})$ and we shall identify these categories.

A box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is called *skeletal* if so is the category A. Then a representation $M \in \mathsf{Rep}(\mathfrak{A}, \mathbf{R})$, where $\mathbf{R}$ is an algebra, is said to be *finite* (or *of finite rank*) if:

- For any object $A \in \mathrm{Ob}\,\mathsf{A}$, $MA \in \mathsf{proj}\text{-}\mathbf{R}$, the category of finitely generated projective (right) $\mathbf{R}$-modules.
- The *support* of M, *i.e.*, the set $\mathrm{supp}\,M = \{A \in \mathrm{Ob}\,\mathsf{A} \mid MA \neq 0\}$, is finite.

If $\mathbf{R} = \mathbf{k}$ (hence, $\mathsf{proj}\text{-}\mathbf{R} = \mathsf{vec}$), they also call finite representations *finite dimensional.* The category of all finite representations of $\mathfrak{A}$ over $\mathbf{R}$ is denoted by $\mathsf{rep}(\mathfrak{A}, \mathbf{R})$ ($\mathsf{rep}(\mathfrak{A})$ if $\mathbf{R} = \mathbf{k}$). Let $|\mathsf{proj}\text{-}\mathbf{R}|$ be the set of isomorphism classes of finitely generated projective $\mathbf{R}$-modules. The function $\mathbf{dim}\,M\colon \mathrm{Ob}\,\mathsf{A} \to |\mathsf{proj}\text{-}\mathbf{R}|$ mapping A to the isomorphism class of MA is called the *vector dimension* of M. If all projective $\mathbf{R}$-modules are free of unique rank (*e.g.* if $\mathbf{R} = \mathbf{k}$), we identify $|\mathsf{proj}\text{-}\mathbf{R}|$ with $\mathbb{N}$, the set of nonnegative integers. If, moreover, the set $\mathrm{Ob}\,\mathsf{A}$ is finite, we consider $\mathbf{dim}\,M$ just as a vector with entries from $\mathbb{N}$. We denote by $\mathrm{ind}(\mathfrak{A}, \mathbf{R})$ the set of isomorphism classes of indecomposable finite representations of $\mathfrak{A}$ over $\mathbf{R}$ and by $\mathrm{ind}_{\mathbf{d}}(\mathfrak{A}, \mathbf{R})$ the subset of $\mathrm{ind}(\mathfrak{A}, \mathbf{R})$ consisting of the classes of representations of vector dimension $\mathbf{d}$. Note that there are boxes such that $\mathrm{ind}_{\mathbf{d}}(\mathfrak{A}) \cap \mathrm{ind}_{\mathbf{d}'}(\mathfrak{A}) \neq \emptyset$ for some vector dimensions $\mathbf{d} \neq \mathbf{d}'$.

If $\mathbf{d}$, $\mathbf{c}$ are two vector dimensions, we write $\mathbf{d} \leq \mathbf{c}$ if $\mathbf{d}(A) \leq \mathbf{c}(A)$ for all objects A.

3. Types of boxes. In the representation theory (especially over algebraically closed fields), as well as in most other applications, they mainly use the so-called *normal free boxes* in the following sense.

A *section* of a box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is, by definition, a set of elements $\omega = \{\omega_A \in \mathsf{V}(A, A) \mid A \in \mathrm{Ob}\,\mathsf{A}\}$ such that $\varepsilon(\omega_A) = 1_A$. This section is said to be *normal* if $\Delta\omega_A = \omega_A \otimes \omega_A$ for all objects A. A box is called *normal* if it has a normal section. Evidently, the element $\partial a = \omega_B a - a\omega_A$ belongs to the kernel $\overline{\mathsf{V}}$ of the box $\mathfrak{A}$. Moreover, if $v \in \overline{\mathsf{V}}(A, B)$, the element $\partial v = \mu(v) - v \otimes \omega_A - \omega_B \otimes v$ belongs to $\overline{\mathsf{V}} \otimes_{\mathsf{A}} \overline{\mathsf{V}}$. We call ∂ the *differential* of the (normal) box $\mathfrak{A}$. Note that it depends on the section. We prolong ∂ to the tensor square $\overline{\mathsf{V}}^{\otimes 2} = \overline{\mathsf{V}} \otimes_{\mathsf{A}} \overline{\mathsf{V}}$ setting

$\partial(u \otimes v) = \partial u \otimes v - u \otimes \partial v \in \overline{\mathsf{V}}^{\otimes 3}$. We often omit the sign $\otimes$ and set $\overline{a} = 0$ for $a \in \operatorname{Mor} \mathsf{A}$, $\overline{v} = 1$ for $v \in \mathsf{V}$. Then the mapping ∂ has the following properties:

- $\partial(xy) = (\partial x)y + (-1)^{\overline{x}}x(\partial y)$ (Leibniz rule);
- $\partial^2 = 0$.

Note that if φ is an isomorphism of representations of a normal box, then $\varphi(\omega_A)$ is an isomorphism for each object A. Especially, the *vector dimensions* of isomorphic finite representations coincide.

A normal box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is called *free* (*semi-free*) if A is a free (semi-free) category, $\partial a = 0$ for each marked loop of A and the kernel $\overline{\mathsf{V}}$ is a free A-bimodule. If Σ_0 is a set of free (semi-free) generators of the category A and Σ_1 is a set of free generators of the A-bimodule $\overline{\mathsf{V}}$, their union $\Sigma = \Sigma_0 \cup \Sigma_1$ is called a set of free (semi-free) generators of the box $\mathfrak{A}$. The elements of Σ_0 are usually called *solid arrows* and those of Σ_1 *dotted arrows* of the box $\mathfrak{A}$. Thus, to a semi-free box we associate a *bigraph*, *i.e.*, a graph whose arrows are of two types: solid and dotted. The *marked loops* and the *marking polynomials* of a semi-free box $\mathfrak{A}$ are just those of the semi-free category A. The morphisms from $\mathsf{A}(A, B)$, or the elements from $\overline{\mathsf{V}}(A, B)$, or from $\overline{\mathsf{V}}^{\otimes 2}$ can be considered as linear combinations of paths of the arrows of the corresponding bigraph and the inverse morphisms $a^* = g_a(a)^{-1}$ for the marked loops a such that all arrows of the paths are solid, respectively, each path contains exactly one, or exactly two dotted arrows.

A semi-free box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is called *so-trivial* if the category A is trivial. If A is a minimal category, we call the box $\mathfrak{A}$ *so-minimal*.

We fix a section $\omega\colon \mathsf{V} \to \mathsf{A}$ and consider the differential ∂ with respect to this section. A semi-free box $\mathfrak{A}$ is called *triangular* if there is a set of semi-free generators Σ and a function $\nu\colon \Sigma \to \mathbb{N}$ such that, for every arrow $a \in \Sigma$, its differential ∂a is a linear combination of paths only containing the arrows b with $\nu(b) < \nu(a)$. Such a set of generators is also called *triangular*. Triangular semi-free boxes have a lot of good features that are not valid in general. Especially, the following important results hold.

PROPOSITION 3.1 (*cf.* [18]).[2] *Let $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a triangular semi-free box with normal section ω and a triangular set of semi-free generators Σ.*

1. *A morphism $f\colon M \to N$ of representations from* $\operatorname{Rep}(\mathfrak{A}, \mathsf{C})$ *is an isomorphism if and only if $f(\omega_A)$ is an isomorphism for every object A.*
2. *If the category C is fully additive, so is* $\operatorname{Rep}(\mathfrak{A}, \mathsf{C})$.
3. *Suppose that $M \in \operatorname{Rep}(\mathfrak{A}, \mathsf{C})$, $\{N_A \mid A \in \operatorname{Ob} \mathsf{A}\}$ is a set of objects from C, for each in $\operatorname{Ob} \mathsf{A}$ an isomorphism $\gamma_A\colon MA \to N_A$ and for each dotted arrow $v\colon A \cdots\!> B$ from Σ_1 a morphism $\gamma_v\colon MA \to N_B$ are given. There is a representation $N \in \operatorname{Rep}(\mathsf{A}, \mathsf{C})$ and an isomorphism $\gamma\colon M \to N$ such that*

[2] In [18] these properties are proved for *free differential graded categories*, but it is well known that this setting is quite equivalent to that of free boxes. Moreover, the proofs for semi-free boxes are the same as for free ones.

$NA = N_A$, $\gamma(\omega_A) = \gamma_A$ for each object A and $\gamma(v) = \gamma_v$ for each dotted arrow v.

Usually we impose additional conditions on the considered boxes. Namely, we say that a box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is *locally finitely generated* if for every object $A \in \operatorname{Ob}\mathsf{A}$ the A-modules $\mathsf{A}(A, \_)$, $\mathsf{V}(A, \_)$ as well as A°-modules $\mathsf{A}(\_, A)$, $\mathsf{V}(\_, A)$ are finitely generated. If the box $\mathfrak{A}$ is semi-free, it means that in the corresponding bigraph there are finitely many arrows ending or starting at each vertex. Denote by s^0_{AB} and s^1_{AB} respectively the number of solid and dotted arrows starting at A and ending at B and set, for every function $\mathbf{d}\colon \operatorname{Ob}\mathsf{A} \to \mathbb{R}$ with finite support $\operatorname{supp}\mathbf{d} = \{A \mid \mathbf{d}(A) \neq 0\}$,

$$Q^+_{\mathfrak{A}}(\mathbf{d}) = \sum_{A \in \operatorname{Ob}\mathsf{A}} \mathbf{d}(A)^2 + \sum_{A,B \in \operatorname{Ob}\mathsf{A}} s^1_{AB}\mathbf{d}(A)\mathbf{d}(B),$$

$$Q^-_{\mathfrak{A}}(\mathbf{d}) = \sum_{A,B \in \operatorname{Ob}\mathsf{A}} s^0_{AB}\mathbf{d}(A)\mathbf{d}(B),$$

$$Q_{\mathfrak{A}}(\mathbf{d}) = Q^+_{\mathfrak{A}}(\mathbf{d}) - Q^-_{\mathfrak{A}}(\mathbf{d}).$$

They call $Q_{\mathfrak{A}}$ the *Tits form* of the box $\mathfrak{A}$.

COROLLARY 3.2. *Let $\mathfrak{A}$ be a locally finitely generated semi-free box. The set of representation $M \in \operatorname{rep}(\mathfrak{A})$ of vector dimension $\mathbf{d}$ can be identified with the points of an affine variety $\operatorname{rep}_{\mathbf{d}}(\mathfrak{A})$ of dimension $Q^-_{\mathfrak{A}}(\mathbf{d})$ over the field $\mathbf{k}$ (actually, with a principal open subset in the affine space $\mathbb{A}_{\mathbf{k}}^{Q^-_{\mathfrak{A}}(\mathbf{d})}$). The isomorphism classes of representations are connected locally closed subsets in $\operatorname{rep}_{\mathbf{d}}(\mathfrak{A})$ of dimensions $d \leq Q^+_{\mathfrak{A}}(\mathbf{d})$.*

Actually, in most applications they only deal with free boxes. Nevertheless, semi-free ones seem unavoidable in the reduction algorithm described in Section 6, especially when we study *tame boxes.*

4. Boxes, bimodules and algebras. Let A be a category, U be an A-bimodule. Define the new category $\mathsf{El}(\mathsf{U})$ of *elements of the bimodule* U (or *matrices* over U) as follows:

- $\operatorname{Ob}\mathsf{El}(\mathsf{U}) = \bigcup_{A \in \operatorname{Ob}\operatorname{add}\mathsf{A}} \mathsf{U}(A, A)$.
- A morphism from $u \in \mathsf{U}(A, A)$ to $u' \in \mathsf{U}(A', A')$ is a morphism $a \in \mathsf{A}(A, A')$ such that $au = u'a$ (both elements are from $\mathsf{U}(A, A')$).

This category is fully additive. The *zero elements* $0 \in \mathsf{U}(A, A)$ form a fully additive subcategory $\mathsf{El}_0(\mathsf{U})$. If the category A is skeletal, each zero element decomposes uniquely into a direct sum of indecomposable zero elements, which belong to $\mathsf{U}(A, A)$, where $A \in \operatorname{Ob}\mathsf{A}$.

We call a category A *locally finite dimensional* if all morphism spaces $\mathsf{A}(A, B)$ are finite dimensional and for every object A the set $\{B \mid \mathsf{A}(A, B) \neq 0$ or $\mathsf{A}(B, A) \neq 0\}$ is finite. A skeletal locally finite dimensional category is called *basic.* For every locally finite dimensional category A there is a unique basic category A_0 such that $\operatorname{add}\mathsf{A} \simeq \operatorname{add}\mathsf{A}_0$. Therefore, in the representation theory of locally finite dimensional categories we may restrict ourselves by basic ones. A bimodule U over a category A is called *locally finite dimensional* if all spaces $\mathsf{U}(A,B)$ are finite dimensional and for every object A the set $\{B \mid \mathsf{U}(A,B) \neq 0$ or $\mathsf{U}(B,A) \neq 0\}$ is finite.

THEOREM 4.1. *Suppose that the field* $\mathbf{k}$ *is algebraically closed. Let* U *be a locally finite dimensional bimodule over a locally finite dimensional category* C. *There is a free triangular locally finitely generated box* $\mathfrak{A}$ *such that* $\mathsf{El}(\mathsf{U}) \simeq \operatorname{rep}(\mathfrak{A})$.

PROOF. Without loss of generality we suppose C basic. Then the set $\mathsf{R} = \operatorname{rad}\mathsf{C}$ of all noninvertible morphisms is an ideal of C called its *radical.* Moreover, it is easy to check that $\bigcap_{n=1}^{\infty} \mathsf{R}^n = 0$. For every nonzero morphism $c \in \mathsf{C}$ set $\nu(c) = \max\{n \mid c \in \mathsf{R}^n\}$. Define sub-bimodules U_n setting $\mathsf{U}_0 = \mathsf{U}$, $\mathsf{U}_{n+1} = \mathsf{R}\mathsf{U}_n + \mathsf{U}_n\mathsf{R}$, and set, for every nonzero $u \in \mathsf{U}$, $\nu(u) = \max\{n \mid u \in \mathsf{U}_n\}$ (again $\bigcap_{n=1}^{\infty} \mathsf{U}_n = 0$). For each two objects A, B and each $n \geq 0$ choose a basis $\mathfrak{E}_n^0(A,B)$ of $\mathsf{R}^n(A,B)$ modulo $\mathsf{R}^{n+1}(A,B)$ and a basis $\mathfrak{E}_n^1(A,B)$ of $\mathsf{U}_n(A,B)$ modulo $\mathsf{U}_{n+1}(A,B)$. Then $\mathfrak{E}^0(A,B) = \bigcup_{n=1}^{\infty} \mathfrak{E}_n^0(A,B)$ and $\mathfrak{E}^1(A,B) = \bigcup_{n=0}^{\infty} \mathfrak{E}_n^1(A,B)$ are bases respectively of $\mathsf{R}(A,B)$ and $\mathsf{U}(A,B)$. Consider the dual spaces $\mathsf{R}(A,B)$ and $\mathsf{U}(A,B)$ with bases $\mathfrak{F}^1(A,B)$ and $\mathfrak{F}^0(A,B)$ dual respectively to $\mathfrak{E}^0(A,B)$ and $\mathfrak{E}^1(A,B)$. For $f \in \mathfrak{F}^i(A,B)$ $(i = 0,1)$ set $\nu(f) = \nu(e)$, where e is the element from $\mathfrak{E}^{1-i}$ dual to f. Let $a \in \mathfrak{E}^i(A,B)$, $b \in \mathfrak{E}^j(C,A)$, where (i,j) is $(0,0)$, or $(0,1)$, or $(1,0)$, and $k = i + j$. Then the elements $\lambda(a,b,c)$ for $c \in \mathfrak{E}^k(C,B)$ are uniquely determined such that

$$ab = \sum_{c \in \mathfrak{E}^k(C,B)} \lambda(a,b,c)c. \tag{1}$$

For elements $a' \in \mathfrak{F}^{1-i}(A,B)$, $b' \in \mathfrak{F}^{1-j}(C,A)$, $c' \in \mathfrak{F}^{1-k}(C,B)$ dual respectively to a, b, c set $\lambda(a',b',c') = \lambda(a,b,c)$.

Consider the free normal box $\mathfrak{A}$ with the set of vertices $\operatorname{Ob}\mathsf{C}$, the solid (dotted) arrows from A to B being $\mathfrak{F}^0(A,B)$ (respectively $\mathfrak{F}^1(A,B)$) and the differential:

$$\partial a = \sum_C \Big(\sum_{b \in \mathfrak{F}^0(C,B), v \in \mathfrak{F}^1(A,C)} \lambda(b,v,a)bv - \sum_{b \in \mathfrak{F}^0(A,C), v \in \mathfrak{F}^1(C,B)} \lambda(v,b,a)vb \Big)$$
$$\text{for } a \in \mathfrak{F}^0(A,B),$$

$$\partial v = \sum_C \sum_{u \in \mathfrak{F}^1(C,B), w \in \mathfrak{F}^1(A,C)} \lambda(u,w,v)u \otimes w \quad \text{for } v \in \mathfrak{F}^1(A,B).$$

One can easily check that $\mathfrak{A}$ is triangular with respect to the function ν defined above, representations of $\mathfrak{A}$ are in a natural one-to-one correspondence with the objects from $\mathsf{El}(\mathsf{U})$ and their morphisms are in one-to-one correspondence with the morphisms from $\mathsf{El}(\mathsf{U})$. ∎

It is often useful to consider an A-B-bimodule U as an (A × B)-bimodule setting $\mathsf{U}((A,B),(A',B')) = \mathsf{U}(A,B')$. Then we call U a *bipartite* bimodule. In particular, every A-bimodule (*e.g.* every ideal of the category A) can be considered as a bipartite (A × A)-bimodule. Such bimodules are especially used in relation with the following result.

THEOREM 4.2. *Let* A *be a locally finite dimensional category,* R = rad A *considered as bipartite* A × A*-bimodule. There is a functor* Cok: El(R) → rep(A) *with the following properties:*

- Cok *is full and dense.*
- *The set* $\operatorname{Ker}\operatorname{Cok} = \{u \in \operatorname{ind}(\mathsf{R}) \mid \operatorname{Cok} u = 0\}$ *only consists of some zero elements.*
- *The restriction of* Cok *onto the full subcategory* El*(R) *consisting of the objects that have no direct summands from* Ker Cok *maps nonisomorphic objects to nonisomorphic ones.*

They often say that the restriction of Cok *onto* El*(R) *is a* representation equivalence.

PROOF. We suppose the category A basic and identify add A with the category dual to that of finitely generated left projective A-modules. Then $\mathsf{R}(P^\circ, Q^\circ)$ can be identified with $\operatorname{Hom}_{\mathsf{A}}(Q, PR)$. For every finite A-module M there is a *minimal projective presentation, i.e.*, a short exact sequence $Q \xrightarrow{\varphi} P \longrightarrow M \longrightarrow 0$ with $\operatorname{Im}\varphi \subseteq \mathsf{R}P$, $\operatorname{Ker}\varphi \subseteq \mathsf{R}Q$. We can consider φ as an element from $\mathsf{R}(P^\circ, Q^\circ)$. Moreover, any two minimal projective presentations give isomorphic elements from El(R). Conversely, if $\varphi \in \mathsf{R}(P^\circ, Q^\circ)$, we can consider it as a homomorphism $Q \to \mathsf{R}P$; thus, setting $\operatorname{Cok}\varphi = \operatorname{Coker}\varphi$, we get a full and dense functor El(R) → rep(A). Note that if the condition $\operatorname{Ker}\varphi \subseteq \mathsf{R}Q$ does not hold, one can decompose $Q = Q_0 \oplus Q_1$ so that $Q_0 \subseteq \operatorname{Ker}\varphi$ and $\operatorname{Ker}\varphi \cap Q_1 \subseteq \mathsf{R}Q_1$. Therefore, as an element from El(R), φ decomposes as $\varphi_0 \oplus \varphi_1$, where φ_1 arises from a minimal projective presentation of $\operatorname{Cok}\varphi$ while φ_0 is a zero morphism $Q_0 \to 0$. ∎

We denote by $\mathfrak{R}_{\mathsf{A}}$ the box corresponding to the bipartite A × A-bimodule R via Theorem 4.1 and by the same symbol Cok the functor $\operatorname{rep}(\mathfrak{R}_{\mathsf{A}}) \to \operatorname{rep}(\mathsf{A})$ that is the composition of the equivalence $\operatorname{rep}(\mathfrak{R}_{\mathsf{A}}) \simeq \mathsf{El}(\mathsf{R})$ and the functor Cok from Theorem 4.2. We also denote by $\operatorname{rep}^*(\mathfrak{R}_{\mathsf{A}})$ the image in $\operatorname{rep}(\mathfrak{R}_{\mathsf{A}})$ of El*(R); thus, the restriction of Cok onto $\operatorname{rep}^*(\mathfrak{R}_{\mathsf{A}})$ is a representation equivalence. Note that El*(R) consists of all representations that have no zero direct summands from $\mathsf{R}(A, 0)$ ($A \in \operatorname{Ob}\mathsf{A}$).

5. Representation types. From now on *we suppose the field* **k** *algebraically closed,* though in the definition of representation finite type it is not necessary and in the definition of representation discrete type we only need that **k** be infinite. Moreover, we suppose that *all boxes are locally finitely generated.*

Let $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a skeletal box. We say that it is *representation (locally) finite* if there is a set $\mathfrak{M} \subseteq \operatorname{rep}(\mathfrak{A})$ of its indecomposable representations such that $\operatorname{add}\mathfrak{M} = \mathsf{rep}(\mathfrak{A})$ and for every object $A \in \operatorname{Ob}\mathsf{A}$ the set $\mathfrak{M}_A = \{M \in \mathfrak{M} \mid MA \neq 0\}$ is finite. If $\mathfrak{A}$ is finitely generated, it just means that the set $\mathfrak{M}$ is finite. (We usually omit the word "locally" and say that $\mathfrak{A}$ is representation finite.)

We say that $\mathfrak{A}$ is *representation discrete* if there is a set $\mathfrak{M} \subseteq \mathsf{rep}(\mathfrak{A})$ such that $\operatorname{add}\mathfrak{M} = \mathsf{rep}(\mathfrak{A})$ and for each vector dimension $\mathbf{d}$ the set $\{M \in \mathfrak{M} \mid \mathbf{dim}\, M = \mathbf{d}\}$ is finite.

Note that if the category $\mathsf{rep}(\mathfrak{A})$ is fully additive (hence, Krull-Schmidt), one can always take for $\mathfrak{M}$ the set $\operatorname{ind}(\mathfrak{A})$ of all nonisomorphic indecomposable representations.

A deep and difficult theorem proved in [2], [3] claims that for finite dimensional algebras over an algebraically closed (hence, over an infinite perfect) field *representation discrete* implies *representation finite* (it had been known before as the Second Brauer-Thrall conjecture). On the other hand, the free category defined by the graph

$$\cdots \longrightarrow A_{-1} \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

is representation discrete but not finite. A problem remains whether a *finite* free box (*i.e.*, with finite bigraph) is representation discrete if and only if it is representation finite.

The following result follows evidently from Corollary 3.2.

COROLLARY 5.1. *If a free box $\mathfrak{A}$ is representation discrete, its Tits form $Q_{\mathfrak{A}}$ is* weakly positive, *i.e., $Q_{\mathfrak{A}}(\mathbf{d}) > 0$ for each nonzero vector $\mathbf{d}$ with nonnegative entries.*

A representation $M \in \mathsf{rep}(\mathfrak{A}, \mathbf{R})$ is said to be *strict* if, for any finite dimensional representations N, N' of the algebra $\mathbf{R}$,

- if $M \otimes_{\mathbf{R}} N \simeq M \otimes_{\mathbf{R}} N'$, then $N \simeq N'$;
- if N is indecomposable, so is also $M \otimes_{\mathbf{R}} N$.

