SEPTEMBER / SEPTEMBRE 2001

IN THIS ISSUE / DANS CE NUMÉRO

23
No 3

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (3), 2001 pp. 65–70

A GENERALIZATION OF BOY'S THEOREM FOR SURFACES IN EUCLIDEAN 3-SPACE

In memory of N. Kuiper and W. Boy

MAREK KOSSOWSKI

Presented by Vlastimil Dlab, FRSC

ABSTRACT. In this paper we generalize Boy's Theorem concerning the "curvatura integra" of a singular surface in Euclidean space. As a consequence, we find that part of the extrinsic topology of the surface is determined by the first fundamental form.

RÉSUMÉ. Dans ce papier nous généralisons le théorème qui concerne la "curvatura integra" d'une surface singulière dans l'espace euclidien. Par conséquent nous trouvons qu'une partie de la topologie extrinsique de la surface est determiné par la première forme fondamentale.

Let M be a C^∞ compact orientable 2-manifold without boundary and let $j: M \to \mathbb{E}^3$ be a C^∞-mapping into Euclidean 3-space. The differential of j will have maximal rank on the complement of a finite set of points $\mathcal{W} \subset M$ where we *assume* j has Whitney singularities (*i.e.*, after C^∞-changes of local coordinates at $w \in \mathcal{W}$, and $j(w) \in \mathbb{E}^3$, j can be represented by $j(u,v) = (u, v^2, uv)$). We also assume that all *double points* and *triple points* are transverse (*i.e.*, if $m \neq m'$, $j(m) = j(m')$ then $m, m' \notin \mathcal{W}$, $T_j(m)M \cap T_j(m')M$ is one-dimensional, and if $m \neq m' \neq m''$, $j(m) = j(m') = j(m'')$ then $m, m', m'' \notin \mathcal{W}$, and these three tangent spaces intersect in the zero subspace). We denote the finite set of triple points by $\mathcal{T} \subset M$ and call such maps $\mathcal{W}$*-generic.* Thus every point of the image $j(M)$ is locally equivalent to one of the diagrams in Figure 1. Note that the cardinality $\#\mathcal{W}$ is even since curves of double points must terminate at Whitney singularities. It follows that $j(M)$ is a $C\mathcal{W}$-complex. Moreover, the 1-skeleton consists of the self-intersection curves and the 0-skeleton consists of $\mathcal{T} \cup \mathcal{W}$.

The local Differential Geometry of Whitney singularities is counter-intuitive, *e.g.*, as $m \to w \in \mathcal{W}$, the normal vectors to the surface n_m, fail to have a limit. Rather, there is a great circle of limiting normals $S^1_w \subset S^2$. The Gauss curvature K becomes unbounded as $m \to w$, and the area 2-form dA vanishes at points of $\mathcal{W}$. We will assume that there exists a global C^∞-normal vector

Received by the editors January 28, 2000.
AMS subject classification: 53Cxx.

field $n: M - \mathcal{W} \to S^2$. We then define the 2-form KdA by pulling back the area 2-form on S^2. It follows that the integral of KdA over M is well defined and finite. (Compare Figure 2 where the j-image of a torus contains a Möbius strip and hence such a normal field does not exist.)

In 1975 Kuiper used height functions to show that

$$\chi(M) = \frac{1}{2\pi} \int_M K\,dA \tag{1}$$

for $\mathcal{W}$-generic surfaces [9]. Now this appears to conflict with Boy's Theorem [2].

$$\chi = \frac{1}{2\pi} \int_M K\,dA + \#\mathcal{W}/2. \tag{2}$$

Due to the passage of time, it is difficult to understand the statement and proof of Boy's Theorem. Specifically, it is unclear if his hypothesis allows triple points. Moreover, he appears to erroneously equate the Euler characteristics of M and $j(M)$. Recent work on another class of singular surfaces will clarify the situation.

There is a class of singular C^∞-mappings $k: M \to \mathbb{E}^3$ which admit a C^∞-global normal mapping $n: M \to S^2$. They have a natural notion of relative *stability* (*i.e.*, C^∞-perturbations must be taken within the set of C^∞-mappings admitting a global normal). The local classification of (relatively) stable singularities for such mappings is referred to as *Lagrangian*, *Legendrian*, or *front singularities* depending upon the group of coordinate changes allowed on $\mathbb{R}^3$. We will say a mapping $k: M \to \mathbb{E}^3$ is *1-resolvable* if the product map $k \times n: M \to \mathbb{E}^3 \times S^2$ into the unit normal bundle has differential with maximal rank, and *stable* if at every singular point $s \in \mathcal{S} \subset M$, the mapping has differential of rank 1, and is locally equivalent to a S_1 or $S_{1,1}$ singularity [1]. It follows that $\mathcal{S} \subset M$ is a disjoint union of closed curves marked with the finite set $\mathcal{S}_{1,1} \subset \mathcal{S}$ of $S_{1,1}$-singularities. These surfaces have a rich and tractable Differential Geometry [5], [6], [7], [8].