Figuratively, it means that the classification of representations of the box $\mathfrak{A}$ "contains" that of algebra $\mathbf{R}$. They often say that the functor $M \otimes_{\mathbf{R}} \_ \colon \mathsf{rep}(\mathbf{R}) \to \mathsf{rep}(\mathfrak{A})$ is a *representation embedding.*

EXAMPLE 5.2. Let $a \colon A \to A$ be a minimal loop of a semi-free box $\mathfrak{A}$ such that there are no marked loops $b \colon A \to A$, $b \neq a$. The following representation $J^a \in \mathsf{rep}(\mathfrak{A}, \mathbf{R})$, where $\mathbf{R} = \mathbf{k}[t, g_a(t)^{-1}]$, is strict:

$$J^a(A) = \mathbf{R}, \quad J^a(B) = 0 \quad \text{if } B \neq A,$$
$$J^a(a) = t, \quad J^a(b) = 0 \quad \text{if } b \neq a$$

(here t is identified with the multiplication by t in $\mathbf{R}$). We denote by $J^a_n(\lambda)$ the representation $J^a \otimes_{\mathbf{R}} \mathbf{R}/(t-\lambda)^n$, where $g_a(\lambda) \neq 0$. All of them are indecomposable and pairwise nonisomorphic. In particular, if there is a minimal loop in a semi-free box, it is *representation strongly infinite*, *i.e.*, the set of vector dimensions $\mathbf{d} \colon \operatorname{Ob}\mathsf{A} \to \mathbb{N}$ such that $\operatorname{ind}_{\mathbf{d}}(\mathfrak{A})$ is infinite is infinite itself.

A box $\mathfrak{A}$ is called *(representation) wild* if, for any finitely generated algebra $\mathbf{R}$, there is a strict representation $M \in \operatorname{rep}(\mathfrak{A}, \mathbf{R})$. The following easy (and well known, *cf.* [17], [10], [12]) results show that to prove the wildness it is enough to construct a strict representation over *one* test algebra.

PROPOSITION 5.3. *Suppose that an algebra $\mathbf{R}_0$ is wild and there is a strict representation of a box $\mathfrak{A}$ over $\mathbf{R}_0$. Then $\mathfrak{A}$ is also wild.*

COROLLARY 5.4. *A box $\mathfrak{A}$ is wild if and only if there is a strict representation $M \in \operatorname{rep}(\mathfrak{A}, \mathbf{R}_0)$, where $\mathbf{R}_0$ is one of the following algebras:*

$\mathbf{k}\langle x, y\rangle$, *the free algebra in 2 generators;*

$\mathbf{k}[x, y]$, *the polynomial algebra in 2 variables;*

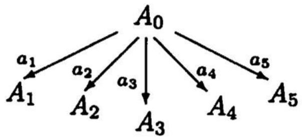
$\mathbf{k}[[x, y]]$, *the power series algebra in 2 variables;*

$$\mathbf{k}[x, y]/(x^2, y^3, xy^2);$$

$\mathbf{k}\Gamma_2$ *or* $\mathbf{k}\Gamma_2^0$, *where Γ_2 is the quiver:*

$$a \circlearrowright A \xrightarrow{b} B;$$

$\mathbf{k}\Gamma_5$ *or* $\mathbf{k}\Gamma_5^0$, *where Γ_5 is the quiver:*



A *rational algebra* is, by definition, an algebra of the form $\mathbf{R} = \mathbf{k}[t, g(t)^{-1}]$, where $g(t)$ is a nonzero polynomial. A strict representation M of a box $\mathfrak{A}$ over such an algebra is called a *rational family* of its representations. They say that the representations $M \otimes_{\mathbf{R}} L$, where $L \in \operatorname{ind}(\mathbf{R})$, *belong to the rational family* M. (Note that any indecomposable representation of a rational algebra $\mathbf{R} = \mathbf{k}[t, f(t)^{-1}]$ is of the form $J_m(\lambda) = \mathbf{R}/(t-\lambda)^m$, where $f(\lambda) \neq 0$.)

Suppose that a box $\mathfrak{A}$ is skeletal. We call it *(representation) tame* if there is a set $\mathfrak{M}$ of its representations such that:

- each $M \in \mathfrak{M}$ is a strict representation of $\mathfrak{A}$ of finite rank over a rational algebra $\mathbf{R}_M$ (it may depend on M);
- for each vector dimension $\mathbf{d}\colon \operatorname{Ob} \mathsf{A} \to \mathbb{N}$ there is only finitely many $M \in \mathfrak{M}$ with $\operatorname{\mathbf{dim}} M = \mathbf{d}$;
- for each vector dimension $\mathbf{d}$ almost all representations from $\operatorname{ind}_{\mathbf{d}} \mathfrak{A}$ (*i.e.*, all but a finite number of them) are isomorphic to $M \otimes_{\mathbf{R}_M} N$ for some $M \in \mathfrak{M}$ and some finite dimensional representation N of $\mathbf{R}_M$.

Such a set $\mathfrak{M}$ is called a *parametrizing set* of representations of the box $\mathfrak{A}$. We denote by $|\mathfrak{M}|$ the set $\{M \otimes_{\mathbf{R}_M} N \mid M \in \mathfrak{M},\ N \in \operatorname{ind}(\mathbf{R}_M)\}$. Note that the set $\mathfrak{M}$ may be empty; thus all representation discrete boxes are by definition also tame.

Let $\mathfrak{M}$ run through all possible parametrizing sets and $\mu_{\mathfrak{A}}(\mathbf{d})$ be the minimum number of elements in the set $\{M \in \mathfrak{M} \mid \mathbf{dim}\, M = \mathbf{d}\}$. Call a tame box $\mathfrak{A}$ *bounded* if there is a constant C such that $\mu_{\mathfrak{A}}(\mathbf{d}) \leq C$ for all $\mathbf{d}$ and *unbounded* otherwise.

If a box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is tame, then for every dimension $\mathbf{d}\colon \mathrm{Ob}\,\mathsf{A} \to \mathbb{N}$ of its finite dimensional representations there is a constructible subset $\mathfrak{R}_{\mathbf{d}} \subseteq \mathsf{rep}_{\mathbf{d}}(\mathfrak{A})$ of dimension at most $|\mathbf{d}| = \sum_{A \in \mathrm{Ob}\,\mathsf{A}} \mathbf{d}(A)$ such that $\mathfrak{R}_{\mathbf{d}}$ intersects all isomorphism classes from $\mathsf{rep}_{\mathbf{d}}(\mathfrak{A})$. Some easy geometrical considerations imply the following result [11].

PROPOSITION 5.5. *Neither skeletal box can be both tame and wild.*

Again, Corollary 3.2 together with some elementary geometrical observations (*cf.* [11]) implies the following result.

COROLLARY 5.6. *If a semi-free box $\mathfrak{A}$ is tame, its Tits form $Q_{\mathfrak{A}}$ is* weakly nonnegative, *i.e., $Q_{\mathfrak{A}}(\mathbf{d}) \geq 0$ for every vector $\mathbf{d}$ with nonnegative entries.*

The relation between the representations of a locally finite dimensional category C and the box $\mathfrak{R}_{\mathsf{C}}$ corresponding to the bipartite $\mathsf{C} \times \mathsf{C}$-bimodule $\mathsf{R} = \mathrm{rad}\,\mathsf{C}$ (*cf.* Section 4) implies the following:

COROLLARY 5.7. *The representation type of a locally finite dimensional category C coincides with that of the box $\mathfrak{R}_{\mathsf{C}}$.*

Let $\mathbf{R} = \mathbf{k}[t, g(t)^{-1}]$ be a rational algebra and $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a skeletal box with a generating set $\mathfrak{G}$ of morphisms from A (for instance, $\mathfrak{A}$ is semi-free and $\mathfrak{G}$ is a set of its solid arrows). A representation $M \in \mathsf{rep}(\mathfrak{A}, \mathbf{R})$ is said to be *linear* if a basis can be chosen in each MA (it is a free $\mathbf{R}$-module) such that all entries of matrices corresponding to the homomorphisms Ma $(a \in \mathfrak{G})$ with respect to these bases are linear polynomials in t. We shall see later that each tame semi-free box or tame locally finite dimensional category has a parametrizing family consisting of linear representations.

6. Reduction algorithm. Theorem 2.1 and especially Corollary 2.2 are used for the so-called "*reduction algorithm.*" The existence of this algorithm is the main advantage of boxes in the representation theory.

Suppose that $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is a semi-free triangular box with a semi-free triangular set of generators $\Sigma = \Sigma_0 \cup \Sigma_1$ with respect to a function $\nu\colon \Sigma \to \mathbb{N}$. Let $a\colon A \to B$ be an element from Σ_0 with the smallest value of $\nu(a)$. There are three possibilities:

(1) $\partial a \neq 0$; then ∂a is just a linear combination of elements from Σ_1: $\partial a = \sum_i \lambda_i v_i$, where $\lambda_i \in \mathbf{k}$, $v_i \in \Sigma_1$ and $\nu(v_i) < \nu(a)$. If $\lambda_j \neq 0$, we can replace v_j by ∂a getting a new triangular semi-free set of generators that contains

∂a. In this case we call a a *superfluous arrow* and always suppose that $\partial a \in \Sigma_1$.[3]

(2) $\partial a = 0$ and $A \neq B$. Then we call a a *minimal edge*.

(3) $\partial a = 0$ and $A = B$. Then we call a a *minimal loop*.

Certainly, these notions depend on the chosen set of generators and the function ν. We often call a a superfluous arrow, or a minimal edge, or a minimal loop if there is a set of generators containing a and a function ν such that a is so with respect to this set and this function.

The following result explains the term "*superfluous.*"

THEOREM 6.1 (*cf.* [18], [12], [6]). *Let a be a superfluous arrow, $\mathsf{B} = \mathsf{A}/(a)$ and $F\colon \mathsf{A} \to \mathsf{B}$ be the natural projection. Then:*

(1) $F^*\colon \mathsf{Rep}(\mathfrak{A}, \mathsf{C}) \to \mathsf{Rep}(\mathfrak{A}^F, \mathsf{C})$ is an equivalence for any category C.

(2) The box $\mathfrak{A}^F$ is again semi-free (free if so is $\mathfrak{A}$) triangular.

(3) The bigraph of the box $\mathfrak{A}^F$ can be obtained from that of the box $\mathfrak{A}$ by deleting the solid arrow a and the dotted arrow ∂a.

(4) The differential of the box $\mathfrak{A}^F$ can be obtained from that of the box $\mathfrak{A}$ by omitting all terms containing a or ∂a.

(5) If $M \simeq F^*N$ and $MA \neq 0$, $MB \neq 0$, then $Q^-_{\mathfrak{B}}(\mathbf{dim}\, N) < Q^-_{\mathfrak{A}}(\mathbf{dim}\, M)$.

Suppose now that a is a *minimal edge* and there are no marked loops in $\mathsf{A}(A, A) \cup \mathsf{A}(B, B)$. Consider the subcategory $\mathsf{A}' \subseteq \mathsf{A}$ consisting of two objects A, B and one arrow a. Denote by B' the trivial category with three objects A_0, B_0, AB and consider the functor $F'\colon \mathsf{A}' \to \operatorname{add} \mathsf{B}'$ that maps $A \mapsto A_0 \oplus AB$, $B \mapsto B_0 \oplus AB$ and $a \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\colon A_0 \oplus AB \to B_0 \oplus AB$. We often write A_1 or B_1 for AB.

PROPOSITION 6.2. *The category $\widetilde{\mathsf{B}} = \operatorname{add}(\mathsf{A} \coprod^{\mathsf{A}'} \mathsf{B}')$ is equivalent to $\operatorname{add} \mathsf{B}$, where B is again a semi-free (free if so is A) category.*

PROOF. The category $\widetilde{\mathsf{B}}$ can be defined up to equivalence as a fully additive category with the following universal property:

- *There is a commutative diagram*

$$\begin{array}{ccc} \mathsf{A}' & \xrightarrow{F'} & \operatorname{add} \mathsf{B}' \\ {\scriptstyle \mathrm{em}}\downarrow & & \downarrow{\scriptstyle \widetilde{E}} \\ \mathsf{A} & \xrightarrow{\widetilde{F}} & \widetilde{\mathsf{B}}, \end{array} \tag{2}$$

where em *is the embedding, such that for any pair of functors $G\colon \mathsf{A} \to \mathsf{C}$, $H'\colon \mathsf{B}' \to \mathsf{C}$, where C is fully additive and $G \cdot \mathrm{em} = H'E$, there is a unique functor $H\colon \widetilde{\mathsf{B}} \to \mathsf{C}$ such that $G = HF$ and $H' = HE$.*

[3] In [18], [12] they call such an arrow *nonregular*, but the word "superfluous" seems more appropriate to the situation.

Consider the semi-free category B and the functors $F\colon \mathsf{A} \to \operatorname{add}\mathsf{B}$, $E\colon \operatorname{add}\mathsf{B}' \to \operatorname{add}\mathsf{B}$ defined as follows:

- $\operatorname{Ob}\mathsf{B} = (\operatorname{Ob}\mathsf{A} \setminus \{A, B\}) \cup \{A_0, B_0, AB\}$.
- The set of arrows of B consists of:
  - the arrows $b\colon C \to D$ from A such that $\{C, D\} \cap \{A, B\} = \emptyset$;
  - for each arrow $b\colon C \to D$ (or $D \to C$), where $C \in \{A, B\}$, $D \notin \{A, B\}$, two arrows $b_0\colon C_0 \to D$ and $b_1\colon C_1 \to D$ (respectively $b_0\colon D \to C_0$ and $b_1\colon D \to C_1$);
  - for any arrow $b\colon C \to D$, where $C, D \in \{A, B\}$ and $b \neq a$, four arrows $b_{ij}\colon C_j \to D_i$ $(i, j = 0, 1)$.
- The marking polynomials for loops in B are the same as in A. (Here we use the assumption that there are no marked loops at A and B).
- E is induced by the natural embedding $\mathsf{B}' \to \mathsf{B}$.
- $F(A) = A_0 \oplus AB$, $F(B) = B_0 \oplus AB$, $F(C) = C$ if $C \notin \{A, B\}$.
- $F(a) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, while $F(b)$ for an arrow $b \neq a$, $b\colon C \to D$, is defined as follows:
  - if $\{C, D\} \cap \{A, B\} = \emptyset$, then $F(b) = b$;
  - if $C \in \{A, B\}$, $D \notin \{A, B\}$ (or $D \in \{A, B\}$, $C \notin \{A, B\}$), then $F(b) = (\, b_0 \; b_1 \,)$ (respectively $F(b) = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$);
  - if $C, D \in \{A, B\}$, then $F(b) = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$.

Obviously, the diagram

$$\begin{array}{ccc} \mathsf{A}' & \xrightarrow{F'} & \operatorname{add}\mathsf{B}' \\ {\scriptstyle \mathrm{em}}\downarrow & & {\scriptstyle E}\downarrow \\ \mathsf{A} & \xrightarrow{F} & \operatorname{add}\mathsf{B} \end{array}$$

commutes and has the same universal property as the diagram (2). Hence, $\widetilde{\mathsf{B}} \simeq \operatorname{add}\mathsf{B}$. ∎

Evidently, every functor $M\colon \mathsf{A}' \to \mathsf{Vec}$ can be factored through F'. Namely, if $M_0 = \operatorname{Ker} Ma$, M_1 is a complement of M_0 in MA and M_2 is a complement of $\operatorname{Im} Ma$ in MB, then $M \simeq NF'$, where $NA_0 = M_0$, $NB_0 = M_2$, $NAB = M_1$. Therefore, Corollary 2.2 implies the first claim of the following theorem.

THEOREM 6.3. *In the above situation*

(1) $F^*\colon \operatorname{Rep}(\mathfrak{A}^F) \to \operatorname{Rep}(\mathfrak{A})$ *is an equivalence.*

(2) The box $\mathfrak{A}^F$ *is equivalent to* $\operatorname{add}\mathfrak{B}$, *where* $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$ *is again a semi-free (free is so is* $\mathfrak{A}$*) triangular box.*

We denote by $\hat{F}$ *the induced equivalence* $\operatorname{Rep}(\mathfrak{B}) \to \operatorname{Rep}(\mathfrak{A})$.

(3) If $M \simeq \hat{F}N$ *and* $MA \neq 0$, $MB \neq 0$, *then* $Q^-_{\mathfrak{B}}(\mathbf{dim}\, N) < Q^-_{\mathfrak{B}}(\mathbf{dim}\, M)$.

PROOF. Certainly, we can take for W the restriction onto B of the coalgebra $\mathsf{V}^F = (\operatorname{add}\mathsf{B})^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^F(\operatorname{add}\mathsf{B})$. We only have to show that the box $\mathfrak{B}$ is indeed semi-free triangular. For every element $v \in \mathsf{V}(C,D)$, consider the matrix presentation of Fv with respect to the decomposition of FC and FD into a direct sum of objects from B. If v runs through a set of generators of V, the matrix elements of such presentations form a set of generators of W. We take the natural set of generators of V consisting of the dotted arrows from Σ and of the elements ω_C ($C \in \operatorname{Ob} \mathsf{A}$). There are the following possibilities for $v: C \dashrightarrow D$:

(1) $\{C,D\} \cap \{A,B\} = \emptyset$. Then $Fv: C \to D$ coincides with its own matrix presentation and we denote it by the same letter v.

(2) $C \in \{A,B\}$, $D \notin \{A,B\}$ (or $D \in \{A,B\}$, $C \notin \{A,B\}$). Then the matrix presentation of Fv is $(\, v_0 \; v_1 \,)$ with $v_0: C_0 \dashrightarrow D$, $v_1: C_1 \dashrightarrow D$ (respectively $\begin{smallmatrix} v_0 \\ v_1 \end{smallmatrix}$ with $v_0: C \dashrightarrow D_0$, $v_1: C \dashrightarrow D_1$).

(3) $C, D \in \{A,B\}$ but $v \neq \omega_C$. Then the matrix presentation of Fv is $\begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{pmatrix}$, where $v_{ij}: C_j \dashrightarrow D_i$.

(4) $v = \omega_A$ or $v = \omega_B$. Then we denote its matrix presentation by $\begin{pmatrix} \xi_{00} & \xi_{01} \\ \xi_{10} & \xi_{11} \end{pmatrix}$, respectively by $\begin{pmatrix} \eta_{00} & \eta_{01} \\ \eta_{10} & \eta_{11} \end{pmatrix}$.

The relations for these generators of W are just the corollaries of those from V, which are only

(3) $$\omega_D b - b\omega_C = \partial b,$$

where b runs through the solid arrows from Σ, $b: C \to D$. Therefore, there are no relations at all for the matrix elements originated from the dotted arrows from Σ. For $b = a$ the relation (3) becomes

$$\begin{pmatrix} \eta_{00} & \eta_{01} \\ \eta_{10} & \eta_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_{00} & \xi_{01} \\ \xi_{10} & \xi_{11} \end{pmatrix},$$

that is $\eta_{01} = \xi_{10} = 0$, $\eta_{11} = \xi_{11}$. Note that $\varepsilon\omega_A = 1_A$ implies $\varepsilon\xi_{ii} = 1_{Ai}$, $\varepsilon\xi_{ij} = 0$ if $i \neq j$ and the same is valid for η. Denote $\eta_{10} = \eta$, $\xi_{01} = \xi$, $\xi_{00} = \omega_{A_0}$, $\eta_{00} = \omega_{B_0}$, $\xi_{11} = \eta_{11} = \omega_{AB}$. Moreover, the matrix equality $\mu(\xi_{ij}) = (\xi_{ij}) \otimes (\xi_{ij})$ means that $\mu\omega_{A_0} = \omega_{A_0} \otimes \omega_{A_0}$, $\mu\omega_{AB} = \omega_{AB} \otimes \omega_{AB}$, $\mu\xi = \omega_{A_0}\xi + \xi\omega_{AB}$, and $\mu(\eta_{ij}) = (\eta_{ij}) \otimes (\eta_{ij})$ means that $\mu\omega_{B_0} = \omega_{B_0} \otimes \omega_{B_0}$, $\mu\eta = \omega_{AB}\eta + \eta\omega_{B_0}$. Hence, ω is a normal section, so $\mathfrak{B}$ is a normal box. As there are no relations for ξ and η, the kernel $\overline{\mathsf{W}}$ of this box is free as B-bimodule with a set of free generators consisting of the matrix elements originated from the dotted arrows from Σ and the elements ξ, η. Moreover, $\partial\xi = \partial\eta = 0$.

If $b: C \to D$ and $\{C,D\} \cap \{A,B\} = \emptyset$, the relation (3) remains unaltered for the corresponding elements from B. If $b: A \to D$, $D \notin \{A,B\}$, it becomes

$$\omega_D (\, b_0 \; b_1 \,) = (\, b_0 \; b_1 \,) \begin{pmatrix} \omega_{A_0} & \xi \\ 0 & \omega_{AB} \end{pmatrix}$$

or $\omega_D b_0 = b_0 \omega_{A_0}$, $\omega_D b_1 = b_1 \omega_{AB} + b_0 \xi$, *i.e.*, $\partial b_0 = 0$, $\partial b_1 = b_0 \xi$. Just in the same way one can calculate the differentials of the other solid arrows from B.

For instance, if $b: B \to A$, then $\partial b_{11} = 0$, $\partial b_{01} = -\xi b_{11}$, $\partial b_{10} = b_{11}\eta$, $\partial b_{00} = b_{01}\eta - \xi b_{10}$.

On the other hand, the equations $\mu(Fv) = Fv \otimes \omega_C + \omega_D Fv + \partial(Fv)$ give the values of ∂w for the matrix components w of Fv. For instance, if $v: C \cdots\!> B$, we get $\partial v_0 = 0$, $\partial v_1 = \eta v_0$, *etc.* These calculations imply immediately that the constructed set of generators of the box $\mathfrak{B}$ is triangular. ∎

Suppose now that $a: A \to A$ is a minimal loop and there is no marked loop $c \neq a$ in $\mathsf{A}(A, A)$. Let $\mathfrak{X}$ be a finite subset of $\mathbf{k}$ and n be a positive integer. Denote by A' the subcategory of A with the unique object A and the algebra of morphisms $\mathbf{k}[a, g_a(a)^{-1}]$. Consider the minimal category B' with the set of objects $\{A_0, A_{m\lambda} \mid 1 \leq m \leq n, \lambda \in \mathfrak{X}\}$ and the unique loop $a_0: A_0 \to A_0$ with the marking polynomial $g_{a_0}(t) = g_a(t)\prod_{\lambda\in\mathfrak{X}}(t-\lambda)$. Define the functor $F': \mathsf{A}' \to \mathsf{B}'$ setting $F'(A) = A_0 \oplus (\bigoplus_{m,\lambda} mA_{m\lambda})$, $F'(a) = \left(\begin{smallmatrix} a_0 & 0 \\ 0 & J\end{smallmatrix}\right)$, where the matrix J is a direct sum of Jordan cells

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix} \qquad (m \times m \text{ matrix}).$$

For any linear mapping $\varphi: V \to V$ of a finite dimensional vector space V, one can consider the Fitting decomposition $V = V_0 \oplus V_1$ such that both V_0 and V_1 are invariant under φ, the restriction $\varphi|_{V_0}$ has no eigenvalues from $\mathfrak{X}$ and the minimal polynomial of the restriction $\varphi|_{V_1}$ is of the form $\prod_{\lambda\in\mathfrak{X}}(t-\lambda)^{k_\lambda(\varphi)}$. Let $M: \mathsf{A} \to \mathsf{vec}$ be a functor. Then the restriction of M onto A' can be factored through F' if and only if $k_\lambda(Ma) \leq n$ for all $\lambda \in \mathfrak{X}$. In particular, it is the case if $\dim MA \leq n$.

Now the calculations quite analogous (though more cumbersome) to those used in the proofs of Proposition 6.2 and Theorem 6.3 give the following result.

THEOREM 6.4. *In the above situation, there is a functor* $F: \mathsf{A} \to \mathsf{add}\,\mathsf{B}$, *where* B *is a semi-free category such that:*

(1) The functor $F^*: \mathsf{rep}(\mathfrak{A}^F) \to \mathsf{rep}(\mathfrak{A})$ *induces an equivalence between* $\mathsf{rep}(\mathfrak{A}^F)$ *and the full subcategory of* $\mathsf{rep}(\mathfrak{A})$ *consisting of all representation* M *such that* $k_\lambda(Ma) \leq n$ *for all* $\lambda \in \mathfrak{X}$. *Especially, the image of* F^* *contains all representation* M *with* $\dim MA \leq n$.

(2) The box $\mathfrak{A}^F$ *is equivalent to* $\mathsf{add}\,\mathfrak{B}$, *where* $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$ *is again a semi-free triangular box.*
We denote by $\hat{F}$ *the induced functor* $\mathsf{rep}(\mathfrak{B}) \to \mathsf{rep}(\mathfrak{A})$.

(3) If $M \simeq \hat{F}N$ *and* Ma *has an eigenvalue from* $\mathfrak{X}$, *then* $Q^-_{\mathfrak{B}}(\mathbf{dim}\,N) < Q^-_{\mathfrak{A}}(\mathbf{dim}\,M)$.

Note that in this case the box $\mathfrak{B}$ is no more free even if so was the box $\mathfrak{A}$. Actually, it was the reason why semi-free boxes were introduced in [12].[4]

One more variant of reduction occurs in studying coverings (*cf.* Section 10) and deals with *minimal lines*. By definition, a *minimal line* L in a semi-free box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is a set of pairwise different vertices $\{A_n \mid n \in \mathbb{Z}\}$ and of arrows $\{a_n : A_n \to A_{n+1}\}$ with $\partial a_n = 0$ for all n. Denote by M_{mn} ($m, n \in \mathbb{Z}$, $m \le n$) the following functor $\mathsf{L} \to \mathrm{vec}$:

$$M_{mn}A = \begin{cases} \mathbf{k} & \text{if } A = A_k,\ m \le k \le n, \\ 0 & \text{otherwise;} \end{cases}$$

$$M_{mn}a = \begin{cases} 1 & \text{if } a = a_k,\ m \le k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Fix an integer r. Let B' be the trivial category with the set of objects $\{B_{mn} \mid |m-n| \le r\}$ and $F' : \mathsf{L} \to \operatorname{add} \mathsf{B}'$ be the functor such that:

- $F'A_k = \bigoplus_{m \le k \le n, |m-n| \le r} B_{mn}$,
- with respect to this decomposition, $F'a_k : \bigoplus_{m \le k \le n} B_{m'n'} \to \bigoplus_{m' \le k+1 \le n'} B_{m'n'}$ is the matrix with the entries

$$\alpha_{mn,m'n'} = \begin{cases} 1 & \text{if } m = m',\ n = n' \\ 0 & \text{otherwise.} \end{cases}$$

The same observations as before give the following result [14].

THEOREM 6.5. *Let* L *be a minimal line in a semi-free box* $\mathfrak{A}$ *such that there are no marked loops at the objects* A_n *belonging to this line. There is a functor* $F : \mathsf{A} \to \operatorname{add} \mathsf{B}$, *where* B *is a semi-free category, such that:*

(1) The functor $F^* : \operatorname{Rep}(\mathfrak{A}^F) \to \operatorname{Rep}(\mathfrak{A})$ *induces an equivalence between* $\operatorname{Rep}(\mathfrak{A}^F)$ *and the full subcategory of* $\operatorname{Rep}(\mathfrak{A})$ *consisting of all representation* M *such that the restriction of* M *onto* L *decomposes into a direct sum of representations* M_{mn} *with* $|m-n| \le r$.

(2) The box $\mathfrak{A}^F$ *is equivalent to* $\operatorname{add} \mathfrak{B}$, *where* $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$ *is again a semi-free triangular box.*
We denote by $\hat{F}$ *the induced functor* $\operatorname{Rep}(\mathfrak{B}) \to \operatorname{Rep}(\mathfrak{A})$.

(3) If $M \simeq \hat{F}N$ *and* $MA_n \ne 0$, $MA_{n+1} \ne 0$ *for some* n, *then* $Q^-_{\mathfrak{B}}(\mathbf{dim}\, N) < Q^-_{\mathfrak{A}}(\mathbf{dim}\, M)$.

The following immediate observation is sometimes useful.

PROPOSITION 6.6. *Let* $\hat{F}$ *be one of the functors from Theorems 6.1, 6.3, 6.4 or 6.5,* M *be a linear representation of* $\mathfrak{B}$ *over a rational algebra* $\mathbf{R}$. *Then* $\hat{F}M$ *is a linear representation of* $\mathfrak{A}$ *over* $\mathbf{R}$.

[4] Their definition in [12] was a bit different and more complicated. The present one is a combination of [12] and [6].

7. **Finite type.** The first application of the reduction algorithm is that to the representation discrete boxes. The following result was proved in [18] (it had been known before as the First Brauer-Thrall conjecture).

THEOREM 7.1. *Suppose that a semi-free triangular box $\mathfrak{A}$ is not representation discrete.*[5]

(1) $\mathfrak{A}$ is representation strongly infinite.

(2) $\mathfrak{A}$ has an indecomposable infinite dimensional representation with finite support.

PROOF. Let the set $\mathrm{ind}_{\mathbf{d}}(\mathfrak{A})$ be infinite for some vector dimension $\mathbf{d}$. We prove the theorem using the induction on $q = Q^{-}_{\mathfrak{A}}(\mathbf{d})$. Obviously, one can suppose that this vector dimension is *sincere*, *i.e.*, $\mathbf{d}(A) \neq 0$ for each object A (especially, $\mathfrak{A}$ only has finitely many objects). If $q = 0$, there are no solid arrows at all, *i.e.*, the box is so-trivial and has finitely many representations of any vector dimension. Thus, the claim is true for $q = 0$. Suppose that it is true for each semi-free box $\mathfrak{B}$ and each vector dimension $\mathbf{c}$ such that $Q^{-}_{\mathfrak{B}}(\mathbf{c}) < q$. If $q > 0$, there are solid arrows, hence, there is either a superfluous arrow, or a minimal edge, or a minimal loop $a: A \to A$. In the latter case the box is representation strongly infinite (*cf.* Example 5.2). Moreover, we can define an indecomposable infinite dimensional representation J_∞ setting

$$J_\infty A = \mathbf{k}(t), \quad J_\infty B = 0 \text{ if } B \neq A,$$
$$J_\infty a \text{ is the multiplication by } t, \quad J_\infty b = 0 \text{ if } b \neq a.$$

If $a: A \to B$ is a minimal edge, consider the functor $\hat{F}$ from Theorem 6.3. For any representation $M \in \mathrm{ind}_{\mathbf{d}}(\mathfrak{A})$ there is a representation $N \in \mathrm{ind}_{\mathbf{d}}(\mathfrak{B})$ such that $M \simeq \hat{F}N$. Set $\mathbf{c} = \mathbf{dim}\, N$. There is finitely many possibilities for $\mathbf{c}$, therefore, there is at least one such dimension with infinite set $\mathrm{ind}_{\mathbf{c}}(\mathfrak{B})$. Since $Q^{-}_{\mathfrak{B}}(\mathbf{c}) < q$, the box $\mathfrak{B}$, hence also $\mathfrak{A}$, is representation strongly infinite and has an indecomposable infinite dimensional representation. Just the same observation works for a superfluous arrow (use Theorem 6.1). ∎

COROLLARY 7.2. *If the Tits form of a semi-free triangular box $\mathfrak{A}$ is not weakly positive, the box $\mathfrak{A}$ is representation strongly infinite.*

Theorem 7.1 and Corollary 5.7 immediately imply the following result.

COROLLARY 7.3. *If a locally finite dimensional category is not representation discrete, it is representation strongly infinite and has an infinite dimensional representation with finite support.*[6]

THEOREM 7.4. *Suppose that a semi-free triangular box $\mathfrak{A}$ is representation (locally) finite. Then every representation $M \in \mathsf{Rep}(\mathfrak{A})$ with a finite support is a direct sum of finite dimensional representations.*

[5] As we have already seen, a representation discrete semi-free box is actually free.

[6] It follows from [2], [3] that one can replace here "representation discrete" by "representation finite."

PROOF. Obviously, we may suppose that $\mathfrak{A}$ only has finitely many objects, hence, finitely many indecomposable finite dimensional representations. Then we can just follow the proof of Theorem 7.1 using the induction on $q = \max\{Q^-_{\mathfrak{A}}(\dim N) \mid N \in \operatorname{ind}(\mathfrak{A})\}$. ∎

COROLLARY 7.5. *Suppose that a locally finite dimensional category* C *is representation (locally) finite. Then every representation* $M \in \operatorname{Rep}(\mathsf{C})$ *with a finite support is a direct sum of finite dimensional representations.*

8. Tame and wild type.

THEOREM 8.1 (TAME-WILD DICHOTOMY, *cf.* [12]). *If a semi-free triangular box is not wild, it is tame. Moreover, it has a parametrizing set consisting of linear representations.*[7]

PROOF. We shall prove that if $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is not wild, then for every dimension $\mathbf{d}: \operatorname{Ob}\mathsf{A} \to \mathbb{N}$ there is a finite set $\mathfrak{M}_{\mathbf{d}}$ such that

- each $M \in \mathfrak{M}_{\mathbf{d}}$ is a strict linear representation of $\mathfrak{A}$ of vector dimension $\mathbf{d}_M \le \mathbf{d}$ over a rational algebra $\mathbf{R}_M$ (it may depend on M);
- if $\mathbf{d}' \le \mathbf{d}$, almost all representations from $\operatorname{ind}_{\mathbf{d}'}\mathfrak{A}$ (*i.e.*, all but a finite number of them) are isomorphic to $M \otimes_{\mathbf{R}_M} N$ for some $M \in \mathfrak{M}_{\mathbf{d}}$ and some finite dimensional representation N of $\mathbf{R}_M$;
- if $\mathbf{d} \le \mathbf{c}$ then $\mathfrak{M}_{\mathbf{d}} \subseteq \mathfrak{M}_{\mathbf{c}}$.

Certainly, then one can put $\mathfrak{M} = \bigcup_{\mathbf{d}} \mathfrak{M}_{\mathbf{d}}$.

We suppose $\mathbf{d}$ sincere and use induction on $q = Q^-_{\mathfrak{A}}(\mathbf{d})$. Again the case $q = 0$ is trivial, so we may suppose that $q > 0$ and the claim is true for all boxes $\mathfrak{B}$ and all dimensions of their representations $\mathbf{c}$ with $Q^-_{\mathfrak{B}}(\mathbf{c}) < q$. If there is a minimal edge or a superfluous arrow in $\mathfrak{A}$, the proof just repeats that of Theorem 7.1 (using Proposition 6.6 for linearity). Hence, we may suppose that there are neither minimal edges nor superfluous arrows in $\mathfrak{A}$, only minimal loops. Note that if there is a minimal loop $a: A \to A$ and a solid arrow $b: A \to B$ or $b: B \to A$ with $\partial b = 0$, the box $\mathfrak{A}$ is wild due to Corollary 5.4. If every solid arrow from $\mathfrak{A}$ is a minimal loop, set $\mathfrak{M} = \{J^a \mid a \text{ is a minimal loop}\}$, where J^a has been defined in Example 5.2. Evidently, $|\mathfrak{M}| = \operatorname{ind}(\mathfrak{A})$ and all representations J^a are linear.

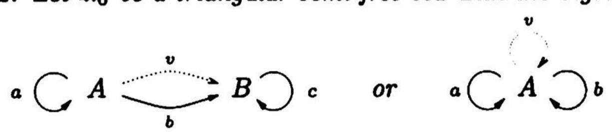
If there are solid arrows that are not minimal loops, the triangularity implies that there is one of them, say $b: A \to B$, such that ∂b only contains minimal loops (and dotted arrows). First suppose that there is a (unique) minimal loop $a: A \to A$ and no minimal loops $c: B \to B$ (or vice versa). Then $\partial b = \sum_{i=1}^{k} v_i f_i(a)$ for some dotted arrows v_i and some nonzero polynomials $f_i(t)$, and, choosing a new set of generators, we may suppose that $k = 1$, *i.e.*, $\partial b = v f(a)$. Use Theorem 6.4 for the set $\mathcal{X} = \{\lambda \in \mathbf{k} \mid f(\lambda) = 0\}$ and $n = \mathbf{d}(A)$. Note that $\operatorname{ind}_{\mathbf{d}}(\mathfrak{A}) = \mathfrak{N}_1 \cup \mathfrak{N}_2$, where $\mathfrak{N}_1$ consists of representations N such that NA has eigenvalues from $\mathcal{X}$

[7] The latter property was first noticed in [15], though it easily follows from the reduction algorithm.

and $\mathfrak{N}_2$ of all other representations. Theorem 6.4 and the induction conjecture implies that almost all representations from $\mathfrak{N}_1$ can be obtained from a finite set of strict linear representation over rational algebras. All representations from $\mathfrak{N}_2$ are isomorphic to G^*N, where $G: \mathsf{A} \to \mathsf{A}[f(a)^{-1}]$. The box $\mathfrak{A}^G$ is also semi-free with the same sets of objects and arrows, but the arrow b is superfluous in $\mathfrak{A}^G$. Hence, we can use again the induction hypothesis and claim that almost all representations from $\mathfrak{N}_2$ (thus almost all representations from $\mathrm{ind_d}(\mathfrak{A})$) can be obtained from a finite set of strict linear representations over rational algebras.

Suppose now that there is both a minimal loop $a \in \mathsf{A}(A, A)$ and a minimal loop $c \in \mathsf{A}(B, B)$, both unique since $\mathfrak{A}$ is not wild (perhaps $A = B$, then $a = c$). We consider $\mathsf{V}(A, B)$ as $\mathbf{k}[x, y]$-bimodule: for $v \in \mathsf{V}(A, B)$ and $f(x, y) = \sum_{ij} \lambda_{ij} x^i y^j$ set $f(x, y)v = \sum_{ij} \lambda_{ij} c^j v a^i$. Then $\partial b = \sum_k f_k(x, y) v_k$ for some dotted arrows $v_k: A \cdots\!> B$). Let $d(x, y)$ be the greatest common divisor of all $f_k(x, y)$. There are polynomials $g_k(x, y)$ and $h(x)$ such that $h(x)d(x, y) = \sum_k g_k(x, y) f_k(x, y)$. Using Theorem 6.4 for the loop a and the set $\mathfrak{X} = \{\lambda \in \mathbf{k} \mid h(\lambda) = 0\}$, we are able to reduce the situation, just as in the preceding paragraph, to the case when $h(a)$ is invertible. Then, changing the set of dotted arrows, we can suppose that $\partial b = d(x, y)v$ for some dotted arrow v. If $d(x, y) = 1$, b is superfluous, so we can use the inductive procedure. Otherwise the following lemma accomplishes the proof.

LEMMA 8.2. *Let $\mathfrak{A}_0$ be a triangular semi-free box with the bigraph*



such that $\partial a = 0$, $\partial c = 0$ and b is not superfluous. Then $\mathfrak{A}_0$ is wild.

The proof of this lemma is just an explicit construction of strict representations of the given boxes over the wild algebra $\mathbf{k}\Gamma_5$ from Corollary 5.4. For instance, if $\mathfrak{A}_0$ is free with the first of the given bigraphs and $\partial b = va - cv$ (it is a typical case), a strict representation M from $\mathsf{rep}(\mathfrak{A}_0, \mathbf{k}\Gamma_5)$ can be defined as follows. We denote by P_i the indecomposable projective module corresponding to the vertex A_i of the graph Γ_5, identify the arrows a_i with homomorphisms $P_0 \to P_i$) and set:

$$MA = 9P_0,$$
$$MB = \bigoplus_{i=1}^{5} (2i-1)P_i,$$
$$Ma = J_9,$$
$$Mc = \bigoplus_{i=1}^{5} J_{2i-1},$$

$$Mb = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_3 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_4 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_5 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

where J_m denotes the $m \times m$ nilpotent Jordan cell. ∎

COROLLARY 8.3. *If a locally finite dimensional category is not wild, it is tame and has a parametrizing family that consists of linear representations.*

COROLLARY 8.4. *If the Tits form of a semi-free triangular box is not weakly nonnegative, this box is wild.*

REMARK 8.5. One can easily see from the proof of Theorem 8.1 that if a semi-free triangular box $\mathfrak{A}$ is wild, it has a strict representation over any free algebra $\mathbf{k}\langle x_1, x_2, \dots, x_m \rangle$ that is also *linear, i.e.*, all entries of matrices corresponding to the homomorphisms Ma (a runs through solid arrows) with respect to some bases chosen in all modules MA (which are free [1, Chapter IV, Section 5]) are linear in $x_1, x_2, \dots, x_n$. The same is true for locally finite dimensional categories.[8]

Indeed, the proof of Theorem 8.1 also gives the following results that are sometimes useful.

PROPOSITION 8.6. *Suppose that a semi-free triangular box $\mathfrak{A}$ is not wild, a is a minimal loop from $\mathfrak{A}$ and $\mathbf{d}$ is a vector dimension of representations of $\mathfrak{A}$.*

[8] It was also first observed in [15].

(1) There is a finite subset $\mathfrak{X} \subseteq \mathbf{k}$ *such that, for each* $M \in \operatorname{ind}_{\mathbf{d}}(\mathfrak{A})$, $M \not\simeq J_n^a(\lambda)$, *the set of eigenvalues of* Ma *is contained in* $\mathfrak{X}$.

(2) There is a morphism $\Phi: \mathfrak{A} \to \operatorname{add} \mathfrak{T}$, *where* $\mathfrak{T}$ *is a so-minimal box, such that the functor* $\Phi^*: \operatorname{rep}(\mathfrak{T}) \to \operatorname{rep}(\mathfrak{A})$ *is full and faithful and its image contains all representations of dimensions* $\mathbf{d}' \leq \mathbf{d}$.

9. **Generic modules.** Let $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a normal box, $M \in \operatorname{Rep}(\mathfrak{A})$ and $\mathcal{E} = \operatorname{Hom}_{\mathfrak{A}}(M, M)$. For each object $A \in \operatorname{Ob} \mathsf{A}$ and each element $\alpha \in \mathcal{E}$, $\alpha(\omega_A)$ is an endomorphism of the vector space MA and $\alpha\beta(\omega_A) = \alpha(\omega_A)\beta(\omega_A)$. Hence, we can consider MA as $\mathcal{E}$-module setting $\alpha u = \alpha(\omega_A)u$. Suppose that $\mathfrak{A}$ is skeletal. They say that M is of *finite endolength* if $\operatorname{supp} M$ is finite and $\operatorname{length}_{\mathcal{E}} MA < \infty$ for each object A. Let $\operatorname{fel}(\mathfrak{A})$ be the category of all representations from $\operatorname{Rep}(\mathfrak{A})$ of finite endolength. A representation $M \in \operatorname{fel}(\mathfrak{A})$ is said to be *generic* if it is indecomposable and infinite dimensional (*i.e.*, $\dim MA = \infty$ for at least one object A). We denote by $\mathbf{e\text{-}len}\, M$ and call the *vector endolength* of M the function $\operatorname{Ob} \mathsf{A} \to \mathbb{N}$ mapping A to $\operatorname{length}_{\mathcal{E}}(MA)$ and set $\operatorname{e-len} M = \sum_{A \in \operatorname{Ob} \mathsf{A}} \operatorname{length}_{\mathcal{E}}(MA)$. Let $\operatorname{gen}(\mathfrak{A})$ denote the set of isomorphism classes of all generic representations of $\mathfrak{A}$ and $\operatorname{gen}_{\mathbf{d}}(\mathfrak{A})$ those of generic representations with vector endolength $\mathbf{d}$. In particular, these definitions are valid if we consider a locally finite dimensional category instead of a box. Thus it contains, in particular, representations of finite dimensional algebras.

EXAMPLE 9.1. Let M be a strict representation of $\mathfrak{A}$ over a rational algebra $\mathbf{R}$. Denote by $M^m(t)$ the representation $M \otimes_{\mathbf{R}} J_m(t)$, where $J_m(t)$ is the $\mathbf{k}(t)$-$\mathbf{R}$-bimodule such that its underlying right $\mathbf{k}(t)$-module is $m\mathbf{k}(t)$ and the left multiplication by t is given by the matrix

$$\begin{pmatrix} t & 1 & 0 & \dots & 0 & 0 \\ 0 & t & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t & 1 \\ 0 & 0 & 0 & \dots & 0 & t \end{pmatrix}.$$

Then $M^m(t)$ is a generic representation of the box $\mathfrak{A}$ with $\mathbf{e\text{-}len}\, M^m = m \,\mathbf{dim}\, M$. Moreover, its endomorphism algebra $\mathcal{E}$ is a finite dimensional $\mathbf{k}(t)$-algebra.

If a box (or an algebra) is representation finite, it has no generic modules (*cf.* Corollary 7.5). Moreover, the following result can be obtained just following the proof of Theorem 7.1.

PROPOSITION 9.2. *If a semi-free triangular box (or a locally finite dimensional category) has a generic representation, it is representation strongly infinite.*

The following refined version of tame–wild dichotomy was actually proved in [7] (though the original formulation was a bit different there).

THEOREM 9.3. *For a semi-free triangular box* $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ *the following conditions are equivalent:*

(1) $\mathfrak{A}$ *is not wild.*

(2) $\mathfrak{A}$ *is tame.*

(3) For each vector dimension $\mathbf{d}: \mathrm{Ob}\,\mathsf{A} \to \mathbb{N}$ *the set* $\mathrm{gen}_{\mathbf{d}}(\mathfrak{A})$ *is finite.*

(4) There is a parametrizing set $\mathfrak{M}$ *of representations of* $\mathfrak{A}$ *such that every generic representation* $N \in \mathrm{gen}(\mathfrak{A})$ *is isomorphic to* $M^m(t)$ *for some* $M \in \mathfrak{M}$.

Moreover, the representations from $\mathfrak{M}$ *can be chosen linear.*

PROOF. (1) $\Leftrightarrow$ (2) is already known. (4) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1): Consider a strict representation M of $\mathfrak{A}$ over $\mathbf{k}[x,y]$. Denote by $N(\lambda)$ $(\lambda \in \mathbf{k})$ the $\mathbf{k}(t)$-$\mathbf{k}[x,y]$-bimodule such that the underlying $\mathbf{k}(t)$-module is just $\mathbf{k}(t)$, x acts as multiplication by λ and y as multiplication by t. Then $M \otimes_{\mathbf{k}[x,y]} N(\lambda)$ are generic pairwise nonisomorphic representations of $\mathfrak{A}$.

(1) $\Rightarrow$ (4). Using induction on $Q^-_{\mathfrak{A}}(\mathbf{d})$, we find finite sets of rational families $\mathfrak{M}_{\mathbf{d}}$ such that

- $\dim M \le \mathbf{d}$ for each $M \in \mathfrak{M}_{\mathbf{d}}$;
- every generic representation of vector endolength $\mathbf{d}' \le \mathbf{d}$ is isomorphic to $M^m(t)$ for some $M \in \mathfrak{M}_{\mathbf{d}}$;
- if $\mathbf{d} < \mathbf{c}$ then $\mathfrak{M}_{\mathbf{d}} \subseteq \mathfrak{M}_{\mathbf{c}}$.

Certainly, then one can put $\mathfrak{M} = \bigcup_{\mathbf{d}} \mathfrak{M}_{\mathbf{d}}$. The inductive procedure uses again the reduction algorithm and is quite analogous to that of the proof of Theorem 8.1. The main new ingredient is to check that vector endolength behave during the reduction in the same way as vector dimension. It follows easily from the fact that if $F: \mathfrak{A} \to \mathrm{add}\,\mathfrak{B}$ is one of the morphisms of boxes described in Section 6 and $FA = \bigoplus_i B_i$, one can arrange indices so that the image $F\omega_A$ be a triangular matrix with the diagonal entries ω_{B_i}. Hence, if $M = F^*N$ and $\mathcal{E} = \mathrm{Hom}_{\mathfrak{A}}(M,M) \simeq \mathrm{Hom}_{\mathfrak{B}}(N,N)$, then $\mathrm{length}_{\mathcal{E}}(MA) = \sum_i \mathrm{length}_{\mathcal{E}}(NB_i)$. We refer to [7] for details. ∎

Theorem 9.3 together with the results from Section 4 implies immediately its analogue for algebras [7].

COROLLARY 9.4. *Let* C *be a locally finite dimensional category (for instance, a finite dimensional algebra). The following conditions are equivalent:*

(1) C *is not wild.*

(2) C *is tame.*

(3) For every n *there is only finitely many generic* C*-modules of endolength* n *(up to equivalence).*

(4) There is a parametrizing set $\mathfrak{M}$ *of representations of* C *such that every generic representation* $N \in \mathrm{gen}(\mathfrak{A})$ *is isomorphic to* $M^m(t)$ *for some* $M \in \mathfrak{M}$.