Such a mapping is *generic 1-resolvable* if the analogous transverse double and triple point conditions are imposed on $k: M - \mathcal{S} \to \mathbb{E}^3$. Specifically there are no triple points in the singular set $\mathcal{S}$ of k. In this setting, every point of the image $k(M)$ is locally equivalent to one of the diagrams in Figure 3, and $k(M)$ is a $C\mathcal{W}$-complex. For this class of surfaces Izumiya and Marar have shown that

$$\chi(k(M)) - \#\mathcal{T} = \chi(M) + \#\mathcal{S}_{1,1}/2. \tag{3}$$

For our purposes the key observation is that the *image* of a $S_{1,1}$-singularity is locally homeomorphic to the image of a Whitney singularity (*i.e.*, a curve of transverse double points has a limit point at the Whitney or $S_{1,1}$-point respectively; see Figure 4). If we triangulate $k(M)$ so that: the set of 0-simplicies include all of the $\mathcal{S}_{1,1}$ and $\mathcal{T}$-points; the 1-simplicies include all of the curves of double points then $\chi(k(M)) = v - e + f$; see the example below.

Because $\mathcal{W}$ and $\mathcal{S}_{1,1}$-singularities have locally homeomorphic images, it follows that (3) also holds for $\mathcal{W}$-generic surfaces. Now combining (1) with (3) we have the following generalization of Boy's Theorem.

THEOREM. *Let $j\colon M \to \mathbb{E}^3$ be $\mathcal{W}$-generic. Then*

$$\chi(j(M)) - \#\mathcal{T} = \frac{1}{2\pi}\int_M K\,dA + \#\mathcal{W}/2. \tag{4}$$

Observe a conceptual resonance between the above and Gauss' "Theorem egregium": the left-hand-side is extrinsic shape information which is determined by the first fundamental form $(M, j^*\langle\ ,\ \rangle = I)$.

EXAMPLE. Consider the $\mathcal{W}$-generic 2-sphere given in Figure 5 and which contains one triple point and two Whitney singularities. By Kuiper's Theorem, the integral of the curvature is $\chi(S^2) = 2$ and by the simplicial decomposition depicted in Figure 6 we have $\chi(j(M)) = v - e + f = 4$, and (4) is verified.