Moreover, the representations from $\mathfrak{M}$ *can be chosen linear.*

10. Coverings. Let $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a box, $\mathbf{G}$ be a group. We say that $\mathbf{G}$ *acts* on $\mathfrak{A}$ if it acts on the sets of objects and morphisms of A as well as on the set of elements of V so that

- if $A, B \in \operatorname{Ob}\mathsf{A}$, $a \in \mathsf{A}(A,B)$, $v \in \mathsf{V}(A,B)$ and $g \in \mathbf{G}$, then $ga \in \mathsf{A}(gA, gB)$, $gv \in \mathsf{V}(gA, gB)$;
- $g(ab) = (ga)(gb)$, $g(a+b) = ga + gb$, $g(\lambda a) = \lambda(ga)$, where $g \in \mathbf{G}$, $\lambda \in \mathbf{k}$ and a, b are elements from $\operatorname{Mor}\mathsf{A}$ or V such that the left parts of the corresponding equations are defined;
- $\mu(gv) = g\mu(v)$, where $g(u_1 \otimes u_2) = (gu_1 \otimes gu_2)$;
- $\varepsilon(gv) = g\varepsilon(v)$.

If, moreover, A is skeletal and $gA \neq A$ for each $A \in \operatorname{Ob}\mathsf{A}$, $g \in \mathbf{G}$, $g \neq 1$, we say that $\mathbf{G}$ *acts freely* on $\mathfrak{A}$.

If $\mathbf{G}$ acts freely on $\mathfrak{A}$, the *orbit box* $\mathbf{G} \setminus \mathfrak{A} = (\mathbf{G} \setminus \mathsf{A}, \mathbf{G} \setminus \mathsf{V})$ is defined in the following way:

- $\operatorname{Ob}(\mathbf{G} \setminus \mathsf{A})$ is the set of orbits of $\mathbf{G}$ on $\operatorname{Ob}\mathsf{A}$;
- $(\mathbf{G}\setminus\mathsf{A})(\mathbf{G}A, \mathbf{G}B) = \bigoplus_{g,h\in\mathbf{G}} \mathsf{A}(gA, hB)/U_{\mathsf{A}}$, where U_{A} is the subspace generated by all differences $a - ga$ $(g \in \mathbf{G})$;
- $(\mathbf{G} \setminus \mathsf{V}) = (\mathbf{G}A, \mathbf{G}B) \bigoplus_{g,h\in\mathbf{G}} \mathsf{V}(gA, hB)/U_{\mathsf{V}}$, where U_{V} is the subspace generated by all differences $v - gv$ $(g \in \mathbf{G})$;
- $(\mathbf{G}a)(\mathbf{G}b) = (\mathbf{G}ab')$, where b' is the unique element from $\mathbf{G}b$ such that its target coincides with the source of a;
- $\varepsilon(\mathbf{G}v) = \mathbf{G}\varepsilon(v)$;
- $\mu(\mathbf{G}v) = \mathbf{G}\mu(v)$.

Let $\Pi\colon \mathfrak{A} \to \mathbf{G} \setminus \mathfrak{A}$ be the natural projection. It defines the inverse image functor $\mathsf{Rep}(\mathbf{G} \setminus \mathfrak{A}, \mathsf{C}) \to \mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ for each category C. On the other hand, if C is additive, the *direct image* functor $\Pi_*\colon \mathsf{rep}(\mathfrak{A}, \mathsf{C}) \to \mathsf{rep}(\mathbf{G} \setminus \mathfrak{A}, \mathsf{C})$ is induced by the tensor product $(\mathbf{G} \setminus \mathsf{A})^{\Pi} \otimes_{\mathsf{A}} \_$. Moreover, if $\mathbf{G}$ is finite or C has infinite direct sums, the functor Π_* is defined for *all* representations, not only finite. The following description of Π and Π_* is straightforward.

PROPOSITION 10.1. *For every objects $\hat{A}$, $\hat{B}$ from $\mathbf{G} \setminus \mathsf{A}$ and for each representatives $A_0 \in \hat{A}$, $B_0 \in \hat{B}$,*

$$(\mathbf{G} \setminus \mathsf{A})(\hat{A}, \hat{B}) \simeq \bigoplus_{A\in\hat{A}} \mathsf{A}(A, B_0) \simeq \bigoplus_{B\in\hat{B}} \mathsf{A}(A_0, B);$$

$$(\mathbf{G} \setminus \mathsf{V})(\hat{A}, \hat{B}) \simeq \bigoplus_{A\in\hat{A}} \mathsf{V}(A, B_0) \simeq \bigoplus_{B\in\hat{B}} \mathsf{V}(A_0, B);$$

$$(\Pi_* M)\hat{A} \simeq \bigoplus_{A\in\hat{A}} MA.$$

If a group $\mathbf{G}$ acts freely on a box $\mathfrak{A}$ and $\overline{\mathfrak{A}} \simeq \mathbf{G} \setminus \mathfrak{A}$, they say that $\mathfrak{A}$ is a *Galois covering* of $\overline{\mathfrak{A}}$ with *Galois group* $\mathbf{G}$.

The construction from Section 4 immediately implies the following result.

PROPOSITION 10.2. *If a group* $\mathbf{G}$ *acts freely on a basic category* C *and* $\mathfrak{R}_{\mathsf{C}}$ *is the box corresponding to the radical of* C *via Theorem 4.1, then* $\mathbf{G}$ *acts freely on* $\mathfrak{R}_{\mathsf{C}}$ *and there is a natural equivalence* $\mathbf{G} \setminus (\mathfrak{R}_{\mathsf{C}}) \simeq \mathfrak{R}_{\mathbf{G}\setminus\mathsf{C}}$*, which commutes with the functors* Cok.

Any action of a group $\mathbf{G}$ on a box $\mathfrak{A}$ induces an action of $\mathbf{G}$ on its representation categories: for any $M \in \mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ and $g \in \mathbf{G}$, gM is the representation such that $(gM)A = M(g^{-1}A)$. In general, this action is not free even if so is the action of $\mathbf{G}$ on $\mathfrak{A}$; nevertheless, it is free on the categories of finite representations if $\mathbf{G}$ is torsion free.

For representation finite and sometimes for tame boxes there are good relations between the representations of a box and those of its Galois coverings.

THEOREM 10.3. *Let a group* $\mathbf{G}$ *acts freely on a semi-free triangular box* $\mathfrak{A}$.

(1) $\mathfrak{A}$ *is representation locally finite if and only if so is* $\overline{\mathfrak{A}} = \mathbf{G} \setminus \mathfrak{A}$.

(2) If these boxes are representation locally finite, $\mathbf{G}$ *acts freely on* $\mathsf{rep}(\mathfrak{A})$ *and the functor* Π_* *induces an equivalence* $\mathbf{G} \setminus \mathsf{rep}(\mathfrak{A}) \simeq \mathsf{rep}(\overline{\mathfrak{A}})$.

COROLLARY 10.4 (*cf.* [4]). *Let a group* $\mathbf{G}$ *acts freely on a locally finite dimensional category* C.

(1) C *is representation locally finite if and only if so is* $\overline{\mathsf{C}} = \mathbf{G} \setminus \mathsf{C}$.

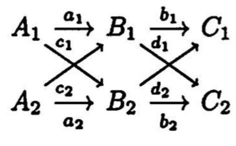
(2) If these categories are representation locally finite, $\mathbf{G}$ *acts freely on* $\mathsf{rep}(\mathsf{C})$ *and the functor* Π_* *induces an equivalence* $\mathbf{G} \setminus \mathsf{rep}(\mathsf{C}) \simeq \mathsf{rep}(\overline{\mathsf{C}})$.

REMARK 10.5. If $\mathfrak{A}$ is *representation discrete*, it may not be the case for $\overline{\mathfrak{A}}$. The easiest example is the free category with the graph

$$\cdots \longrightarrow A_{-1} \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

and the obvious action of the group $\mathbf{Z}$. The orbit category consists of one loop, hence, is representation strongly infinite.

If $\mathfrak{A}$ is tame, $\mathbf{G} \setminus \mathfrak{A}$ may not be so. The easiest example is perhaps that of finite dimensional category $\mathsf{C} = \mathbf{k}\Gamma/I$, where Γ is the graph



and I is the ideal generated by the set

$$\{d_1a_1 - b_2c_1, d_2a_2 - b_1c_2, b_1a_1 - d_2c_1, b_2a_2 - d_1c_2\},$$

with the evident free action of the group $\mathbf{G}$ of order 2. It is not difficult to check that C is tame, but if $\operatorname{char}\mathbf{k} = 2$ the orbit category $\mathbf{G} \setminus \mathsf{C}$ is wild [16].

Nevertheless, the situation becomes much better if the group $\mathbf{G}$ is torsion free.

THEOREM 10.6 (*cf.* [14]). *Suppose that a torsion free group* $\mathbf{G}$ *acts freely on a semi-free triangular box* $\mathfrak{A}$.

(1) $\mathfrak{A}$ *is tame if and only if so is* $\overline{\mathfrak{A}} = \mathbf{G} \setminus \mathfrak{A}$.

(2) If these boxes are tame, then:

(a) $\mathrm{ind}(\overline{\mathfrak{A}}) = \mathrm{ind}_0 \sqcup \mathrm{ind}_1$, *where* $\mathrm{ind}_0 = \Pi_*(\mathrm{ind}(\mathfrak{A})) \simeq \mathbf{G} \setminus \mathsf{rep}(\mathfrak{A})$ *and* $\mathrm{ind}_1 = |\mathfrak{N}|$, *where* $\mathfrak{N}$ *is a set of strict linear representations of* $\mathfrak{A}$ *over the algebra* $\mathbf{T} = \mathbf{k}[t, t^{-1}]$.

(b) $\mathrm{Hom}_{\mathfrak{A}}(M, M') \subseteq \mathrm{rad}^{\infty}(\overline{\mathfrak{A}})$ *if* $M \in \mathrm{ind}_0$, $M' \in \mathrm{ind}_1$, *or vice versa, or* M, M' *belong to different rational families from* $\mathfrak{N}$.

(c) If $M \simeq N \otimes_{\mathbf{T}} L$, $M' \simeq N \otimes_{\mathbf{T}} L'$ *for some* $N \in \mathfrak{N}$ *and* $L, L' \in \mathrm{ind}(\mathbf{T})$, *then* $\mathrm{Hom}_{\mathfrak{A}}(M, M') = 1 \otimes \mathrm{Hom}_{\mathbf{T}}(L, L') \oplus H$, *where* $H = \mathrm{Hom}_{\mathfrak{A}}(M, M') \cap \mathrm{rad}^{\infty}(\overline{\mathfrak{A}})$.

Here rad^{∞} *denotes the intersection of all powers of the radical of the category of representations.*

COROLLARY 10.7.[9] *Suppose that a torsion free group* $\mathbf{G}$ *acts freely on a locally finite dimensional category* C.

(1) C *is tame if and only if so is* $\overline{\mathsf{C}} = \mathbf{G} \setminus \mathsf{C}$.

(2) If these boxes are tame, then

(a) $\mathrm{ind}(\overline{\mathsf{C}}) = \mathrm{ind}_0 \sqcup \mathrm{ind}_1$, *where* $\mathrm{ind}_0 = \Pi_*(\mathrm{ind}(\mathsf{C})) \simeq \mathbf{G} \setminus \mathsf{rep}(\mathsf{C})$ *and* $\mathrm{ind}_1 = |\mathfrak{N}|$, *where* $\mathfrak{N}$ *is a set of strict linear representations of* C *over the algebra* $\mathbf{T} = \mathbf{k}[t, t^{-1}]$.

(b) $\mathrm{Hom}_{\mathsf{C}}(M, M') \subseteq \mathrm{rad}^{\infty}(\overline{\mathsf{C}})$ *if* $M \in \mathrm{ind}_0$, $M' \in \mathrm{ind}_1$, *or vice versa, or* M, M' *belong to different rational families from* $\mathfrak{N}$.

(c) If $M \simeq N \otimes_{\mathbf{T}} L$, $M' \simeq N \otimes_{\mathbf{T}} L'$ *for some* $N \in \mathfrak{N}$ *and* $L, L' \in \mathrm{ind}(\mathbf{T})$, *then* $\mathrm{Hom}_{\mathsf{C}}(M, M') = 1 \otimes \mathrm{Hom}_{\mathbf{T}}(L, L') \oplus H$, *where* $H = \mathrm{Hom}_{\mathsf{C}}(M, M') \cap \mathrm{rad}^{\infty}(\overline{\mathsf{C}})$.

PROOF. The proofs of Theorems 10.3 and 10.6 are based on the procedure of "equivariant reduction". Namely, we find a solid arrow a of the box $\overline{\mathfrak{A}}$ that is either superfluous, or a minimal edge, or a minimal loop. In the first two cases one can lift a to a set $\tilde{a}$ of superfluous arrows, respectively minimal edges of the box $\mathfrak{A}$. Then we apply the corresponding step of the reduction algorithm (*cf.* Section 6) both to the arrow a and to all arrows of the set $\tilde{a}$. As the result, we obtain a new box $\mathfrak{B}$ with a free action of the same group $\mathbf{G}$ and functors $F\colon \mathfrak{A} \to \mathsf{add}\,\mathfrak{B}$, $\overline{F}\colon \mathfrak{B} \to \mathsf{add}\,\overline{\mathfrak{B}}$, where $\overline{\mathfrak{B}} = \mathbf{G} \setminus \mathfrak{B}$, such that $F(gA) = g(FA)$ and both F and $\overline{F}$ induce equivalences $\mathsf{rep}(\mathfrak{B}) \simeq \mathsf{rep}(\mathfrak{A})$, $\mathsf{rep}(\overline{\mathfrak{B}}) \simeq \mathsf{rep}(\overline{\mathfrak{A}})$ so that the diagram

$$\begin{array}{ccc} \mathsf{rep}(\mathfrak{B}) & \xrightarrow{\sim} & \mathsf{rep}(\mathfrak{A}) \\ \Pi_* \downarrow & & \downarrow \Pi_* v \\ \mathsf{rep}(\overline{\mathfrak{B}}) & \xrightarrow{\sim} & \mathsf{rep}(\overline{\mathfrak{A}}) \end{array}$$

[9] Partial cases of this theorem were proved in [8], [9].

is commutative. Therefore, we can proceed inductively as in the proofs of Sections 7 and 8.

If a is a minimal loop, there are two possibilities: either a is lifted to $\mathfrak{A}$ as a set of minimal loops or as a set of *minimal lines* (*cf.* Section 6). In the former case we again use equivariant reduction and induction, while in the latter case the following lemma works.

LEMMA 10.8. *If the box $\mathfrak{A}$ is not wild and a minimal loop a of $\overline{\mathfrak{A}}$ is lifted to minimal lines, then, for every indecomposable representation $M \in \mathrm{ind}(\overline{\mathfrak{A}})$, either Ma is nilpotent or $M \simeq M'$ such that $M'b = 0$ for each arrow $b \neq a$.*

The representations of the second kind belong to the rational family $J^a \in \mathrm{rep}(\mathfrak{A}, \mathbf{T})$. For those of the first kind we again use an equivariant reduction, namely, apply Theorem 6.4 to the loop a (setting $\mathfrak{X} = \{0\}$) and Theorem 6.5 to the minimal lines that form the preimage of a.

The proof of Lemma 10.8 is the most intricate. Here we use a new class of boxes called *quasi-triangular.* Roughly speaking, a box is quasi-triangular if it becomes semi-free triangular after making invertible the arrows of several lines that become minimal after this procedure. Actually, in [14, Lemma 8.4] we prove a generalization of Lemma 10.8 for a quasi-triangular box $\mathfrak{A}$ using a generalized version of reduction algorithm to arrange an equivariant reduction (we refer to [14] for technical details). ∎

As we have already noticed, if $\mathbf{G}$ has elements of finite order, there is not a simple relation between representations of $\mathfrak{A}$ and $\mathbf{G} \setminus \mathfrak{A}$. Nevertheless, there is evidence that the following result might hold.

CONJECTURE 10.9. *Suppose that a group $\mathbf{G}$, which has no elements of order equal to* $\mathrm{char}\,\mathbf{k}$, *acts freely on a semi-free triangular box $\mathfrak{A}$.*

(1) $\mathfrak{A}$ is tame if and only if so is $\overline{\mathfrak{A}} = \mathbf{G} \setminus \mathfrak{A}$.

(2) If these boxes are tame, then

- *(a)* $\mathrm{ind}(\overline{\mathfrak{A}}) = \mathrm{ind}_0 \sqcup \mathrm{ind}_1$, *where* ind_0 *consists of direct summands of images* $\{\Pi_* M \mid M \in \mathrm{ind}(\mathfrak{A})\}$ *and* $\mathrm{ind}_1 = |\mathfrak{N}|$, *where $\mathfrak{N}$ is a set of strict representations of $\mathfrak{A}$ over the algebra $\mathbf{T} = \mathbf{k}[t, t^{-1}]$.*
- *(b)* $\mathrm{Hom}_{\mathfrak{A}}(M, M') \subseteq \mathrm{rad}^{\infty}(\overline{\mathfrak{A}})$ *if $M \in \mathrm{ind}_0$, $M' \in \mathrm{ind}_1$, or vice versa, or M, M' belong to different rational families from $\mathfrak{N}$.*
- *(c) If $M \simeq N \otimes_{\mathbf{T}} L$, $M' \simeq N \otimes_{\mathbf{T}} L'$ for some $N \in \mathfrak{N}$ and $L, L' \in$* $\mathrm{ind}(\mathbf{T})$, *then* $\mathrm{Hom}_{\mathfrak{A}}(M, M') = 1 \otimes \mathrm{Hom}_{\mathbf{T}}(L, L') \oplus H$, *where* $H =$ $\mathrm{Hom}_{\mathfrak{A}}(M, M') \cap \mathrm{rad}^{\infty}(\overline{\mathfrak{A}})$.

The same is true for representations of locally finite dimensional categories.

(For instance, these properties always hold if $\mathrm{char}\,\mathbf{k} = 0$.*)*

REFERENCES

1. H. Bass, *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

2. R. Bautista, *On algebras of strongly unbounded representation type.* Comment. Math. Helv. **60**(1985), 392–399.
3. K. Bongartz, *Indecomposable modules are standard.* Comment. Math. Helv. **60**(1985), 400–410.
4. K. Bongartz and P. Gabriel, *Covering spaces in representation theory.* Invent. Math. **65**(1982), 331–378.
5. H. Cartan and S. Eilenberg, *Homological Algebra.* Princeton University Press, Princeton, New Jersey, 1956.
6. W. Crawley-Boevey, *On tame algebras and BOCS's.* Proc. London Math. Soc. **56**(1988), 451–483.
7. ———, *Tame algebras and generic modules.* Proc. London Math. Soc. **63**(1991), 241–265.
8. P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras.* Arch. Math. **44**(1985), 522–529.
9. ———, *Galois coverings of representation-infinite algebras.* Comment. Math. Helv. **62**(1987), 311–337.
10. Y. A. Drozd, *Representations of commutative algebras.* Funkts. Anal. Prilozh. (4) **6**(1972), 41–43.
11. ———, *On tame and wild matrix problems.* In: Matrix Problems, Institute of Mathematics, Kiev, 1977, 104–114.
12. ———, *Tame and wild matrix problems.* In: Representations and Quadratic Forms, Institute of Mathematics, Kiev, 1979, 39–74. (English translation: Amer. Math. Soc. Transl. **128**(1986), 31–55.)
13. ———, *Representations of bocses and algebras.* Contemp. Math. **131**(1992), Part 2, 301–316.
14. Y. A. Drozd and S. A. Ovsienko, *Coverings of tame boxes and algebras.* Preprint MPI 00-26, Max-Plank-Institut für Mathematik, Bonn, 2000.
15. P. Gabriel, L. A. Nazarova, A. V. Roiter, V. V. Sergejchuk and D. Vossiek, *Tame and wild subspace problems.* Ukrainian Math. J. **45**(1993), 313–352.
16. C. Geiss and J. A. de la Peña, *An interesting family of algebras.* Arch. Math. **60**(1993), 25–35.
17. I. M. Gelfand and V. A. Ponomarev, *Remarks on the classification of a pair of commuting linear transformations in a finite-dimensional space.* Funkts. Anal. Prilozh. (4) **3**(1969), 81–82.
18. M. M. Kleiner and A. V. Roiter, *Representations of differential graded categories.* In: Representations of Algebras, Lecture Notes in Math. **488**, Springer-Verlag, Berlin, 1975, 316–339.
19. S. MacLane, *Homology.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
20. Y. V. Roganov, *Heredity of the tensor algebra and semi-heredity of the completed tensor algebra.* Dopovidi Akad. Nauk Ukr. SSR. Ser. A, **4**(1977), 410–413.

Kiev Taras Shevchenko University
Department of Mechanics and Mathematics
01033 Kiev
Ukraine
email: yuriy@drozd.org

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (4), 2001 pp. 126–129

ANTI-MORPHISME INVOLUTIF ET ALGÈBRES p-BANACH HERMITIENNES

A. EL KINANI ET A. IFZARNE

Présenté par Vlastimil Dlab, FRSC

ABSTRACT. We consider p-Banach algebras A endowed with an involutive anti-morphism $x \mapsto x^*$, *i.e.*, a vector involution such that $(xy)^* = x^*y^*$, for every $x, y \in A$. We show that if such algebras are hermitian, then they are commutative modulo their Jacobson radicals.

RÉSUMÉ. Nous considérons des algèbres p-Banach munies d'un anti-morphisme involutif $x \mapsto x^*$, *i.e.*, une involution d'espace vectoriel telle que $(xy)^* = x^*y^*$, pour tous $x, y \in A$. Nous montrons que si A est hermitienne, alors A est commutative modulo son radical de Jacobson.

Un anti-morphisme involutif, sur une algèbre complexe A, est une involution d'espace vectoriel $x \mapsto x^*$ [1] telle que $(xy)^* = x^*y^*$, pour tous $x, y \in A$. Soient A une algèbre complexe et $\|.\|_p$, $0 < p \leq 1$, une p-norme d'espace vectoriel sur A [7]. On dit que $\|.\|_p$ est une p-norme d'algèbre si $\|xy\|_p \leq \|x\|_p \|y\|_p$, pour tous $x, y \in A$. Une algèbre est dite p-normée si elle est munie d'une p-norme d'algèbre. Signalons que les algèbres p-normées considérées ici ne sont pas nécéssairement complètes comme c'est le cas pour W. Zelazko dans [7]. Ici une algèbre p-normée complète sera dite p-Banach. Soit $(A, \|.\|_p)$, $0 < p \leq 1$, une algèbre p-Banach complexe munie d'un anti-morphisme involutif $x \mapsto x^*$. Un élément a de A est dit hermitien (resp. normal) si $a = a^*$ (resp. $a^*a = aa^*$). On désigne par $H(A)$ (resp. $N(A)$) l'ensemble des éléments hermitiens (resp. normaux) de A. Une algèbre p-Banach complexe munie d'un anti-morphisme involutif est dite hermitienne si le spectre de tout élément hermitien est réel.

Dans cet article, nous montrons que si $(A, \|.\|_p)$, $0 < p \leq 1$, est une algèbre p-Banach hermitienne, alors $A/\operatorname{Rad}(A)$ est commutative. Comme conséquence, nous obtenons que toute algèbre p-Banach complexe $(A, \|.\|_p)$ munie d'un anti-morphisme involutif $x \mapsto x^*$ telle que $\|x\|_p^2 \leq c\|x^*x\|_p$, pour tout $x \in A$, où $c > 0$ est une constante positive, est en fait une C^*-algèbre pour une norme équivalente.

Dans toute la suite, $\rho(a) = \sup\{|\lambda| : \lambda \in \operatorname{Sp}(a)\}$ désignera le rayon spectral de a, où $\operatorname{Sp}(a)$ est le specte de a. Rappelons aussi que le théorème de Johnson [4], qui assure la continuité de l'involution dans les algèbres de Banach involutives

Reçu par les éditeurs le 7 août, 2000.
Classification (AMS) par sujet : 16W20, 46K05.
Mots clés: algèbre p-Banach, anti-morphisme involutif, algèbre hermitienne, C^*-algèbre.

semi-simples, reste valable pour les algèbres p-Banach semi-simples munies d'un anti-morphisme involutif.

Le lemme suivant nous sera utile par la suite.

LEMME 1. *Soit* $(A, \|.\|_p)$, $0 < p \leq 1$, *une algèbre p-Banach complexe unifère munie d'un anti-morphisme involutif* $x \mapsto x^*$. *Alors* A *est hermitienne si, et seulement si,* $e + ih$ *est inversible pour tout* $h \in H(A)$.

PREUVE. Si A est hermitienne, il est clair que $e + ih$ est inversible, pour tout $h \in H(A)$. Pour la réciproque, soient $x \in H(A)$ et $\alpha + i\beta \in \mathrm{Sp}_A(x)$, où $\alpha, \beta \in R$. Alors $h = x - \alpha e \in H(A)$ et $i\beta \in \mathrm{Sp}_A(h)$. Si $\beta \neq 0$, alors $e + i(\beta^{-1}h)$ serait non inversible; contradiction.

Une algèbre de Banach complexe munie d'un anti-morphisme involutif $x \mapsto x^*$, hermitienne n'est pas nécessairement commutative. En effet, si A est une algèbre de Banach réelle non commutative et strictement réelle, *i.e.*, le spectre de tout élément de A est réel (par exemple $A = T_2(R)$, l'algèbre des matrices carrées triangulaires supérieures), alors sa complexifiée A_C est une algèbre de Banach complexe non commutative et hermitienne pour l'anti-morphisme involutif $a + ib \mapsto a - ib$. On est alors amené à chercher des exemples d'algèbres p-Banach munies d'un anti-morphisme involutif hermitiennes semi-simples et qui ne sont pas commutatives. En fait, comme nous allons le voir, de telles algèbres n'existent pas.

THÉORÈME 2. *Soit* $(A, \|.\|_p)$, $0 < p \leq 1$, *une algèbre p-Banach complexe munie d'un anti-morphisme involutif* $x \mapsto x^*$ *hermitienne. Alors* $A/\mathrm{Rad}(A)$ *est hermitienne et commutative.*

PREUVE. Quitte à considérer l'algèbre $A^1 = A \oplus C$ obtenue par adjonction d'une unité à A, munie de l'anti-morphisme involutif $x + \lambda \mapsto x^* + \bar{\lambda}$, $x \in A$, $\lambda \in C$, on peut supposer que A est unifère. Posons $B = A/\mathrm{Rad}(A)$ et s la surjection canonique de A sur B. On a $H(B) = s\big(H(A)\big)$. De plus,

$$\mathrm{Sp}_B\, s(h) = \mathrm{Sp}_A(h) \subset R, \quad \text{pour tout } h \in H(A).$$

Donc $A/\mathrm{Rad}(A)$ est hermitienne. Montrons maintenant que $A/\mathrm{Rad}(A)$ est commutative. Quitte à remplacer A par $A/\mathrm{Rad}(A)$, on peut supposer que l'algèbre est semi-simple. Donc l'anti-morphisme involutif $x \mapsto x^*$ est continu. Par conséquent l'algèbre réelle $H(A)$ est p-Banach telle que

$$\mathrm{Sp}_{H(A)}(h) = \mathrm{Sp}_A(h) \subset R, \quad \text{pour tout } h \in H(A).$$

Par ailleurs, en utilisant le théorème 3.10 de [7] et le fait que le quotient d'une algèbre p-Banach par un idéal primitif est une algèbre primitive, on montre que le résultat de Kaplansky (Theorem 4.8 de [5]) s'étend au cas p-Banach. Ainsi $H(A)/\mathrm{Rad}\big(H(A)\big)$ est commutative. Pour finir, montrons que $H(A)$ est semi-simple. Soient $h \in \mathrm{Rad}\big(H(A)\big)$ et $a = u + iv \in A$, avec $u, v \in H(A)$. Alors

$\rho(hu) = 0$ et $\rho(hv) = 0$. Par conséquent $\rho(ha) = 0$. En effet soit $\lambda = \alpha + i\beta \in \mathrm{Sp}(ha)$, avec $\alpha, \beta \in R$. Si $hu - \alpha$ est inversible, alors

$$ha - \lambda = (hu - \alpha)[e + i(hu - \alpha)^{-1}(hv - \beta)].$$

D'où, par le lemme 1, $e + i(hu - \alpha)^{-1}(hv - \beta)$ est inversible car

$$(hu - \alpha)^{-1}(hv - \beta) \in H(A);$$

contradiction. Donc $hu - \alpha$ est non inversible, *i.e.*, $\alpha \in \mathrm{Sp}(hu)$. D'où $\alpha = 0$. De même, on montre que $\beta = 0$. Ainsi $\rho(ha) = 0$ et par suite $h \in \mathrm{Rad}(A) = \{0\}$.

Comme conséquences, on obtient ce qui suit.

COROLLAIRE 3. *Soit $(A, \|.\|_p)$, $0 < p \leq 1$, une algèbre p-Banach complexe munie d'un anti-morphisme involutif $x \mapsto x^*$ et posons $|x| = \rho(xx^*)^{\frac{1}{2}}$, pour $x \in A$. Alors les assertions suivantes sont équivalentes*

1) *A est hermitienne.*

2) $|.|$ *est une semi-norme d'algèbre.*

3) $|.|$ *est sous-additive.*

4) $\rho(a) = |a|$, *pour tout* $a \in A$.

5) $\rho(a) \leq |a|$, *pour tout* $a \in A$.

6) $\mathrm{Sp}(a^*a) \subset R_+$, *pour tout* $a \in A$.

7) $a^*a + aa^* \geq 0$, *pour tout* $a \in A$.

PREUVE. Quitte à remplacer A par $A/\mathrm{Rad}(A)$, ce qui ne change pas le spectre et la fonction de Pták, on peut supposer que A est semi-simple. Par le théorème précédent, l'algèbre A est commutative. On est ainsi ramené au cas d'une involution d'algèbre hermitienne. Et on conclut par [6, 5.4 et le théorème 5.10].

On sait que toute algèbre de Banach complexe $(A, \|.\|)$, munie d'un anti-morphisme involutif $x \mapsto x^*$ telle que $\|x^*x\| = \|x\|^2$, pour tout $x \in A$, est une C^*-algèbre commutative [3]. Plus généralement, on a le résultat suivant.

COROLLAIRE 4. *Soit $(A, \|.\|_p)$, $0 < p \leq 1$, une algèbre p-Banach complexe munie d'un anti-morphisme involutif $x \mapsto x^*$, telle que $\|x\|_p^2 \leq c\|x^*x\|_p$, pour tout $x \in A$, où $c > 0$ est une constante positive. Alors A est une C^*-algèbre commutative pour une norme équivalente.*

PREUVE. Montrons tout d'abord que A est commutative et donc que l'anti-morphisme involutif $x \mapsto x^*$ est en fait une involution d'algèbre. Par hypothèse, on obtient

$$\|a^n\|_p^2 \leq c\|(a^*a)^n\|_p, \quad \text{pour tout } a \in N(A), n = 1, 2, \ldots .$$

D'où, par passage à la limite, $\rho(a)^2 \leq \rho(a^*a)$, pour tout $a \in N(A)$. Donc, comme dans le théorème 5.10 de [6], on montre que A est hermitienne. De plus $\mathrm{Rad}(A) = \{0\}$ vu que

$$\|h\|_p \leq c\rho(h)^p, \quad \text{pour tout } h \in H(A).$$

D'après le théorème précédent, A est commutative. Montrons que ρ est une norme de C^*-algèbre, sur A, équivalente à $\|.\|_p$. Soit $x \in A$. D'après ce qui précéde, on a $\|x\|_p^2 \leq c^2\rho(x)^{2p}$. De plus $\rho(x)^p \leq \|x\|_p$. Ainsi ρ est une norme d'algèbre, sur A, équivalente à $\|.\|_p$. D'où le résultat vu que $\rho(x)^2 = \rho(xx^*)$, pour tout $x \in A$.

REFERENCES

1. F. F. Bonsall and J. Duncan, *Complete normed algebras*. Ergebrise der Mathematik, Band **80**, Springer-Verlag, 1973.
2. A. El Kinani, A. Ifzarne et M. Oudadess, *Commutativité de certaines algèbres de Banach à anti-morphisme involutif*. Rev. Acad. Cienc. Zaragoza (2) **53**(1998), 165–173.
3. A. El Kinani et M. Oudadess, *Involution généralisée et structure de C^*-algèbre*. Rev. Acad. Cienc. Zaragoza (2) **52**(1997), 15–16.
4. B. E. Johnson, *The uniqueness of the complete norm topology*. Bull. Amer. Math. Soc. **73**(1967), 537–539.
5. I. Kaplansky, *Normed algebras*. Duke. Math. J. **16**(1949), 399–418.
6. V. Ptak, *Banach algebras with involution*. Manuscripta Math. **6**(1972), 245–290.
7. W. Zelazko, *Selected topics on topological algebras*. Lecture notes series **31**, Aarhus, 1971.

Département de Mathématiques
École Normale Supérieure Takaddoum
B.P. 5118
Rabat 10105
Morocco

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (4), 2001 pp. 130–135

A NOTE ON ARITHMETIC TOPOLOGY

NIRANJAN RAMACHANDRAN

Presented by M. Ram Murty, FRSC

RÉSUMÉ. Mazur, Manin et autres ont indiqué des analogies frappantes entre les corps de nombres et les variétés de dimension 3. On montre ici que l'analogue arithmetique de la conjecture du Poincaré est fausse.

0. **Introduction.** B. Mazur and Yu. I. Manin (see [7], [9]) have indicated deep analogies between number fields and 3-manifolds. This analogy has been baptized *Arithmetic Topology* by A. Reznikov [13], [14]. Let us briefly review this analogy (*cf.* the dictionary by him and M. Kapranov in §14 of [14]). We shall write **A** for Arithmetic and **T** for topology. The torsion subgroup (resp. the maximal torsion-free quotient) of any abelian group N is denoted by $\mathrm{tors}(N)$ (resp. N/tors).

T1. A closed connected 3-manifold M.
A1. $X = \operatorname{Spec}\mathcal{O}$, the ring of algebraic integers of a number field L.
T2. S^3, the 3-dimensional sphere.
A2. $\operatorname{Spec}\mathbb{Z}$, the ring of integers.
T3. $\mathrm{tors}(H_1(M;\mathbb{Z}))$.
A3. $\mathrm{Pic}(X)$, the ideal class group of $\mathcal{O}$.
T4. $H_1(M;\mathbb{Z})/\mathrm{tors}$.
A4. $\mathcal{O}^*/\mathrm{tors}$; here $\mathcal{O}^*$ denotes the units of $\mathcal{O}$.
T5. The fundamental group $\pi_1(M)$ of M.
A5. The Galois group G_L^{ur} of the maximal extension of L (contained in a fixed algebraic closure $\bar{L}$ of L) which is unramified at all primes (both finite and infinite); this is not always equal to the étale $\pi_1(X)$.
T6. The group $\pi_0(\mathrm{Diff}(M))$ of connected components of the group $\mathrm{Diff}(M)$ of diffeomorphisms of M.
A6. The group $K_1(X)$ of X.

Since M is a closed manifold, there is no essential difference between cohomology, cohomology with compact support, and homology of M (Poincaré duality). The singular (co)homology groups of M vanish beyond dimension three. The scheme X is analogous to a 3-dimensional manifold since X is a smooth scheme and its étale cohomology groups $H^i(X;\mathbb{G}_m)$ are non-zero only for $i \leq 3$.[1]

Received by the editors January 5, 2001.
AMS subject classification: 11R37, 57M40.

[1] Strictly speaking, this is true only up to 2-torsion (see Section 1).

Let us recall that a closed orientable connected 3-manifold is said to be an integral homology 3-sphere if $H_1(M;\mathbb{Z}) = 0$, a homotopy 3-sphere if $\pi_1(M) = 0$, and a rational homology 3-sphere if $H_1(M;\mathbb{Q}) = 0$.

The purpose of this note is to prove the following:[2]

THEOREM 1. *Every integral homology 3-sphere (in number theory) is a homotopy 3-sphere.*

This is in utter contrast to the situation in topology. First, the topological analog of Theorem 1 is false (pp. 244–246 of [15]). Second, there are infinitely many (mutually non-homeomorphic) integral homology 3-spheres. The Poincaré conjecture [12] states that every homotopy 3-sphere is homeomorphic to S^3. (Consequently, there should be only one homotopy 3-sphere, up to homeomorphism.)

From Theorem 1, we shall deduce that there are several mutually distinct integral homology 3-spheres (in number theory) and that the arithmetic analog of the Poincaré conjecture is *false* (Corollary 9).

1. Étale cohomology of number fields. The étale cohomology $H^*(X;\mathbb{G}_m)$ of X with coefficients in the sheaf $\mathbb{G}_m$ is the guiding light of arithmetic topology. Good references for this are [1], [6] and [8, (II, §2)]. Let r_1 and $2r_2$ be the number of real and complex embeddings of L. We shall denote these infinite primes of L by v. The completion of L with respect to a prime v is denoted L_v. The groups $H^*(X;\mathbb{G}_m)$ can be calculated explicitly (Proposition 2.1 of [8]):

$$H^0(X;\mathbb{G}_m) = \mathcal{O}^*, \quad H^1(X;\mathbb{G}_m) = \mathrm{Pic}(X) \tag{1}$$

$$H^2(X;\mathbb{G}_m) = \begin{cases} 0 & r_1 = 0 \\ (\mathbb{Z}/2\mathbb{Z})^{r_1-1} & r_1 > 0 \end{cases} \tag{2}$$

$$H^3(X;\mathbb{G}_m) = \mathbb{Q}/\mathbb{Z} \tag{3}$$

$$H^n(X;\mathbb{G}_m) = \bigoplus_{v\,\mathrm{real}} H^n(L_v;\mathbb{G}_m) \quad (n \geq 4). \tag{4}$$

In the above, note that, for v real, $L_v = \mathbb{R}$ and $H^*(L_v;\mathbb{G}_m) = H^*(G;\mathbb{G}_m)$ where $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. We recall the identities

$$H^{2m}(G;\mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}, \quad H^{2m-1}(G;\mathbb{G}_m) = 0 \quad (\text{for } m > 0). \tag{5}$$

With the étale cohomology in hand, we can now turn to our main question.

2. When is X an integral homology 3-sphere? As remarked earlier, we shall use cohomology (!) to define an integral homology sphere in arithmetic:

DEFINITION 2. We say that X is an integral homology 3-sphere if it satisfies

(a) $H^i(X;\mathbb{G}_m) = 0$ for $i \neq 0, 3$[3].

[2] Reznikov recently informed me that he and M. Kapranov also had independently discovered this result.

[3] See p. 137 of [9].

(b) $H^0(X;\mathbb{G}_m)$ is torsion.

THEOREM 3. *X is an integral homology 3-sphere if and only if L is an imaginary quadratic field of class number one.*

PROOF. By Dirichlet's theorem, $H^0(X;\mathbb{G}_m) = \mathcal{O}^*$ is a finitely generated abelian group of rank r_1+r_2-1. Condition (b) says that $r_1+r_2=1$. So we find that L must be an imaginary quadratic field or $\mathbb{Q}$. Let us turn now to (a). When $i=1$, it means that the ideal class group Pic(X) of $\mathcal{O}$ must be trivial, *i.e.*, L has class number one. If L has a real embedding ($r_1>0$), then $H^{2m}(X;\mathbb{G}_m)$ will be nonzero for all $m>1$. Therefore, (a) says that L is a totally complex field of class number one. We have proved our claim. ∎

The imaginary quadratic fields of class number one have been the subject of intense study. The conjecture of Gauss (proved by Baker, Heegner and Stark [2]) states that there are exactly nine of them. These are the fields $\mathbb{Q}(\sqrt{-d})$ where d runs over the set $\{1,2,3,7,11,19,43,67,163\}$. Thus, there are exactly *nine* distinct integral homology 3-spheres in number theory.

DEFINITION 4. X is a rational homology 3-sphere if it satisfies

(a′) $H^*(X;\mathbb{G}_m)\otimes\mathbb{Q}=0$.

PROPOSITION 5. *X is a rational homology 3-sphere exactly when L is an imaginary quadratic field or $\mathbb{Q}$.*

PROOF. Since $H^i(X;\mathbb{G}_m)$ is torsion for $i\neq 0$, the restriction comes from the requirement $H^0(X;\mathbb{G}_m)\otimes\mathbb{Q}=0$ (see the proof of Theorem 3). ∎

COROLLARY 6. Spec $\mathbb{Z}$ *is not an integral homology 3-sphere. It is, however, a rational homology 3-sphere.*

3. Homotopy 3-spheres.

We need an arithmetic analog of a homotopy 3-sphere in order to discuss a version of the Poincaré conjecture. We say that X is a homotopy 3-sphere if the group $G_L^{ur}=0$; *cf.* A4. So our Theorem 1 translates to:

THEOREM 7. *Let L be an imaginary quadratic field of class number one. Then L has no nontrivial finite unramified extension.*

PROOF. It can be found in [19, Exercise 11.2], [20, §2]. The ingredients are:

(i) the smallest perfect group has order 60 (it is the alternating group A_5 of degree five).

(ii) A. Odlyzko's [10], [11] bounds on discriminants of number fields (see Theorem 11.20 and Table on p. 223 of [19]). ∎

COROLLARY 8. *In number theory, there are exactly ten rational homology 3-spheres which are homotopy 3-spheres.*

By Minkowski's theorem, Spec $\mathbb{Z}$ is simply connected. So, it is also a homotopy 3-sphere.

COROLLARY 9. *In number theory, there exist (at least) ten mutually non-isomorphic homotopy 3-spheres.*

The previous corollary shows that the arithmetic analog of the Poincaré conjecture is *false.*

REMARK 10.
(i) Each of the integral homology 3-spheres has an involution with one fixed point.[4] The quotient by the involution is always Spec $\mathbb{Z}$.
(ii) (B. Mazur) Every X comes equipped with a canonical projection π (the structure morphism) to Spec $\mathbb{Z}$. One is led to enriched 3-manifolds, namely, pairs (M, π) of a 3-manifold M and a map $\pi: M \to S^3$. The enriched Poincaré conjecture is false: there exist many self-maps of S^3. This (and the analogy with curves over finite fields) shall be pursued elsewhere.

4. **Variants.** Since X is not compact (it is an affine scheme), there is a difference between homology, cohomology, and cohomology with compact support of X. Here we consider the following two variants which could be used in Definitions 2 and 4.

4.1. *Étale cohomology with compact support.* The groups $H_c^*(X; \mathbb{G}_m)$ are the right objects for duality theorems; they are defined in [8, pp. 207–208]. From Proposition 2.6 and Remark 2.8a of [8], we obtain
(a) $H_c^i(X; \mathbb{G}_m) = 0$ for $i \neq 0, 1, 3$,
(b) $H_c^3(X; \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$,
(c) the exact sequence

$$0 \to H_c^0(X; \mathbb{G}_m) \to \mathcal{O}^* \to \bigoplus_{v \text{ real}} L_v^*/(L_v^*)^2 \to H_c^1(X; \mathbb{G}_m) \to \mathrm{Pic}(X) \to 0.$$

Hence $H_c^0(X; \mathbb{G}_m)$ is the group of *totally positive* units of $\mathcal{O}$ and $H_c^1(X; \mathbb{G}_m)$ is the *narrow* class group of $\mathcal{O}$.

We should view the étale fundamental group $\pi_1(X)$ of the scheme X as the *narrow* fundamental group of X since its abelianization $\pi_1^{ab}(X)$ is $H_c^1(X; \mathbb{G}_m)$. Utilizing $H_c^*(X)$ as our defining cohomology theory, we obtain only one new integral homology sphere, namely, Spec $\mathbb{Z}$. Theorem 1, Proposition 5 and Corollary 9 remain true in this definition.

4.2. *Artin-Verdier theory.* Consider the space $X_\infty = X(\mathbb{C})/G$ where $X(\mathbb{C})$ is the set of points of X with values in $\mathbb{C}$ and $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ operates by complex conjugation. M. Artin and J.-L. Verdier [1] have defined an étale topology on the space $\bar{X} = X \coprod X_\infty$. The étale fundamental group of $\bar{X}$ is the group G_L^{ur} of **A4**.

They define $\mathbb{G}_m$ on $\bar{X}$ to be $j_*\mathbb{G}_m$ where $j: X \hookrightarrow \bar{X}$ denotes the open immersion. Theorem 2.4 of [1] shows that the étale cohomology groups $H^i(\bar{X}; \mathbb{G}_m)$

[4] Note that S^3 is a 2-fold branched covering of S^3 in an essentially unique manner [18].

are zero for $i \neq 0, 1, 3$, and that $H^0(\bar{X}; \mathbb{G}_m) = \mathcal{O}^*$, $H^1(\bar{X}; \mathbb{G}_m) = \mathrm{Pic}(X)$ and $H^3(X; \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$.

In this variant as well, it turns out that we acquire only one new integral homology sphere, namely, $\mathrm{Spec}\,\mathbb{Z}$. Theorem 1, Proposition 5 and Corollary 9 remain true in this definition as well. Perhaps the Artin-Verdier definition is the most appropriate.

REMARK 11. The three different cohomology theories agree for totally complex fields ($r_1 = 0$). For these fields, it is also the case that the étale fundamental group $\pi_1(X)$ is the same as G_L^{ur}.

REMARK 12. We wish to contribute a few more points to the analogy.

T7. $\pi_0(\mathrm{Diff}(S^3)) = \mathbb{Z}/2\mathbb{Z}$ (see [3]).

A7. $K_1(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ since $K_1(\mathcal{O}) = \mathcal{O}^*$.

T8. The ends of a (possibly non-compact) manifold.

A8. X_∞ (see 4.2).

Mazur and Manin have suggested that primes correspond to knots. A knot is given by a closed immersion $S^1 \hookrightarrow M$ and a finite prime is given by a closed immersion $\mathrm{Spec}\,\mathbb{F} \hookrightarrow X$ of a finite field $\mathbb{F}$.

T9. S^1 is an Eilenberg-Mac Lane space $K(\mathbb{Z}, 1)$.

A9. For any finite field $\mathbb{F}$, the scheme $\mathrm{Spec}\,\mathbb{F}$ is an Eilenberg-Mac Lane space $K(\hat{\mathbb{Z}}, 1)$ where $\hat{\mathbb{Z}}$ is the profinite completion of the group $\mathbb{Z}$.

Let T denote the tubular neighbourhood of a knot K in M. Let $\mathcal{O}_\mathfrak{p}$ (resp. $L_\mathfrak{p}$) denote the local ring (resp. field) corresponding to a finite prime $\mathfrak{p}$ of X. As usual, $\mathbb{N}\mathfrak{p}$ is the cardinality of the residue field $\mathbb{F}_\mathfrak{p} = \mathcal{O}/\mathfrak{p}$.

T10. The surface $S = \partial T$ (homotopy equivalent to $T - K$).

A10. $\mathrm{Spec}\, L_\mathfrak{p} = \mathrm{Spec}\,\mathcal{O}_\mathfrak{p} - \mathrm{Spec}\,\mathbb{F}_\mathfrak{p}$.

T11. $\pi_1(S)$ has a presentation $\langle x, y \mid xyx^{-1} = y \rangle$.

A11. The Galois group[5] of the maximal tamely ramified extension [5, p. 173] of $L_\mathfrak{p}$ has a presentation $\langle x, y \mid xyx^{-1} = y^{\mathbb{N}\mathfrak{p}} \rangle$.

Since there is no wild ramification in characteristic zero, it is the tame Galois group of $L_\mathfrak{p}$ that enters into the analogy. Should we consider the residue field of a knot to be $\mathbb{F}_1$ (the field with one element!) [17]?

T12. Every closed 3-manifold M is (non-canonically) a branched covering of S^3 [4].

A12. Every X is (canonically) a branched covering of $\mathrm{Spec}\,\mathbb{Z}$.

To conclude, in number theory, there are nine legitimate (= insensitive to the variation of definition) integral homology 3-spheres and ten homotopy 3-spheres. Even if $\mathrm{Spec}\,\mathbb{Z}$ is adopted as the analog of S^3 (and it should be), the Poincaré conjecture is still *false* in number theory.

ACKNOWLEDGEMENTS. I have benefited greatly from discussions with A. Giacobbe, J. Gordon, I. Kofman, M. Marcolli, A. Sikora, and L. Washington.

[5] This group has cohomological dimension two [16, p. 86].