Figure 1

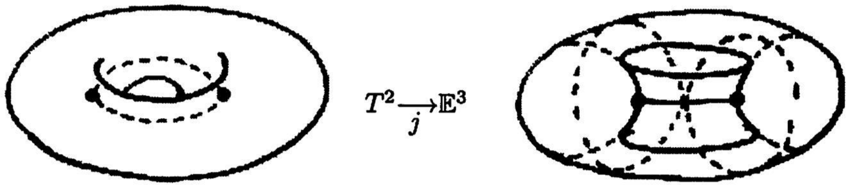


Figure 2

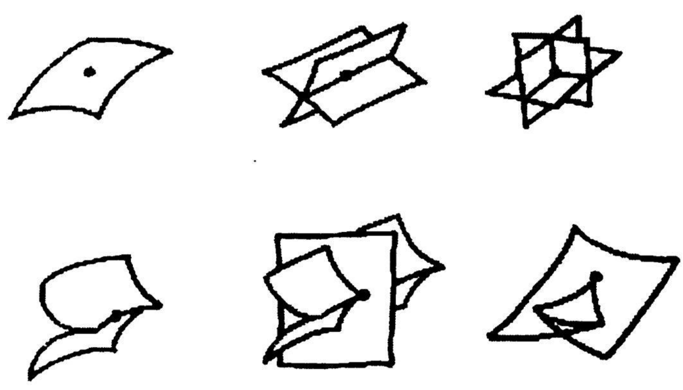


Figure 3

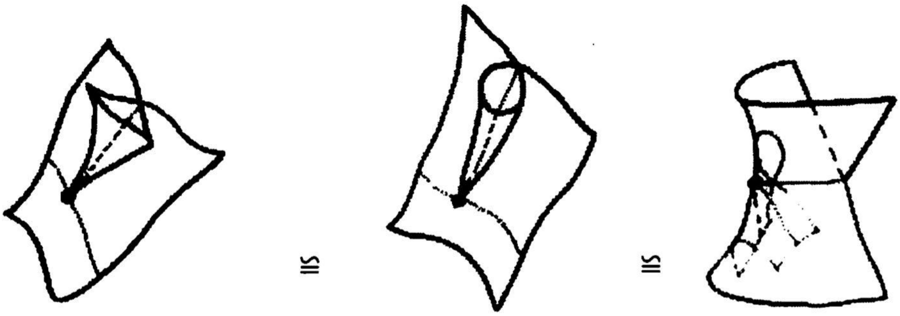


Figure 4

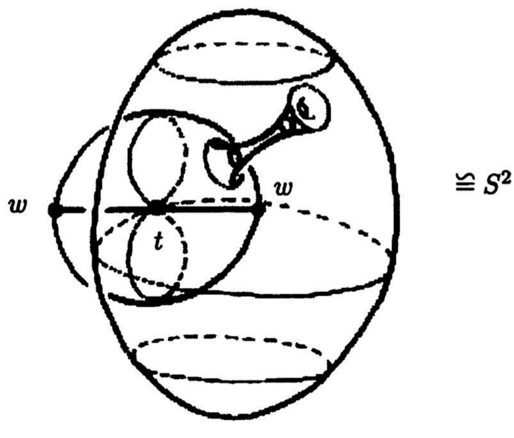
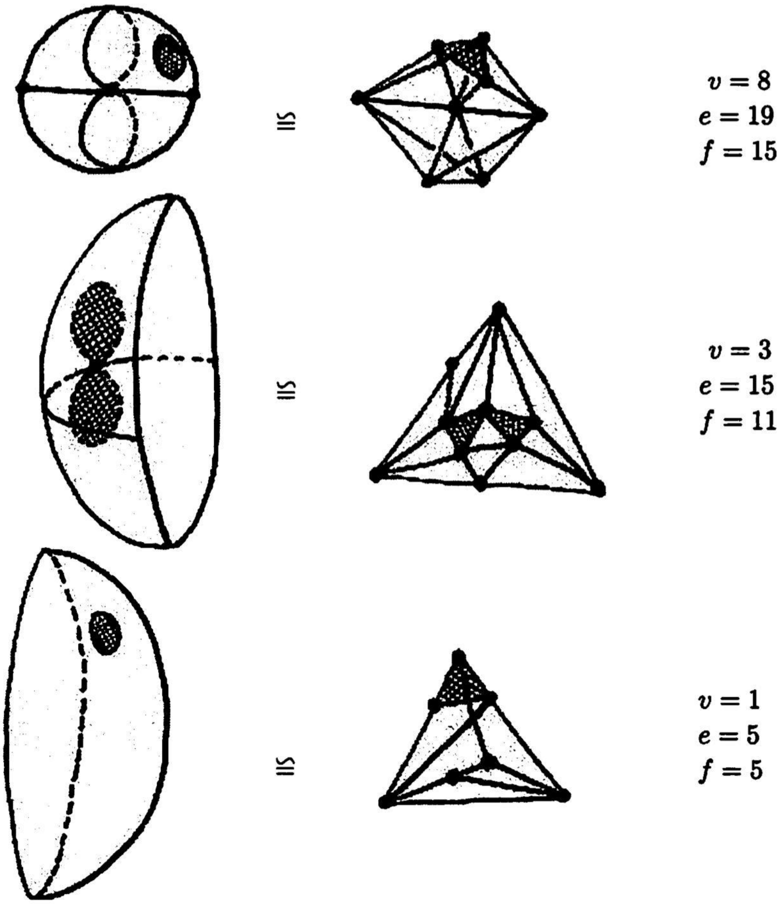


Figure 5



$$\chi(j(M)) = (8+3+1) - (19+15+5) + 15 + 11 + 5 = 4$$
$$\chi(j(M)) - \#\tau = 4 - 1 = 3$$
$$\chi(M) + \tfrac{\#\mathcal{W}}{2} = 2 + \tfrac{2}{2}$$

Figure 6

REFERENCES

1. V. Arnold, *Singularity Theory*. London Math. Soc. Lecture Note Ser. **53**, Cambridge University Press, 1981.
2. W. Boy, *Über die Curvatura Integral und die Topologie geschlossner*. Math. Ann. **57**(1903).
3. M. Golubusky and V. Guillemin, *Stable mappings and their singularities*. Graduate Texts in Math. **14**, Springer-Verlag, 1973.
4. S. Izumiya and W. Marar, *The Euler Characteristic of a Generic Wavefront in a 3-Manifold*. Proc. Amer. Math. Soc. (4) **118**(1993), 1347–1350.
5. M. Kossowski, *The Lagrangian Gauss image of a surface in Euclidean 3-space*. Trans. Amer. Math. Soc. (2) **335**(1993), 791–803.