REFERENCES

1. M. Artin and J.-L. Verdier, *Seminar on étale cohomology of number fields.* Woods Hole, 1964.
2. D. Goldfeld, *Gauss's class number problem for imaginary quadratic fields.* Bull. Amer. Math. Soc. (N.S.) (1) **13**(1985), 23–37.
3. A. Hatcher, *A proof of a Smale conjecture,* Diff(S^3) $\simeq$ O(4). Ann. of Math. (2) **117**(1983), 553–607.
4. H. M. Hilden and J. M. Montesinos, *A method of constructing 3-manifolds and its application to the computation of the μ-invariant.* In: Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Proc. Sympos. Pure Math. **XXXII**, Amer. Math. Soc., Providence, R.I., 1978, Part 2, 61–69.
5. H. Koch, *Algebraic Number Theory.* Springer-Verlag, Berlin, 1997.
6. B. Mazur, *Notes on étale cohomology of number fields.* Ann. Sci. École Norm. Sup. (4) **6**(1974), 521–552.
7. ———, *The theme of p-adic variation.* In: Mathematics: Frontiers and Perspectives (eds. V. Arnold, M. Atiyah, P. Laxa and B. Mazur), Amer. Math. Soc., 2000, 433-459.
8. J. S. Milne, *Arithmetic Duality Theorems.* Perspectives in Mathematics **1**, Academic Press, Inc., Boston, Mass., 1986.
9. C. McMullen, *From dynamics on surfaces to rational points on curves.* Bull. Amer. Math. Soc. (N.S.) (2) **37**(2000), 119–140.
10. A. M. Odlyzko, *Some analytic estimates of class numbers and discriminants.* Invent. Math. (3) **29**(1975), 275–286.
11. ———, *Lower bounds for discriminants of number fields.* Acta Arith. (3) **29**(1976), 275–297.
12. H. Poincaré, *Cinquième complément à l'Analysis Situs.* Rend. Circ. Mat. Palermo. **18**(1904), 45–110.
13. A. Reznikov, *Three manifolds class field theory (homology of coverings for a nonvirtually b_1-positive manifold).* Selecta Math. (N.S). (3) **3**(1997), 361–399.
14. ———, *Hakenness and b_1.* Preprint math.DG/9903064 at xxx.lanl.gov.
15. D. Rolfsen, *Knots and Links.* Publish or Perish. Berkeley, 1976.
16. J.-P. Serre, *Galois Cohomology.* Springer-Verlag, Berlin, 1997.
17. C. Soulé, *On the field with one element.* Preprint available at http://www.ihes.fr/PREPRINTS/M99/Resu/resu-M99-55.html.
18. F. Waldhausen, *Über Involutionen der 3-Sphäre.* Topology **8**(1969), 81–91.
19. L. Washington, *Introduction to Cyclotomic Fields.* Second edition, Graduate Texts in Math. **83**, Springer-Verlag, New York, 1997.
20. K. Yamamura, *The maximal unramified extensions of the imaginary quadratic number fields with class number two.* J. Number Theory (1) **60**(1996), 42–50.

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (4), 2001 pp. 136–140

ALGEBRAS OVER THE FOCK SPACE

YUN GAO AND NAIHUAN JING

Presented by Robert V. Moody, FRSC

ABSTRACT. We find that the Fock space for the affine Lie algebra $\widehat{\mathfrak{gl}}_N$ allows both the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ and the quantum toroidal algebra $U_q(\mathfrak{sl}_{N,\mathrm{tor}})$ actions.

RÉSUMÉ. Nous trouvons pous l'espace de Fock de l'algèbre de Lie affine $\widehat{\mathfrak{gl}}_N$ accepte deux actions: une de l'algèbre affine quantique $U_q(\widehat{\mathfrak{gl}}_N)$, l'autre de l'algèbre torique quantique $U_q(\mathfrak{sl}_{N,\mathrm{tor}})$.

A quantum Kac-Moody algebra $U_q(\mathfrak{g})$ is a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Kac-Moody Lie algebra $\mathfrak{g}$. Lusztig [L] has shown that the character formula for dominant highest weight representations is preserved when $U_q(\mathfrak{g})$ is deformed to $U(\mathfrak{g})$. Our purpose is to give an irreducible vertex representation for the newly developed quantum toroidal algebra $U_q(\mathfrak{sl}_{N,\mathrm{tor}})$. However, this leads us to discover one surprising fact: the irreducible basic $\widehat{\mathfrak{gl}}_N$-module allows an irreducible $U_q(\widehat{\mathfrak{gl}}_N)$ action. Moreover, each weight space as both $U_q(\widehat{\mathfrak{gl}}_N)$-module and $\widehat{\mathfrak{gl}}_N$-module coincides (does not depend on q!). Therefore, this phenomena visualizes the above mentioned Lusztig theorem in the case $\mathfrak{g} = \widehat{\mathfrak{gl}}_N$.

Quantum toroidal algebras were introduced by Ginzburg-Kapranov-Vasserot [GKV] in the study of the Langlands reciprocity for algebraic surfaces. These algebras are quantized analogues for toroidal Lie algebras of Moody-Rao-Yokonuma [MRY].

A full description of this work is in preparations. Preprints may be obtained from the authors.

We always assume that the complex number q is generic and N is a positive integer with $N \geq 3$. Let d be a nonzero complex number. The quantum toroidal algebra $U_q(\mathfrak{sl}_{N,\mathrm{tor}})$ is the unital associative algebra over $\mathbb{C}$ generated by $e_{i,k}$, $f_{i,k}$, $h_{i,l}$, $k_i^{\pm 1}$, where $i = 0, 1, \ldots, N-1$, $k \in \mathbb{Z}$, $l \in \mathbb{Z} \setminus \{0\}$, and the central elements

Received by the editors March 21, 2001.

The first author gratefully acknowledges the grant support from the Natural Sciences and Engineering Research Council of Canada. The second author gratefully acknowledges the support from NSF Grant DMS-9970493.

AMS subject classification: 17B37, 17B67, 17B69.

$c^{\pm 1}$. The relations are expressed in terms of the formal series

$$e_i(z) = \sum_{k \in \mathbb{Z}} e_{i,k} z^{-k}, \quad f_i(z) = \sum_{k \in \mathbb{Z}} f_{i,k} z^{-k},$$

and

$$k_i^{\pm}(z) = k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{k=1}^{\infty} h_{i,\pm k} z^{\mp k}\right),$$

as follows

$$k_i k_i^{-1} = k_i^{-1} k_i = cc^{-1} = 1,$$
$$[k_i^{\pm}(z), k_j^{\pm}(w)] = 0,$$
$$\theta_{-a_{ij}}\left(c^2 d^{-m_{ij}} \frac{w}{z}\right) k_i^+(z) k_j^-(w) = \theta_{-a_{ij}}\left(c^{-2} d^{-m_{ij}} \frac{w}{z}\right) k_j^-(w) k_i^+(z),$$
$$k_i^{\pm}(z) e_j(w) = \theta_{\mp a_{ij}}\left(c^{-1} d^{\mp m_{ij}} \left(\frac{w}{z}\right)^{\pm 1}\right) e_j(w) k_i^{\pm}(z),$$
$$k_i^{\pm}(z) f_j(w) = \theta_{\pm a_{ij}}\left(c d^{\mp m_{ij}} \left(\frac{w}{z}\right)^{\pm 1}\right) f_j(w) k_i^{\pm}(z),$$
$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left(\delta\left(c^{-2} \frac{z}{w}\right) k_i^+(cw) - \delta\left(c^2 \frac{z}{w}\right) k_i^-(cz)\right),$$
$$(d^{m_{ij}} z - q^{a_{ij}} w) e_i(z) e_j(w) = (q^{a_{ij}} d^{m_{ij}} z - w) e_j(w) e_i(z),$$

$$(q^{a_{ij}} d^{m_{ij}} z - w) f_i(z) f_j(w) = (d^{m_{ij}} z - q^{a_{ij}} w) f_j(w) f_i(z),$$
$$\{e_i(z_1) e_i(z_2) e_j(w) - (q + q^{-1}) e_i(z_1) e_j(w) e_i(z_2)$$
$$+ e_j(w) e_i(z_1) e_i(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0, \quad \text{if } a_{ij} = -1,$$

$$\{f_i(z_1) f_i(z_2) f_j(w) - (q + q^{-1}) f_i(z_1) f_j(w) f_i(z_2)$$
$$+ f_j(w) f_i(z_1) f_i(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0, \quad \text{if } a_{ij} = -1,$$
$$[e_i(z), e_j(w)] = [f_i(z), f_j(w)] = 0, \quad \text{if } a_{ij} = 0,$$

where $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$ is the δ-function and

$$\theta_m(z) = \frac{q^m z - 1}{z - q^m} \in \mathbb{C}[[z]]$$

is understood as the Taylor series expansion, $A = (a_{ij})$ is the Cartan matrix of affine type $A_{N-1}^{(1)}$ and $M = (m_{ij})$ is a skew symmetric matrix given by $m_{ij} = \delta_{i,j+1} - \delta_{j,i+1}$ for $1 \le i, j \le N-1$, and $\delta_{0N} = \delta_{N0} = 1$.

The reason to call $U_q(sl_{N,\mathrm{tor}})$ the quantum toroidal algebra is that it is a two-parameter deformation of the enveloping algebra of the toroidal Lie algebra $\hat{sl}_N([x^{\pm 1}, y^{\pm 1}])$ (the universal central extension of the double loop algebra $sl_N([x^{\pm 1}, y^{\pm 1}])$, see [MRY]).

Let $P = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_N$ be a rank N free abelian group provided with a $\mathbb{Z}$-bilinear form $(\cdot,\cdot)$ defined by $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $1 \le i, j \le N$. Let $Q = \mathbb{Z}(\epsilon_1 - \epsilon_2) \oplus \cdots \oplus \mathbb{Z}(\epsilon_{N-1} - \epsilon_N)$ be the rank $(N-1)$ free subgroup of P. Then

$$\Delta = \{\alpha \in Q : (\alpha, \alpha) = 2\} = \{\epsilon_i - \epsilon_j : 1 \le i \neq j \le N\}$$

is the root system of type A_{N-1}.

Let $\varepsilon: Q \times Q \to \{\pm 1\}$ be a bimultiplicative function such that $\varepsilon(\alpha,\alpha) = (-1)^{\frac{1}{2}(\alpha,\alpha)}$, for $\alpha \in Q$. Let $\mathbb{C}[Q] = \sum \oplus \mathbb{C}e^{\alpha}$ be the group algebra of Q. For $\beta \in Q$, define $e_\beta \in \operatorname{End}\mathbb{C}[Q]$ by $e_\beta e^\alpha = \varepsilon(\beta,\alpha)e^{\alpha+\beta}$, for $\alpha \in Q$. Also, for $\beta \in H = Q \otimes_{\mathbb{Z}} \mathbb{C}$, define $\beta(0) \in \operatorname{End}\mathbb{C}[Q]$ by $\beta(0)e^\alpha = (\beta,\alpha)e^\alpha$, for $\alpha \in Q$.

Next let $\epsilon_i(n)$ be the generators of the Heisenberg algebra $\mathcal{H}$, $1 \le i \le N$, $n \in \mathbb{Z} \setminus \{0\}$, subject to relations:

$$[\epsilon_i(m), \epsilon_j(n)] = m\delta_{ij}\delta_{m+n,0}C.$$

Let

$$S(\mathcal{H}^-) = \mathbb{C}[\epsilon_i(n) : 1 \le i \le N, n \in -\mathbb{Z}_+]$$

denote the symmetric algebra of $\mathcal{H}^-$ and set

$$V_Q = S(\mathcal{H}^-) \otimes \mathbb{C}[Q].$$

The operator $z^\alpha \in (\operatorname{End}\mathbb{C}[Q])[z, z^{-1}]$ is defined as $z^{\alpha(0)} = \exp\big(\alpha(0)\ln z\big)$.

Set $\epsilon_{i+N} = \epsilon_i$, for $i \in \mathbb{Z}$. Let p be a non-zero complex number. For $r, i, j \in \mathbb{Z}$, we define the vertex operator $X_{ij}(r,z)$ as follows.

$$X_{ij}(r,z) = \;:\exp\Big(-\sum_{n\neq 0} \frac{\big(\epsilon_i(n) - p^{-rn}q^{(i-j)|n|}\epsilon_j(n)\big)}{n} z^{-n}\Big):$$
$$e_{\epsilon_i-\epsilon_j} z^{\epsilon_i-\epsilon_j+\frac{(\epsilon_i-\epsilon_j,\epsilon_i-\epsilon_j)}{2}} p^{-r\epsilon_j-\frac{(\epsilon_j,\epsilon_i-\epsilon_j)}{2}r}$$

Let

$$\tilde{\mathfrak{gl}}_N = \widehat{\mathfrak{gl}}_N \oplus \mathbb{C}D = \mathfrak{gl}_N(\mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}C \oplus \mathbb{C}D$$

be the affine Lie algebra, where C is the central element and D is the degree operator.

$$\mathfrak{h} = H \oplus \mathbb{C}C \oplus \mathbb{C}D$$

is its Cartan subalgebra. It is known that (see [F]) V_Q is an irreducible $\tilde{\mathfrak{gl}}_N$ module.

Next, for $r, i, j \in \mathbb{Z}$, we define

$$u_{ij}(r,z) = q^{(j-i)(\epsilon_i-\epsilon_j)}$$
$$\cdot \exp\Big(\sum_{n\geq 1} \frac{q^{(j-i)n} - q^{(i-j)n}}{n} \big(q^{\frac{j-i}{2}n}\epsilon_i(n) - p^{-nr}q^{\frac{i-j}{2}n}\epsilon_j(n)z^{-n}\big)\Big)$$

$$v_{ij}(r,z)= q^{(i-j)(\epsilon_i-\epsilon_j)} \cdot \exp\Big(\sum_{n\geq 1}\frac{q^{(i-j)n}-q^{(j-i)n}}{n}\big(q^{\frac{i-j}{2}n}\epsilon_i(-n)-p^{nr}q^{\frac{i-j}{2}n}\epsilon_j(-n)\big)z^n\Big).$$

From now on we assume that $p=d^{-N}$.

THEOREM. *The linear map* π *given by*

$$\pi(e_i(z))=X_{i,i+1}(0,p^{\frac12}d^iz),\quad \pi(f_i(z))=X_{i+1,i}(0,p^{\frac12}d^iz),\quad i=1,2,\dots,N-1$$
$$\pi(e_0(z))=X_{01}(1,p^{-\frac12}z),\quad \pi(f_0(z))=X_{10}(-1,p^{\frac12}z),$$
$$\pi(k_i^+(z))=u_{i,i+1}(0,p^{\frac12}d^iz),\quad \pi(k_i^-(z))=v_{i,i+1}(0,p^{\frac12}d^iz),\quad i=1,2,\dots,N-1$$
$$\pi(k_0^+(z))=u_{01}(1,p^{-\frac12}z),\quad \pi(k_0^-(z))=v_{01}(1,p^{-\frac12}z),$$
$$\pi(c)=q^{\frac12},\pi(k_i^{\pm})=q^{\pm(\epsilon_i-\epsilon_{i+1})},\quad i=0,1,\dots,N-1$$

yields a representation of $U_q(sl_{N,\mathrm{tor}})$. *If* $pq^{\pm N}$ *is not a root of unity, then* V_Q *is an irreducible* $U_q(sl_{N,\mathrm{tor}})$ *module.*

Let $\mathcal{B}$ be the unital associative algebra generated by the coefficients of $X_{i,i+1}(0,z)$ and $X_{i+1,i}(0,z)$, for $1\leq i\leq N-1$ and $\epsilon_i(n)$, $q^{\pm\epsilon_i}$, $1\leq i\leq N$, $n\in\mathbb{Z}\setminus\{0\}$. In order to let V_Q have a weight space decomposition, we need to add the operator q^D to $U_q(\hat{gl}_N)$ and $\mathcal{B}$. Then we have:

PROPOSITION. $\mathcal{B}$ *is a homomorphic image of* $U_q(\hat{\mathfrak{gl}}_N)$. *Moreover,* V_Q *is an irreducible* $U_q(\hat{\mathfrak{gl}}_N)$*-module.*

As a $U_q(\hat{\mathfrak{gl}}_N)$-module, we have $V_Q=\bigoplus V^\mu$, where

$$V^\mu=\{v\in V_Q : q^h.v=q^{\mu(h)}v,\ \text{for } h\in\mathfrak{h}\}.$$

It is easy to see that $V^\mu=V_\mu$, where $V_\mu=\{v\in V_Q : h.v=\mu(h)v, \text{ for } h\in\mathfrak{h}\}$.

The proof for the Proposition is based on a new set of generators within the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ with the advantage that they are orthogonal (mutually commutative) in the usual sense. In [1] the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ was defined in the context of the quantum Yang-Baxter equation, which contains the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_N)$ in a canonical way.

REFERENCES

1. J. Ding and I. B. Frenkel, *Isomorphism of two realizations of quantum affine algebra* $U_q(\widehat{\mathfrak{gl}(n)})$. Comm. Math. Phys. **156**(1993), 277–300.
2. I. B. Frenkel, *Representations of affine Lie algebras, Hecke modular forms and Korteweg-De Vries type equations.* Lecture Notes in Math. **933**(1982), 71–110.
3. I. B. Frenkel and N. Jing, *Vertex representations of quantum affine algebras.* Proc. Nat. Acad. Sci. USA **85**(1988), 9373–9377.
4. Y. Gao, *Vertex operators arising from the homogeneous realization for* $\widehat{gl}_N$. Comm. Math. Phys. **211**(2000), 745–777.

5. V. Ginzburg, M. Kapranov and E. Vasserot, *Langlands reciprocity for algebraic surfaces.* Math. Res. Lett. **2**(1995), 147–160.
6. N. Jing, *Twisted vertex representations of quantum affine algebras.* Invent. Math. **102**(1990), 663–690.
7. G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras.* Adv. Math. **70**(1988), 237–249.
8. R. V. Moody, S. E. Rao and T. Yokonuma, *Toroidal Lie algebras and vertex representations.* Geom. Dedicata **35**(1990), 283–307.

Department of Mathematics and Statistics
York University
Toronto, Ontario
M3J 1P3
email: ygao@yorku.ca

Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205
USA
email: jing@math.ncsu.edu

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (4), 2001 pp. 141–147

ON APPROXIMATELY JENSEN-CONVEX AND WRIGHT-CONVEX FUNCTIONS

KAZIMIERZ NIKODEM AND ZSOLT PÁLES

Presented by J. Aczél, FRSC

ABSTRACT. The aim of this note is to characterize functions that differ from convex, Jensen-convex and Wright-convex functions in a bounded term.

RÉSUMÉ. L'objectif de ce travail est la caractérisation des fonctions qui diffèrent des fonctions convexes, des fonctions convexes de Jensen et celles de Wright par un terme limité.

1. Introduction. Let X be a real vector space, D be a convex subset of X and ε be a nonnegative constant. A function $f\colon D \to \mathbb{R}$ is said to be:

- *ε-convex* if
$$f\big(tx+(1-t)y\big) \le tf(x)+(1-t)f(y)+\varepsilon \quad \text{for } x,y\in D, t\in[0,1];$$
- *ε-Wright-convex* if
$$f\big(tx+(1-t)y\big)+f\big((1-t)x+ty\big) \le f(x)+f(y)+2\varepsilon \quad \text{for } x,y\in D, t\in[0,1];$$
- *ε-Jensen-convex* if
$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2}+\varepsilon \quad \text{for } x,y\in D.$$

If the function f admits the above properties for some $\varepsilon \ge 0$, then it is called *approximately* convex, Jensen-convex, and Wright-convex, respectively. One can observe that every ε-convex function is ε-Wright-convex and every ε-Wright-convex function is ε-Jensen-convex, but not conversely. In the case $\varepsilon = 0$ we get the standard definitions of convex, Wright-convex and Jensen-convex functions, respectively (*cf.* [23]). The classical Hyers-Ulam theorem [9] states that for every $n \in \mathbb{N}$, there is a constant c_n such that if D is a convex subset of $\mathbb{R}^n$ and $f\colon D \to \mathbb{R}$

Received by the editors April 12, 2001.

The research of the second author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant T-030082, by the Higher Education, Research and Development Fund (FKFP) Grant 0215/2001, and also by a visiting research grant received from the Department of Mathematics, Technical University of Łódź, Branch in Bielsko-Biała.

AMS subject classification: 39B72.

Key words and phrases: convexity, Jensen-convexity, Wright-convexity, ε-convexity, ε-Jensen-convexity, ε-Wright-convexity, stability of Jensen-convexity, stability of Wright-convexity.

is ε-convex, then there exists a convex function $g: D \to \mathbb{R}$ satisfying $\|f-g\| \le c_n\varepsilon$, where the $\|\cdot\|$ denotes the supremum norm defined by

$$\|\phi\| := \sup_{x\in D} |\phi(x)|$$

for a bounded function $\phi: D \to \mathbb{R}$.

The value of c_n was shown to be of order $\log_2(n)/2$ by Green [8], Cholewa [3]. Recently, Laczkovich [13] proved that, for convex sets D with $\dim D = n$, the infimum of these constants, for which the Hyers-Ulam theorem is true, is not less then $(\log_2(n/2))/4$. This implies that the Hyers-Ulam theorem cannot be extended to arbitrary infinite dimensional spaces. Earlier a counterexample of this type was given by Casini and Papini [4].

Using a condition stronger than ε-convexity, we can characterize functions which are uniformly close to a convex function. Namely, as a consequence of the sandwich theorem presented in [2, Theorem 1b] (with $g = f + 2\varepsilon$), we obtain the following result.

THEOREM BMN. *Let D be a convex subset of a real vector space, let $f: D \to \mathbb{R}$, and $\varepsilon \ge 0$. Then there exists a convex function $g: D \to \mathbb{R}$ such that $\|f-g\| \le \varepsilon$ if and only if f satisfies the inequality*

$$f(t_1x_1 + \cdots + t_nx_n) \le t_1f(x_1) + \cdots + t_nf(x_n) + 2\varepsilon \tag{1}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$, and $t_1, \ldots, t_n \in [0,1]$ with $t_1 + \cdots + t_n = 1$.

The stability problem for Jensen-convexity and Wright-convexity is not solved completely. Cholewa [3] gave an example of an ε-Jensen-convex function f such that there is no Jensen-convex function uniformly close to f. However, the domain D of f in that example is only a $\mathbb{Q}$-convex subset of $\mathbb{R}$ (*i.e.*, $tx + (1-t)y \in D$ for all $x, y \in D$ and $t \in [0,1] \cap \mathbb{Q}$). A similar counterexample with a function defined on a convex subset of an infinite dimensional space is given by Ger [6]. But for Jensen-convex functions defined on a convex subset of $\mathbb{R}^n$ the stability question is still open. A thorough study of ε-Jensen-convex functions is made in the recent paper of Dilworth, Howard and Roberts [5]. In particular, they present a stability result for Jensen-convex functions defined on a convex subset of $\mathbb{R}^n$ and bounded above on compact sets. Some further properties of ε-Jensen-convex and ε-Wright-convex functions are also investigated by Ng and Nikodem [16] and by Mrowiec [14].

The aim of this paper is to characterize functions which differ from Jensen-convex and Wright-convex functions in a bounded term. Unfortunately, we are not able to show that the ε-Jensen- and ε-Wright-convexity properties characterize these functions. Nevertheless, we obtain necessary and sufficient conditions for both classes of functions.

2. Approximately Jensen-convex functions. In this section, we characterize functions of the form $f = g + h$, where g is a Jensen-convex function and h is a bounded function on a midpoint-convex subset of vector space X over $\mathbb{Q}$. This result is also a consequence of [18, Corollary 5]. However, it is more convenient to have a direct proof here. We recall that a subset D of a vector space is called *midpoint-convex* if $(x + y)/2$ belongs to D for all $x, y \in D$. Clearly, midpoint convex sets are the natural domains for Jensen-convex functions.

THEOREM 1. *Let X be a vector space over the field of rational numbers, let $D \subset X$ be a midpoint-convex subset of X, and let $f: D \to \mathbb{R}$, $\varepsilon \geq 0$. Then there exists a Jensen-convex function $g: D \to \mathbb{R}$ such that $\|f - g\| \leq \varepsilon$ if and only if f satisfies the inequality*

$$f\left(\frac{x_1 + \cdots + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + \cdots + f(x_{2^n})}{2^n} + 2\varepsilon \quad \textit{for } n \in \mathbb{N}, \quad x_1, \ldots, x_{2^n} \in D. \tag{2}$$

PROOF. Assume that there exists a Jensen-convex function $g: D \to \mathbb{R}$ such that $\|f - g\| \leq \varepsilon$. Then the Jensen-convexity of g yields (*cf.* [12], [23])

$$g\left(\frac{x_1 + \cdots + x_{2^n}}{2^n}\right) \leq \frac{g(x_1) + \cdots + g(x_{2^n})}{2^n} \quad \text{for } n \in \mathbb{N}, \quad x_1, \ldots, x_{2^n} \in D$$

and $\|f - g\| \leq \varepsilon$ implies

$$f(x) \leq g(x) + \varepsilon \quad \text{and} \quad g(x) \leq f(x) + \varepsilon \quad \text{for } x \in D.$$

Combining these inequalities, we get that, for $n \in \mathbb{N}$ and $x_1, \ldots, x_{2^n} \in D$,

$$\begin{aligned} f\left(\frac{x_1 + \cdots + x_{2^n}}{2^n}\right) &\leq g\left(\frac{x_1 + \cdots + x_{2^n}}{2^n}\right) + \varepsilon \\ &\leq \frac{g(x_1) + \cdots + g(x_{2^n})}{2^n} + \varepsilon \\ &\leq \frac{f(x_1) + \cdots + f(x_{2^n})}{2^n} + 2\varepsilon, \end{aligned}$$

i.e., (2) is valid.

Now assume that f satisfies (2) and define $g: D \to \mathbb{R}$ by

$$g(x) := \inf\left\{\frac{f(x_1) + \cdots + f(x_{2^n})}{2^n} + \varepsilon \;\middle|\; n \in \mathbb{N}, x_1, \ldots, x_{2^n} \in D, x = \frac{x_1 + \cdots + x_{2^n}}{2^n}\right\}.$$

By taking $n = 1$, $x_1 = x_2 = x$, the definition of g yields that $g(x) \leq f(x) + \varepsilon$ for all $x \in D$. On the other hand, due to inequality (2), we have that $g(x) \geq f(x) - \varepsilon$. Therefore, $\|f - g\| \leq \varepsilon$.

To complete the proof of the theorem, it suffices to show that g is Jensen-convex. For, let $x, y \in D$ be fixed and let c be an arbitrary positive number. Then there exist $n, m \in \mathbb{N}$ and $x_1, \ldots, x_{2^n}$, $y_1, \ldots, y_{2^m} \in D$ such that

$$x = \frac{x_1 + \cdots + x_{2^n}}{2^n}, \quad y = \frac{y_1 + \cdots + y_{2^m}}{2^m},$$

and

$$\frac{f(x_1) + \cdots + f(x_{2^n})}{2^n} + \varepsilon < g(x) + c, \quad \frac{f(y_1) + \cdots + f(y_{2^m})}{2^m} + \varepsilon < g(y) + c.$$

Then

$$\frac{x+y}{2} = \frac{1}{2^{n+m+1}} \sum_{i=1}^{2^n} \sum_{j=1}^{2^m} (x_i + y_j),$$

therefore, by the definition of g,

$$\begin{aligned} g\left(\frac{x+y}{2}\right) &\leq \frac{1}{2^{n+m+1}} \sum_{i=1}^{2^n} \sum_{j=1}^{2^m} (f(x_i) + f(y_j)) + \varepsilon \\ &= \frac{1}{2}\left(\frac{1}{2^n} \sum_{i=1}^{2^n} f(x_i) + \varepsilon\right) + \frac{1}{2}\left(\frac{1}{2^m} \sum_{j=1}^{2^m} f(y_j) + \varepsilon\right) \\ &< \frac{1}{2}(g(x) + c) + \frac{1}{2}(g(y) + c). \end{aligned}$$

Taking the limit $c \to 0$, we get that g is Jensen-convex on D. ∎

REMARK 1. One can easily see that (2) implies the inequality (1) for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$, and $t_1, \ldots, t_n \in [0,1] \cap \mathbb{D}$ with $t_1 + \cdots + t_n = 1$, where $\mathbb{D}$ denotes the set of dyadic rational numbers. Thus, (2) is a special case of (1) which characterizes (by Theorem BMN) functions of the form $f = g + h$, where g is convex, h is bounded.

REMARK 2. If a function f satisfies (2), then it is automatically 2ε-Jensen-convex. Conversely, if f is 2ε-Jensen-convex, then an induction argument yields that it satisfies an inequality analogous to (2) where 2ε is replaced by $2n\varepsilon$.

3. Approximately Wright-convex functions. In [15], Ng proved that a function f defined on a convex subset of $\mathbb{R}^n$ is Wright-convex if and only if it can be represented in the form $f = a + k$, where a is an additive and k is a convex function (see also [17]). Kominek [10] extended this result to functions defined on algebraically open subsets of a linear space. In this section, we characterize functions of the form $f = a + k + h$, where a is an additive, k is a convex, and h is a bounded function on a convex subset of a linear space.

THEOREM 2. *Let X be a (real) linear space, let $D \subset X$ be a convex subset of X such that the algebraic interior of D is nonempty, let $f: D \to \mathbb{R}$, and let $\varepsilon \geq 0$. Then there exist an additive function $a: X \to \mathbb{R}$ and a convex function $k: D \to \mathbb{R}$ such that $\|f - (a + k)\| \leq \varepsilon$ if and only if f satisfies (2) and the inequality*

$$f\big(tx + (1-t)y\big) + f\big((1-t)x + ty\big) \leq f(x) + f(y) + 4\varepsilon \quad \text{for } x, y \in D, t \in [0,1]. \tag{3}$$

PROOF. Assume that there exist $a: X \to D$ additive and $k: D \to \mathbb{R}$ convex such that $\|f - (a + k)\| \leq \varepsilon$. Then $g := a + k$ is a Jensen-convex function and $\|f - g\| \leq \varepsilon$. Therefore, by Theorem 1, f satisfies (2). On the other hand, using the additivity of a and the convexity of k, we get

$$\begin{aligned}
&f\big(tx + (1-t)y\big) + f\big((1-t)x + ty\big) \\
&\quad \leq a\big(tx + (1-t)y\big) + k\big(tx + (1-t)y\big) + a\big((1-t)x + ty\big) \\
&\qquad + k\big((1-t)x + ty\big) + 2\varepsilon \\
&\quad \leq a(x + y) + tk(x) + (1-t)k(y) + (1-t)k(x) + tk(y) + 2\varepsilon \\
&\quad = a(x) + k(x) + a(y) + k(y) + 2\varepsilon \\
&\quad \leq f(x) + f(y) + 4\varepsilon
\end{aligned}$$

for $x, y \in D$, $t \in [0,1]$. Thus (3) holds.

Now assume that $f: D \to \mathbb{R}$ satisfies both (2) and (3). Then, by Theorem 1, there exists a Jensen-convex function $g: D \to \mathbb{R}$ such that $\|f - g\| \leq \varepsilon$. Let x_0 be a an arbitrary point in the algebraic interior of D. By Rode's theorem (see [22], [11]), there exists a Jensen function $\alpha: D \to [-\infty, \infty[$ such that $\alpha \leq g$ on D and $\alpha(x_0) = g(x_0)$. Since x_0 is in the algebraic interior of D, we have that $\alpha(x) \neq -\infty$ for all $x \in D$. It follows from the extension theorems of Ger [7] (*cf.* also [20]) that α is of the form $\alpha = a + c$, where $a: X \to \mathbb{R}$ is an additive function and $c \in \mathbb{R}$ is a constant.

In the rest of the proof we are going to show that $g - a = k$ is a convex function on D. Let $x, y \in D$ be fixed. Then, using (3), we get for $t \in [0,1]$,

$$\begin{aligned}
&k\big(tx + (1-t)y\big) \\
&\quad = g\big(tx + (1-t)y\big) - a\big(tx + (1-t)y\big) \\
&\quad \leq f\big(tx + (1-t)y\big) - \big[a(x + y) - a\big((1-t)x + ty\big)\big] + \varepsilon \\
&\quad \leq f(x) + f(y) - f\big((1-t)x + ty\big) - \big[\alpha(x + y) - \alpha\big((1-t)x + ty\big)\big] + 5\varepsilon \\
&\quad \leq f(x) + f(y) - \alpha(x + y) - \big[g\big((1-t)x + ty\big) - \alpha\big((1-t)x + ty\big)\big] + 6\varepsilon \\
&\quad \leq f(x) + f(y) - \alpha(x + y) + 6\varepsilon
\end{aligned}$$

for all $t \in [0,1]$. Therefore, the function

$$t \mapsto k\big(tx + (1-t)y\big) =: \chi(t) \quad (t \in [0,1])$$

is bounded from above on $[0,1]$. The function k is Jensen-convex, hence χ is also Jensen-convex. Thus, by the famous Bernstein-Doetsch theorem (see [1], [12], and [19]), χ is continuous, and hence convex, on $]0,1[$. Being Jensen-convex on $[0,1]$, it is also convex on $[0,1]$ (*cf.* [16, Lemma 6] with $\varepsilon = 0$). Hence we get that $\chi(t) \le t\chi(1) + (1-t)\chi(0)$, *i.e.*,

$$k\big(tx + (1-t)y\big) \le tk(x) + (1-t)k(y) \quad \text{for} \quad t \in [0,1].$$

This proves the convexity of k and thus the proof of the theorem is complete. ∎

REMARK 3. It is clear from the proof of Theorem 2, that the condition (3) can be replaced by the following weaker one: *There exists a two variable function* $\Phi\colon D \times D \to \mathbb{R}$ *such that*

$$f\big(tx + (1-t)y\big) + f\big((1-t)x + ty\big) \le \Phi(x,y) \quad (x, y \in D, t \in [0,1]). \tag{4}$$

For the necessity of this condition, one can take $\Phi(x,y) := f(x) + f(y) + 4\varepsilon$. The proof of the sufficiency is analogous to the argument followed above, except that in the last chain of inequalities, we get

$$k\big(tx + (1-t)y\big) \le \Phi(x,y) - \alpha(x+y) + 2\varepsilon,$$

which is also enough (together with the Bernstein-Doetsch theorem) to obtain the conclusion that k is convex.

REMARK 4. If $\varepsilon = 0$ in Theorem 2, then (3) with $t = 1/2$ yields the Jensen-convexity of f. Thus (2) holds automatically and the result obtained states that f is of the form $f = a + k$, where a is additive and k is convex. Thus the result of Ng [15] is also a consequence of Theorem 2. It is not clear (for the authors) if one could deduce (2) from (3) also for positive ε. An affirmative answer to this question would solve the stability problem of Wright-convex functions completely.

REFERENCES

1. F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*. Math. Ann. **76**(1915), 514–526.
2. K. Baron, J. Matkowski and K. Nikodem, *A sandwich with convexity*. Math. Pannon. **5**(1994), 139–144.
3. P. W. Cholewa, *Remarks on the stability of functional equations*. Aequationes Math. **27**(1984), 76–86.
4. E. Casini and P. L. Papini, *A counterexample to the infinity version of the Hyers-Ulam stability theorem*. Proc. Amer. Math. Soc. **118**(1993), 885–890.
5. S. J. Dilworth, R. Howard and J. W. Roberts, *Extremal approximately convex functions and estimating the size of convex hulls*. Adv. Math. **148**(1999), 1–43.
6. R. Ger, *Almost approximately convex functions*. Math. Slovaca **38**(1988), 61–78.
7. ———, *On extensions of polynomial functions*. Results Math. **26**(1994), 281–289.
8. J. W. Green, *Approximately convex functions*. Duke Math. J. **19**(1952), 499–504.
9. D. H. Hyers and S. M. Ulam, *Approximately convex functions*. Proc. Amer. Math. Soc. **3**(1952), 821–828.
10. Z. Kominek, *On additive and convex functionals*. Radovi Mat. **3**(1987), 267–279.

11. H. König, *On the abstract Hahn–Banach theorem due to Rodé*. Aequationes Math. **34**(1987), 89–95.
12. M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*. Państwowe Wydawnictwo Naukowe—Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.
13. M. Laczkovich, *The local stability of convexity, affinity and of the Jensen equation*. Aequationes Math. **58**(1999), 135–142.
14. J. Mrowiec, *Remark on approximately Jensen-convex functions*. C. R. Math. Rep. Acad. Sci. Canada **23**(2001), 16–21.
15. C. T. Ng, *Functions generating Schur-convex sums*. In: General Inequalities 5 (Oberwolfach, 1986) (ed. W. Walter), Internat. Ser. Numer. Math. vol. **80**, Birkhäuser Verlag, Basel-Boston, 1987, 433–438.
16. C. T. Ng and K. Nikodem, *On approximately convex functions*. Proc. Amer. Math. Soc. **118**(1993), 103–108.
17. K. Nikodem, *On some class of midconvex functions*. Ann. Polon. Math. **50**(1989), 145–151.
18. K. Nikodem, Zs. Páles and Sz. Wąsowicz, *Abstract separation theorems of Rodé type and their applications*. Ann. Polon. Math. **72**(1999), 207–217.
19. Zs. Páles, *Bernstein-Doetsch-type results for general functional inequalities*. (Dedicated to Professor Zenon Moszner on his 70th birthday.) Rocznik Nauk.-Dydakt. Prace Mat. **17**(2000), 197–206.
20. ———, *Extension theorems for functional equations with bisymmetric operations*. Aequationes Math., to appear.
21. ———, *On approximately convex functions*. Proc. Amer. Math. Soc., submitted.
22. G. Rodé, *Eine abstrakte Version des Satzes von Hahn-Banach*. Arch. Math. **31**(1978), 474–481.
23. A. W. Roberts and D. E. Varberg, *Convex Functions*. Academic Press, New York-London, 1973.

Department of Mathematics
Technical University of Łódź
Branch in Bielsko–Biała
ul. Willowa 2
PL-43-309 Bielsko–Biała
Poland
email: knik@aristo.pb.bielsko.pl

Institute of Mathematics and Informatics
University of Debrecen
H-4010 Debrecen, Pf. 12
Hungary
email: pales@math.klte.hu

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (4), 2001 pp. 148–152

TRACES OF WAVELET MULTIPLIERS

JINGDE DU AND M. W. WONG

Presented by L. A. Lorch, FRSC

ABSTRACT. We give a trace formula for wavelet multipliers as bounded linear operators in the trace class from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ and use it to compute the trace of the n-dimensional Landau-Pollak-Slepian operator arising in signal analysis.

RÉSUMÉ. On présente une formule de trace pour les multiplicateurs d'ondelettes comme opérateurs de trace bornés de $L^2(\mathbb{R}^n)$ dans $L^2(\mathbb{R}^n)$. On emploie cette formule pour calculer la trace de l'opérateur de Landau-Pollak-Slepian en dimension n qu'on retrouve en analyse du signal.

1. Wavelet multipliers. Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then we define the linear operator $T_\sigma\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$T_\sigma u = \mathcal{F}^{-1}\sigma\mathcal{F}u, \quad u \in L^2(\mathbb{R}^n),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transformation and inverse Fourier transformation respectively. The Fourier transform $\mathcal{F}u$, sometimes denoted by $\hat{u}$, of a function u in $L^2(\mathbb{R}^n)$ is given by $\mathcal{F}u = \lim_{R\to\infty}(\chi_R u)^\wedge$, where χ_R is the characteristic function of the ball with center at the origin and radius R,

$$(\chi_R u)^\wedge(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\chi_R(x)u(x)\,dx, \quad \xi \in \mathbb{R}^n,$$

and the convergence of $(\chi_R u)^\wedge$ to $\mathcal{F}u$ is understood to be in $L^2(\mathbb{R}^n)$. It is a consequence of Plancherel's theorem that $T_\sigma\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator.

Let $\pi\colon \mathbb{R}^n \to B\big(L^2(\mathbb{R}^n)\big)$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$\big(\pi(\xi)u\big)(x) = e^{ix\cdot\xi}u(x), \quad x,\xi \in \mathbb{R}^n,$$

for all functions u in $L^2(\mathbb{R}^n)$, where $B\big(L^2(\mathbb{R}^n)\big)$ is the C^*-algebra of all bounded linear operators from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Let φ be any function in $L^2(\mathbb{R}^n)\cap$

Received by the editors July 3, 2001.

This research has been partially supported by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0008562.

AMS subject classification: 47G10.

$L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_2 = 1$. Then it has been proved in the paper [3] by He and Wong that

$$(1.1)\qquad \langle \varphi u, \varphi v\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n} \langle u, \pi(\xi)\varphi\rangle\langle \pi(\xi)\varphi, v\rangle\, d\xi$$

for all functions u and v in the Schwartz space $\mathcal{S}$, where $\langle\ ,\ \rangle$ is the inner product in $L^2(\mathbb{R}^n)$.

Let σ be a function in $L^1(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$. Then for all functions u in $\mathcal{S}$, we define $P_{\sigma,\varphi}u$ by

$$(1.2)\qquad \langle P_{\sigma,\varphi}u, v\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n} \sigma(\xi)\langle u, \pi(\xi)\varphi\rangle\langle \pi(\xi)\varphi, v\rangle\, d\xi$$

for all functions v in $\mathcal{S}$. As has been proved in the paper [3] by He and Wong, $P_{\sigma,\varphi}u \in L^2(\mathbb{R}^n)$ for all functions u in $\mathcal{S}$, and $P_{\sigma,\varphi}$, initially defined on $\mathcal{S}$, can be extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

REMARK 1.1. Let $\sigma \in L^\infty(\mathbb{R}^n)$ and let φ be any function in $L^2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_2 = 1$. Then it has been proved in the paper [3] by He and Wong that the bounded linear operators $P_{\sigma,\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\varphi T_\sigma \bar{\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are equal. By (1.1) and (1.2), the bounded linear operator $P_{\sigma,\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a variant of a localization operator studied in the papers [1], [2]. Had the "admissible wavelet" φ in (1.2) been replaced by the function φ_0 on $\mathbb{R}^n$ given by $\varphi_0(x) = 1$ for all x in $\mathbb{R}^n$, we would have obtained $T_\sigma: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ instead of $P_{\sigma,\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. In other words, the bounded linear operator $P_{\sigma,\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ would have been a "constant coefficient" pseudo-differential operator or a Fourier multiplier studied in, say, the book [10] by Wong. In view of the fact that the function φ in the bounded linear operator $P_{\sigma,\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ plays the role of the admissible wavelet in a localization operator, we call the bounded linear operator $P_{\sigma,\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a wavelet multiplier.

2. The trace class. Let X be a complex and separable Hilbert space of infinite dimension in which the inner product is denoted by $(\ ,\ )$. Let $A: X \to X$ be a compact operator. If we denote by $A^*: X \to X$ the adjoint of $A: X \to X$, then the linear operator $(A^*A)^{1/2}: X \to X$ is positive and compact. Let $\{\psi_k : k = 1, 2, \ldots\}$ be an orthonormal basis for X consisting of eigenvectors of $(A^*A)^{1/2}: X \to X$, and let $s_k(A)$ be the eigenvalue corresponding to the eigenvector ψ_k, $k = 1, 2, \ldots$. We say that the compact operator $A: X \to X$ is in the trace class S_1 if $\sum_{k=1}^\infty s_k(A) < \infty$. It can be shown that S_1 is a Banach space in which the norm $\|\ \ \|_{S_1}$ is given by

$$\|A\|_{S_1} = \sum_{k=1}^\infty s_k(A), \quad A \in S_1.$$

PROPOSITION 2.1. *Let $A: X \to X$ be a bounded linear operator in S_1 and let $\{\varphi_k : k = 1, 2, \ldots\}$ be any orthonormal basis for X. Then the series $\sum_{k=1}^{\infty}(A\varphi_k, \varphi_k)$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $\{\varphi_k : k = 1, 2, \ldots\}$.*

Proposition 2.1 is well-known and can be found on, say, p. 211 of the book [6] by Reed and Simon.

In view of Proposition 2.1, we can define the trace $\mathrm{tr}(A)$ of a bounded linear operator $A: X \to X$ in S_1 by

$$\mathrm{tr}(A) = \sum_{k=1}^{\infty}(A\varphi_k, \varphi_k), \tag{2.1}$$

where $\{\varphi_k : k = 1, 2, \ldots\}$ is any orthonormal basis for X.

The following theorem is a special case of Theorem 4.3 in the paper [3] by He and Wong.

THEOREM 2.2. *Let $\sigma \in L^1(\mathbb{R}^n)$ and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_2 = 1$. Then the wavelet multiplier $P_{\sigma,\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in S_1*

3. A trace formula.

The main result in this paper is the following theorem.

THEOREM 3.1. *Let $\sigma \in L^1(\mathbb{R}^n)$ and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_2 = 1$. Then*

$$\mathrm{tr}(P_{\sigma,\varphi}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)\, d\xi.$$

PROOF. Let $\{\varphi_k : k = 1, 2, \ldots\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then, using (1.2), (2.1), Fubini's theorem, Parseval's identity, $\|\varphi\|_2 = 1$ and the fact that $\pi(\xi): L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator for all ξ in $\mathbb{R}^n$, we get

$$\begin{aligned}
\mathrm{tr}(P_{\sigma,\varphi}) &= \sum_{k=1}^{\infty}\langle P_{\sigma,\varphi}\varphi_k, \varphi_k\rangle = \sum_{k=1}^{\infty}(2\pi)^{-n}\int_{\mathbb{R}^n}\sigma(\xi)|\langle\varphi_k, \pi(\xi)\varphi\rangle|^2\, d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\sigma(\xi)\sum_{k=1}^{\infty}|\langle\varphi_k, \pi(\xi)\varphi\rangle|^2\, d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\sigma(\xi)\|\pi(\xi)\varphi\|_2^2\, d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n}\sigma(\xi)\, d\xi.
\end{aligned}$$

∎

REMARK 3.2. A trace formula for localization operators can be found in the paper [1] by Du and Wong.

4. The Landau-Pollak-Slepian operator. Let Ω and T be positive numbers. Then we define the linear operators $P_\Omega: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and Q_T: $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$(P_\Omega f)^\wedge(\xi) = \begin{cases} \hat{f}(\xi), & |\xi| \le \Omega, \\ 0, & |\xi| > \Omega, \end{cases}$$

and

$$(Q_T f)(x) = \begin{cases} f(x), & |x| \le T, \\ 0, & |x| > T, \end{cases}$$

for all functions f in $L^2(\mathbb{R}^n)$. It can be checked easily that $P_\Omega: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $Q_T: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are self-adjoint projections.

In signal analysis, a signal is a function in $L^2(\mathbb{R}^n)$. Thus, for any function f in $L^2(\mathbb{R}^n)$, the function $Q_T P_\Omega f$ can be considered to be a time and band-limited signal. Therefore it is of interest to compare the energy $\|Q_T P_\Omega f\|_2^2$ of the time and band-limited signal $Q_T P_\Omega f$ with the energy $\|f\|_2^2$ of the original signal f. Using the fact that $P_\Omega: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $Q_T: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are self-adjoint and the fact that $Q_T: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a projection, we get

$$\sup\left\{ \frac{\|Q_T P_\Omega f\|_2^2}{\|f\|_2^2} : f \in L^2(\mathbb{R}^n), f \neq 0 \right\} = \|P_\Omega Q_T P_\Omega\|_*.$$

The bounded linear operator $P_\Omega Q_T P_\Omega: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ that we have just seen in the context of time and band-limited signals is the n-dimensional analogue of the Landau-Pollak-Slepian operator studied in the fundamental papers [4], [5] by Landau and Pollak, [7], [8] by Slepian, and [9] by Slepian and Pollak.

The following theorem is an n-dimensional analogue of a result in Section 5 of the paper [3] by He and Wong.

THEOREM 4.1. *Let φ be the function on $\mathbb{R}^n$ defined by*

$$\varphi(x) = \begin{cases} \frac{1}{\sqrt{\mu(B_\Omega)}}, & x \in B_\Omega, \\ 0, & x \notin B_\Omega, \end{cases}$$

where

$$B_\Omega = \{x \in \mathbb{R}^n : |x| \le \Omega\},$$

and let σ be the characteristic function on the set B_T, where

$$B_T = \{\xi \in \mathbb{R}^n : |\xi| \le T\}.$$

Then the n-dimensional Landau-Pollak-Slepian operator $P_\Omega Q_T P_\Omega: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitarily equivalent to a scalar multiple of the wavelet multiplier $P_{\sigma,\varphi}$: $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. In fact,

$$P_\Omega Q_T P_\Omega = \mu(B_\Omega)\mathcal{F}^{-1} P_{\sigma,\varphi}\mathcal{F}.$$

We have the following result on the trace of the n-dimensional Landau-Pollak-Slepian operator $P_\Omega Q_T P_\Omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

THEOREM 4.2. $\operatorname{tr}(P_\Omega Q_T P_\Omega) = \left\{\frac{n}{2}\Gamma(\frac{n}{2})\right\}^{-2}(\frac{T\Omega}{2})^n$.