6. ———, *Realizing a singular first fundamental as a nonimmersed surface in Euclidean 3-space*. Preprint.
7. ———, *The "Boy-Gauss-Bonett" and "Boy" theorems for C^∞-singular hypersurfaces with limiting tangent bundle*. Preprint.
8. ———, *Cohomology classes for the extrinsic geometry of 1-resolvable surfaces in Euclidean 3-space*. In preparation.
9. N. Kuiper, *Stable surfaces in Euclidean three space*. Math. Scand. **36**(1975), 83–96.

Department of Mathematics
The University of South Carolina
Columbia, SC 29208
USA
email: kossowsk@math.sc.edu

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (3), 2001 pp. 71–77

SALEM NUMBERS, PV NUMBERS AND SPECTRAL RADII OF COXETER TRANSFORMATIONS

PIROSKA LAKATOS

Presented by Vlastimil Dlab, FRSC

RÉSUMÉ. Dans [MRS], une méthode est donnée pour construire des nombres de Salem et des PV-nombres. Il est montré que les plus grandes valeurs propres d'étoiles exceptionnelles (qui sont appelées étoiles sauvages en théorie des représentations) déterminent des nombres de Salem. Nous généralisons cette construction dans le cas appelé des étoiles sauvages généralisées et nous montrons que les nombres de Salem ainsi obtenus sont les rayons spectraux des transformations de Coxeter correspondantes. Cela nous donne un ensemble de nombres de Salem plus grand que celui précédemment obtenu mais qui est cependant un sous-ensemble propre de l'ensemble des nombres de Salem. Nous montrons également que les limites des rayons spectraux d'étoiles sont des PV-nombres, un fait annoncé dans [MRS].

1. Introduction. A real algebraic integer $\alpha > 1$ is called a *PV* (Pisot-Vijayaraghavan) *number* if all of its other conjugates lie inside the unit circle. Furthermore, a real algebraic integer α is called a *Salem number* if all its other conjugates lie inside or on the unit circle with at least one conjugate on the unit circle. The monic irreducible polynomial over $\mathbb{Q}$ having a Salem number as a zero is called a *Salem polynomial.* Denote by T the set of all Salem numbers.

A *star* is a (finite) tree (with simple edges) which has a unique branching vertex i_0 (called the *centre* of the star). Here, introduce a more general type of connected graphs without loops which have again exactly one vertex i_0 of (arbitrary) degree $d \geq 3$ and every other vertex of degree ≤ 2, *i.e.*, we allow also proper cycles (excluding loops) having one of the vertices the vertex i_0. We call such a graph a *generalized star.* A generalized star is called *wild* if one of the eigenvalues of its adjacency matrix is greater than 2, *i.e.*, if it is neither of Dynkin nor of Euclidean type.

Salem [S] has shown that each PV number is the limit of Salem numbers. McKee, Rowlinson and Smyth have constructed in [MRS] a set of Salem numbers and showed that their limits form a proper closed subset of the set of all PV

Received by the editors September 5, 2000.

Research partially supported by Hungarian NFSR grant no.TO29525 and by NSERC of Canada grant no. A-7257.

AMS subject classification: Primary: 11R06; secondary: 05C05, 20F55, 05C50.

Key words and phrases: PV number, Salem number, Coxeter transformation, Coxeter polynomial.

numbers. These Salem numbers have been also constructed independently in [L1]. Here, we shall prove that the related PV numbers are limits of spectral radii of Coxeter transformations of wild stars. Furthermore, we shall construct a set of Salem numbers which contains the set presented in [MRS] and [L1] as a proper subset.