Theorem 4.2 is an immediate consequence of Theorem 3.1, Theorem 4.1 and the fact that the volume of the ball in $\mathbb{R}^n$ with radius r is equal to $\frac{\pi^{n/2}r^n}{\frac{n}{2}\Gamma(\frac{n}{2})}$.

REFERENCES

1. J. Du and M. W. Wong, *Traces of localization operators.* C. R. Math. Rep. Acad. Sci. Canada **22**(2000), 92–96.
2. Z. He and M. W. Wong, *Localization operators associated to square integrable group representations.* Panamer. Math. J. (1) **6**(1996), 93–104.
3. ———, *Wavelet multipliers and signals.* J. Austral. Math. Soc. Ser. B **40**(1999), 437–446.
4. H. J. Landau and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty, II.* Bell Syst. Tech. J. **40**(1961), 65–84.
5. ———, *Prolate spheroidal wave functions, Fourier analysis and uncertainty, III.* Bell Syst. Tech. J. **41**(1962), 1295–1336.
6. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis, Revised and Enlarged Edition.* Academic Press, 1980.
7. D. Slepian, *On bandwidth.* Proc. IEEE **64**(1976), 292–300.
8. ———, *Some comments on Fourier analysis, uncertainty and modeling.* SIAM Rev. **25**(1983), 379–393.
9. D. Slepian and H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty, I.* Bell Syst. Tech. J. **40**(1961), 43–64.
10. M. W. Wong, *An Introduction to Pseudo-Differential Operators.* 2nd edition, World Scientific, 1999.

Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario
M3J 1P3
email: jingde@pascal.math.yorku.ca
mwwong@pascal.math.yorku.ca

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (4), 2001 pp. 153–159

HILBERT AND POINCARÉ SERIES OF KOSZUL ALGEBRAS

ISTVÁN ÁGOSTON, VLASTIMIL DLAB, FRSC, AND ERZSÉBET LUKÁCS

ABSTRACT. This note reflects some of the recent results of the authors concerning the Hilbert and Poincaré series of a graded algebra. In particular, their results show that a graded Koszul algebra is a standard Koszul quasi-hereditary algebra if and only if its (Yoneda) extension algebra is a standard Koszul quasi-hereditary algebra.

RÉSUMÉ. Cet article s'agit de quelques résultats récentes des auteurs en langage des séries de Hilbert et Poincaré d'une algèbre graduée. En particulier, on montre qu'une algèbre graduée Koszul est une algèbre quasi-héréditaire Koszul standarde si et seulement si son algèbre (Yoneda) d'extension est une algèbre quasi-héréditaire Koszul standarde.

Throughout the paper, $A = \bigoplus_{r\geq 0} A_r$ is a basic *positively graded* connected K-algebra, with $\dim_K A_r < \infty$ for all r. Unless otherwise stated, we shall also assume that A is *tightly graded*. Thus A_0 is a finite dimensional semi-simple algebra and $A_r \cdot A_s = A_{r+s}$ for all integers $r, s \geq 0$. Obviously the (graded) radical of A is $\operatorname{rad} A = \bigoplus_{r\geq 1} A_r$. Let us fix a primitive orthogonal decomposition of the identity element $1 = e_1 + e_2 + \cdots + e_n$ so that $e_i \in A_0$ and keep the order $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of this complete set of primitive orthogonal idempotents throughout the paper. By a *graded (right) A-module X* we shall always mean a vector space $X = \bigoplus_{r\geq k} X_r$ for some $k \in \mathbb{Z}$ with $X_r \cdot A_s \subseteq X_{r+s}$ where all X_r are finite dimensional. The module X is said to be *generated in degree k* if $X = X_k \cdot A$ (*i.e.*, $X_k \cdot A_r = X_{k+r}$ for every $r \geq 0$). Note that in this case $X_r = 0$ for all $r < k$. A submodule Y of X is said to be a *graded submodule* of X if $Y = \bigoplus_{r\geq k} Y_r$ with $Y_r = Y \cap X_r$. If both X and Y are generated in the same degree k then Y is called a *top submodule* of X. If $X = \bigoplus_{r\geq k} X_r$ is a graded module generated in degree k, then the graded submodule $\operatorname{rad}^t X = \bigoplus_{r\geq t+k} X_r$ is generated in degree $t + k$ for all $t > 0$.

By a *graded morphism* $f: X \to Y$ between two graded A-modules we shall understand a module homomorphism of degree 0, given by a family of linear maps $f_r: X_r \to Y_r$ with obvious commuting properties. It is easy to see that, for a graded morphism $f: X \to Y$, the morphism $\operatorname{Ker} f \to X$ and $Y \to \operatorname{Coker} f$ are

Received by the editors June 5, 2001.

The first and third authors' research was partially supported by Hungarian NFSR grants no. T029525 and T034530; part of this research was done while the first author held the Széchenyi Fellowship, and while the third author held the Bolyai Research Fellowship. The second author's research was partially supported by NSERC of Canada grant no. A-7257.

AMS subject classification: 16E10.

also graded, and if Y is generated in degree k, so is Coker f. Every graded module X generated in degree k has a graded projective cover $P \to X \to 0$, where P is also generated in degree k.

Note that the right regular representation A_A and its indecomposable direct summands $P(i) = e_i A$ are graded modules generated in degree 0 and the graded submodules $\operatorname{rad}^t A_A$ and $\operatorname{rad}^t P(i)$ in degree $t \geq 1$. Similarly, the irreducible A-modules $S(i) = P(i)/\operatorname{rad} P(i)$ are generated in degree 0. Furthermore, let us recall that the *(right) standard module* $\Delta(i)$ is the largest quotient of the projective module $P(i)$ with no composition factors isomorphic to $S(j)$ for $j > i$. Hence

$$\Delta(i) = e_i A / e_i A(e_{i+1} + e_{i+2} + \cdots + e_n)A$$

for $1 \leq i \leq n$ is a graded module generated in degree 0.

DEFINITION 1. If $X = \bigoplus_{r \geq k} X_r$ is a graded right A-module over a (not necessarily tightly) graded algebra A, then the *Hilbert vector of* X is defined as the vector $H_A^X(q) = \big(H_1^X(q), \ldots, H_n^X(q)\big) \in \mathbb{Z}[[q, q^{-1}]]^n$ such that

$$H_i^X(q) = \sum_{r \geq k} [X_r : S(i)] \cdot q^r,$$

where $[X_r : S(i)]$ denotes the multiplicity of the simple module $S(i)$ in X_r, both considered as A_0-modules.

DEFINITION 2. For a given ordered system $\mathbf{X} = \big(X(1), \ldots, X(m)\big)$ of graded A-modules $X(j)$ generated in degree 0, the *Hilbert matrix of* $\mathbf{X}$, denoted by $H_A^{\mathbf{X}}(q)$ is the matrix whose j-th column is the Hilbert vector $H_A^{X(j)}(q)$ of $X(j)$. In particular, $H_A(q) = H_A^{\mathbf{P}}(q)$ stands for the Hilbert matrix of the system $\mathbf{P} = \big(P(1), \ldots, P(n)\big)$, where $P(j)$ is the j-th indecomposable projective module. Similarly, $H_A^{\boldsymbol{\Delta}}(q)$ is the Hilbert matrix corresponding to the system $\boldsymbol{\Delta} = \big(\Delta(1), \ldots, \Delta(n)\big)$ of the standard modules $\Delta(j)$.

Let us point out that the above defined vectors appear as columns in the forthcoming matrix calculations. Observe that

$$H_{A^{\mathrm{opp}}}(q) = W_A^{-1} \cdot \big(H_A(q)\big)^T \cdot W_A,$$

where $W_A = (w_{ij})$ is the diagonal matrix with $w_{ii} = [\operatorname{End}_A S(i) : K]$; we shall use the symbol W_A to denote this matrix throughout the note. Let us also mention that $H_A^{\boldsymbol{\Delta}}(q)$ is an upper triangular matrix. Moreover, $H_A^{\mathbf{S}}(q) = I_n$, the $n \times n$ identity matrix, for the system of simple modules $\mathbf{S} = \big(S(1), \ldots, S(n)\big)$. Finally, note that $H_A(1)$ is the Cartan matrix of the algebra A.

Recall that a finite dimensional algebra $(A, \mathbf{e})$ (*i.e.*, the algebra A with respect to the given order $\mathbf{e}$ of the primitive idempotents e_i) is said to be a *quasi-hereditary algebra* if every indecomposable projective module $P(i)$ has a filtration whose factors are isomorphic to some $\Delta(j)$, $j \geq i$, and all standard modules $\Delta(i)$

are *Schurian* (*i.e.*, the multiplicity $[\Delta(i) : S(i)] = 1$ for $1 \leq i \leq n$). Note that the latter condition is equivalent to the requirement that the diagonal elements of the Hilbert matrix $H_A^{\mathbf{\Delta}}(q)$ are all 1.

It is well-known that these properties imply the same properties for the corresponding (right) projective, standard and simple A-modules $P^\circ(i)$, $\Delta^\circ(i)$ and $S^\circ(i)$ of the opposite algebra $A^{\rm opp}$.

PROPOSITION 1. *A finite dimensional algebra* $(A, \mathbf{e})$ *is quasi-hereditary if and only if*

$$H_A(q) = H_A^{\mathbf{\Delta}}(q) \cdot \left(W_A \cdot H_{A^{\rm opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T \quad \textit{and} \quad \det H_A(q) = 1,$$

where $W_A = (w_{ij})$ *is the diagonal matrix with* $w_{ii} = [\operatorname{End}_A S(i) : K]$.

PROOF. The matrix equality ensures that the regular module has a filtration by standard modules, while the condition on the determinant of the Hilbert matrix gives that all standard modules are Schurian. This is a straightforward modification of the well-known numerical characterization of quasi-hereditary algebras (*cf.* [4, Theorem 2.4]). ∎

REMARK 1. In fact, the above paper shows that the statement is true in a more general form: the algebra $(A, \mathbf{e})$ is *standardly stratified* (*i.e.*, ${}_AA$ has a Δ°-filtration) if and only if

$$H_A(q) = H_A^{\bar{\mathbf{\Delta}}}(q) \cdot \left(W_A \cdot H_{A^{\rm opp}}^{\mathbf{\Delta}^\circ}(q) \cdot W_A^{-1}\right)^T.$$

Here, $\bar{\mathbf{\Delta}} = \left(\bar{\Delta}(1), \dots, \bar{\Delta}(n)\right)$ is a sequence of the proper standard modules $\bar{\Delta}(i)$, defined as the largest quotient of $\Delta(i)$ with $[\bar{\Delta}(i) : S(i)] = 1$ (see also [1]).

REMARK 2. Proposition 1 also shows that if $(A, \mathbf{e})$ is quasi-hereditary then its Hilbert matrix is the product of a lower and an upper unitriangular matrix. Simple examples show that this property is far from characterizing quasi-heredity of algebras.

Recall that the Yoneda extension algebra A^* of the algebra A is the vector space $\bigoplus_{h \geq 0} \operatorname{Ext}_A^h(\hat{S}, \hat{S})$ together with multiplication given by the Yoneda composition of extensions; here, $\hat{S} = \bigoplus_{i=1}^n S(i)$. Thus A^* has a natural grading given by the degree of Ext-modules. This grading is tight if and only if the algebra is Koszul. Let us recall that, by definition, a graded A-module X, generated in degree k is *Koszul* if and only if X has a *linear projective resolution*, *i.e.*, in the graded minimal projective resolution

$$\cdots \to P_h \to \cdots \to P_1 \to P_0 \to X \to 0$$

the projective modules P_h are generated in degree $h + k$ for each $h \geq 0$. This is equivalent to the fact that all syzygies $\Omega_{h+1}(X)$ are top submodules of $\operatorname{rad} P_h(X)$.

The finite dimensional algebra A is called *Koszul* if all simple modules are Koszul. Thus A is Koszul if and only if, for all $1 \leq i, j \leq n$,

$$\mathrm{Ext}_A^h\big(S(i)[s], S(j)[r]\big) \neq 0 \text{ implies } r - s = h.$$

Here, $S(k)[t]$ denotes the graded module $S(k)$, shifted by t.

The natural contravariant functor Ext_A^*: mod-$A \to A^*$-mod is defined by

$$\mathrm{Ext}_A^*(X) = \bigoplus_{h\geq 0} \mathrm{Ext}^h(X, \hat{S}) \quad \text{for every } X \in \text{mod-}A,$$

where this decomposition provides also a grading of $\mathrm{Ext}_A^*(X)$. Note that the graded simple and indecomposable projective left A^*-modules can be obtained as $S^{*\circ}(i) = \mathrm{Ext}_A^*\big(P(i)\big)$ and $P^{*\circ}(i) = \mathrm{Ext}_A^*\big(S(i)\big)$, respectively.

DEFINITION 3. The *Poincaré vector* $P_A^X(q)$ of a graded A-module $X = \bigoplus_{r\geq 0} X_r$, generated in degree 0 is the vector $P_A^X(q) = \big(P_1^X(q), \ldots, P_n^X(q)\big)$, where

$$P_i^X(q) = \sum_{h\geq 0}(-1)^h[\mathrm{Ext}_A^h(X, \hat{S}) : S^{*\circ}(i)]q^h.$$

One may define, as in the case of Hilbert vectors, the *Poincaré matrix* $P_A^{\mathbf{X}}(q)$ of a system $\mathbf{X}$ of modules.

Note that $P_A^X(q) \in \mathbb{Z}[[q]]$ can be defined, equivalently, in terms of a minimal projective resolution of the A-module X, and that it is a polynomial vector if and only if the projective dimension of X is finite. Let us observe that $P_A^X(q) = H_{A^{*\mathrm{opp}}}^{\mathrm{Ext}^*(X)}(-q)$. In particular, $P_A^{\mathbf{S}}(q) = H_{A^{*\mathrm{opp}}}(-q) = W_A^{-1} \cdot \big(H_{A^*}(-q)\big)^T \cdot W_A$. For simplicity, we shall denote this matrix by $P_A(q)$. Let us note that, again,

$$P_{A^{\mathrm{opp}}}(q) = W_A^{-1} \cdot \big(P_A(q)\big)^T \cdot W_A.$$

We can now formulate the following result.

PROPOSITION 2 ([3]). *The graded A-module X, generated in degree* 0 *is Koszul if and only if*

$$H_A(q) \cdot P_A^X(q) = H_A^X(q).$$

PROOF. Consider a minimal graded projective resolution

$$\cdots \to P_h \to \cdots \to P_1 \to P_0 \to X \to 0$$

of the module X. Define the power series vector (formally) by

$$R_A^X(q) = \big(R_1^X(q), \ldots, R_n^X(q)\big) = \sum_{h\geq 0}(-1)^h \cdot H_A^{\mathrm{top}\, P_h}(q);$$

here top P_h is the semisimple graded module $P_h/\operatorname{rad} P_h$. Note that the r-th component $(P_h)_r$ of P_h is 0 for all $r < h$. Consequently, each component of $H_A^{\operatorname{top} P_h}(q)$ is divisible by q^h. Using Euler's argument we get easily that

$$H_A(q) \cdot R_A^X(q) = H_A^{P_0}(q) - H_A^{P_1}(q) + \cdots + (-1)^h H_A^{P_h}(q) + \cdots = H_A^X(q).$$

On the other hand, it is clear that $P_i^X(q) = \sum_{h\geq 0}[\operatorname{top} P_h : S_i]\cdot(-q)^h$. So one can see easily that X is Koszul if and only if $R_A^X(q) = P_A^X(q)$. Finally, the statement follows by observing that $H_A(q)$ is an invertible matrix, since $H_A(0) = I_n$. ∎

COROLLARY 1. *A graded algebra A is Koszul if and only if*

$$H_A(q) \cdot P_A(q) = I_n.$$

Note that $H_A(q)\cdot R_A^{\mathbf{S}}(q) = H_A^{\mathbf{S}}(q) = I_n$. Thus, if gl.dim $A < \infty$, then $\det R_A^{\mathbf{S}}(q)$ and $\det H_A(q)$ belong to $\mathbb{Z}[q]$, and therefore $\det H_A(q) = \pm 1$. On the other hand, $\det H_A(0) = 1$; hence we obtain the following well-known result.

PROPOSITION 3 ([5]). *If the graded algebra A has finite global dimension, then* $\det H_A(q) = 1$, *consequently the Cartan determinant of A is* 1. ∎

Recall that an algebra $(A, \mathbf{e})$ is said to be *standard Koszul* if all right and left standard modules are Koszul. It is one of the main results of [2] that a standard Koszul quasi-hereditary algebra is Koszul. We can reformulate this result in terms of Hilbert and Poincaré matrices in the following way.

THEOREM 1 ([2]). *Let us assume that for the algebra $(A, \mathbf{e})$ the following conditions hold:*
(i) $H_A(q) = H_A^{\Delta}(q) \cdot \left(W_A \cdot H_{A^{\mathrm{opp}}}^{\Delta^\circ}(q) \cdot W_A^{-1}\right)^T$, *and* $\det H_A(q) = 1$.
(ii) $H_A(q) \cdot P_A^{\Delta}(q) = H_A^{\Delta}(q)$ *and* $H_{A^{\mathrm{opp}}}(q) \cdot P_{A^{\mathrm{opp}}}^{\Delta^\circ}(q) = H_{A^{\mathrm{opp}}}^{\Delta^\circ}(q)$.
Then, $H_A(q) \cdot P_A(q) = I_n$.

A simple matrix calculation leads to the following statement.

COROLLARY 2. *Let $(A, \mathbf{e})$ be a graded algebra satisfying the conditions (i) and (ii) of Theorem 1. Then the condition (i) holds also for the Poincaré matrices:*

$$P_A(q) = P_A^{\Delta}(q) \cdot \left(W_A \cdot P_{A^{\mathrm{opp}}}^{\Delta^\circ}(q) \cdot W_A^{-1}\right)^T \textit{and } \det P_A(q) = 1.$$

PROOF. Substituting the expressions in *(ii)* into *(i)*, we get

$$H_A(q) = \left(H_A(q) \cdot P_A^{\Delta}(q)\right) \cdot \left(W_A \cdot H_{A^{\mathrm{opp}}}(q) \cdot P_{A^{\mathrm{opp}}}^{\Delta^\circ}(q) \cdot W_A^{-1}\right)^T.$$

Rewriting this equality,

$$I_n = P_A^{\Delta}(q) \cdot W_A^{-1} \cdot \left(P_{A^{\mathrm{opp}}}^{\Delta^\circ}(q)\right)^T \cdot W_A \cdot W_A^{-1} \cdot \left(H_{A^{\mathrm{opp}}}(q)\right)^T \cdot W_A,$$

and thus

$$I_n = P_A^{\Delta}(q) \cdot \left(W_A \cdot P_{A^{\mathrm{opp}}}^{\Delta^\circ}(q) \cdot W_A^{-1}\right)^T \cdot H_A(q).$$

By Theorem 1, the inverse of the matrix $H_A(q)$ is $P_A(q)$. This yields the desired formula. Clearly $\det P_A(q) = \det\left(H_A(q)\right)^{-1} = 1$. ∎

Now we can give an alternative proof of one of the main statements of [2].

THEOREM 2. *Let* $(A, \mathbf{e})$ *be a graded standard Koszul quasi-hereditary algebra. Then the extension algebra* $(A^*, \mathbf{f})$ *(with respect to the opposite order of the idempotents* $\mathbf{f} = (f_n, f_{n-1}, \dots, f_1)$*, where* $f_i = \mathrm{id}_{S(i)}$*) is also a standard Koszul quasi-hereditary algebra.*

PROOF. In order to identify the standard modules over A^*, we need the following characterization of standard modules in terms of Hilbert and Poincaré matrices.

PROPOSITION 4. *A system* $\mathbf{X} = (X(1), \dots, X(n))$ *of* A*-modules generated in degree* 0 *is isomorphic to the system of the standard modules* $\mathbf{\Delta} = (\Delta(1), \dots, \Delta(n))$ *of* $(A, \mathbf{e})$ *if and only if the matrix* $H_A^{\mathbf{X}}(q)$ *is an upper triangular matrix with* $H_A^{\mathbf{X}}(0) = I_n$*, while in the matrix* $P_A^{\mathbf{X}}(q) - I_n$ *all elements above and on the main diagonal are divisible by* q^2*. In particular, if* $(A, \mathbf{e})$ *is quasi-hereditary, then the matrices* $H_A^{\mathbf{\Delta}}(q)$ *and* $H_{A^{\mathrm{opp}}}^{\mathbf{\Delta}^\circ}(q)$ *are upper unitriangular and the matrices* $P_A^{\mathbf{\Delta}}(q)$ *and* $P_{A^{\mathrm{opp}}}^{\mathbf{\Delta}^\circ}(q)$ *are lower unitriangular.*

PROOF. It is clear that the system of standard modules satisfies the given conditions. For the converse, the fact that the Hilbert matrix of $\mathbf{X}$ is upper triangular means that $X(i)$ has no composition factors of index larger than i while the condition on the constant part of the matrix ensures that each module $X(i)$ is local. Finally, the last condition yields $\mathrm{Ext}_A^1(X(i), S(j)) = 0$ for $j \leq i$, implying that $X(i)$ is a maximal quotient of $\Delta(i)$, *i.e.*, $X(i) = \Delta(i)$. The last statement follows from well-known properties of standard modules over quasi-hereditary algebras. ∎

COROLLARY 3. *If* $(A, \mathbf{e})$ *is a graded standard Koszul quasi-hereditary algebra, then* $\mathrm{Ext}_A^*(\Delta(i)) = \Delta^{*\circ}(i)$ *and* $\mathrm{Ext}_A^*(\Delta^\circ(i)) = \Delta^*(i)$ *are the left and right standard modules over* $(A^*, \mathbf{f})$*, respectively.*

PROOF. We make use of the following facts. The left A^*-module $\mathrm{Ext}^*(X)$ of a right graded Koszul module X over a Koszul algebra A, is Koszul, and furthermore, $X \cong \mathrm{Ext}^*(\mathrm{Ext}^*(X))$ if we identify A with the isomorphic algebra A^{**}.

Thus, while $P_A^{\mathbf{\Delta}}(q) = H_{A^{*\mathrm{opp}}}^{\mathrm{Ext}^*(\mathbf{\Delta})}(-q)$ is true in general, the Koszul property of the standard modules now implies that $P_{A^{*\mathrm{opp}}}^{\mathrm{Ext}^*(\mathbf{\Delta})}(q) = H_{A^{**}}^{\mathrm{Ext}^*(\mathrm{Ext}^*(\mathbf{\Delta}))}(-q) = H_A^{\mathbf{\Delta}}(-q)$, and the same relations hold for the left standard modules. Thus by Proposition 4, we get that $\mathrm{Ext}^*(\Delta^\circ(1)), \dots, \mathrm{Ext}^*(\Delta^\circ(n))$ are the right, while $\mathrm{Ext}^*(\Delta(1)), \dots, \mathrm{Ext}^*(\Delta(n))$ are the left standard modules over $(A^*, \mathbf{f})$. ∎

In order to finish the proof of Theorem 2, we need only to note that, by our earlier observation, Corollary 3 implies that the left and right standard modules over $(A^*, \mathbf{f})$ are Koszul. Furthermore, by Corollary 2 and Proposition 1, we get that $(A^*, \mathbf{f})$ is quasi-hereditary. Thus A^* is a standard Koszul quasi-hereditary algebra, as required. ∎

REFERENCES

1. I. Ágoston, V. Dlab and E. Lukács, *Stratified algebras.* C. R. Math. Rep. Acad. Sci. Canada **20**(1998), 20–25.
2. ———, *Quasi-hereditary extension algebras*, In: Algebras and Representation Theory, to appear.
3. A. Beilinson and V. Ginsburg, *Mixed categories, Ext-duality and representations (results and conjectures).* Preprint, Moscow, 1986.
4. V. Dlab, *Quasi-hereditary algebras revisited.* An. St. Univ. Ovidius Constantza **4**(1996), 43–54.
5. G. V. Wilson, *The Cartan map on categories of graded modules.* J. Algebra **85**(1983), 390–398.

Department of Algebra and Number Theory
Eötvös University
P.O. Box 120
H-1518 Budapest
Hungary
email: agoston@cs.elte.hu

School of Mathematics and Statistics
Carleton University
Ottawa, Ontario
K1S 5B6
email: vdlab@math.carleton.ca

Department of Algebra
Technical University of Budapest
P.O. Box 91
H-1521 Budapest
Hungary
email: lukacs@math.bme.hu

Index—Volume 23, 2001