Let Δ be a finite non-oriented connected graph (multiple edges are not allowed); let $\{1,2,\ldots,n\}$ be the set of its vertices. The *spectrum* $\mathrm{Spec}(\Delta)$ of Δ is the set of the eigenvalues of the adjacency matrix $A = A(\Delta) = (a_{ij})$ of Δ; here a_{ij} is the number of edges between the vertices i and j, and thus A is an integral symmetric matrix. Denote the *spectral radius* of Δ (*i.e.*, the largest eigenvalue of A) by $\rho(\Delta)$. Let Ω be such an orientation of the graph Δ that the oriented graph does not have an oriented cycle and $\mathcal{C} = \mathcal{C}_{\Omega(\Delta)}\colon \mathbb{C}^n \to \mathbb{C}^n$ the corresponding Coxeter transformation. Recall that the matrix $\Phi = \Phi_{\Omega(\Delta)}$ of $\mathcal{C}$ with respect to the standard basis can be written as $\Phi = -C^{-1}C^t$, where $C = C_{\Omega(\Delta)} = (c_{ij})$ is an integral $n \times n$ matrix with c_{ij} equal to the number of paths from the vertex j to the vertex i in $\Omega(\Delta)$. The characteristic polynomial of Φ is called the *Coxeter polynomial* of $\mathcal{C}$. The *spectrum* $\mathrm{Spec}(\mathcal{C})$ is the set of all eigenvalues of Φ and the *spectral radius* of $\mathcal{C}$ is

$$\rho(\mathcal{C}_{\Omega(\Delta)}) = \max\{|\lambda| : \lambda \in \mathrm{Spec}(\mathcal{C}_{\Omega(\Delta)})\}.$$

It is well-known (see [C]) that the characteristic polynomial of the Coxeter transformation is reciprocal and does not depend on the orientation of the tree. This allows us to speak about the Coxeter polynomial of a (non-oriented) tree without fixing a concrete orientation. In this paper we shall always assume that the graphs are equipped with bipartite orientation, *i.e.*, every vertex is either a sink or a source. Although such a bipartite orientation is not unique, the spectrum of a Coxeter transformation is independent of a particular choice of it (see *e.g.* [DR]).

2. Stars, Salem and PV numbers. Denote by

$$\Delta = \Delta_{[p_1,p_2,\ldots,p_s,C_{q_1},C_{q_2},\ldots,C_{q_r}]}$$

a wild generalized star consisting of s arms of lengths $p_1, p_2, \ldots, p_s$ and r cycles with $q_1, q_2, \ldots, q_r$ vertices. Suppose, that each cycle has an even number of vertices. Thus $n = \sum_{i=1}^{s} p_i + \sum_{j=1}^{r} q_j - r + 1$ is the number of vertices of Δ. Let Ω be a bipartite orientation of Δ and denote by $\rho(\mathcal{C}_{[p_1,p_2,\ldots,p_s,C_{q_1},C_{q_2},\ldots,C_{q_r}]})$ and by $\chi_{[p_1,\ldots,p_s,C_{q_1},C_{q_2},\ldots,C_{q_r}]}$ the spectral radius and the characteristic polynomial of $\mathcal{C}_{\Omega(\Delta_{[p_1,p_2,\ldots,p_s,C_{q_1},C_{q_2},\ldots,C_{q_r}]})}$, respectively.

The following theorem will be repeatedly used in the course of our arguments.

THEOREM [C]. *Let Δ be a graph and $\Omega(\Delta)$ its bipartite orientation. Then the following statements hold:*

a) *Given* $0 \neq \lambda \in \mathbb{C}$ *then* $\lambda + \lambda^{-1} \in \operatorname{Spec}(\Delta)$ *if and only if* $\lambda^2 \in \operatorname{Spec}(\mathcal{C}_{\Omega(\Delta)})$.
b) $\operatorname{Spec}(\mathcal{C}_{\Omega(\Delta)}) \subseteq S^1 \cup \mathbb{R}^+$, *where* $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.
c) *If* Δ *is not Dynkin, then there exists a real number* $\lambda \geq 1$ *such that* $\rho(\Delta) = \lambda + \lambda^{-1}$ *and* $\rho(\mathcal{C}_{(\Omega(\Delta))}) = \lambda^2$. *Moreover,* Δ *is Euclidean if and only if* $\lambda = 1$.

From this statement it follows that the *reciprocal polynomial* of a tree introduced in [MRS] is identical to the Coxeter polynomial of the same tree.

The smallest known Salem number τ is one of degree 10 which was discovered by Lehmer in 1933. In fact, $\tau \sim 1.17628\cdots$ is the spectral radius $\rho(\mathcal{C}_{[1,2,6]})$ and $\lim_{m\to\infty} \rho(\mathcal{C}_{[1,2,m]})(\sim 1.3241\cdots)$ is equal to the real zero of the polynomial $x^3 - x - 1$, which happens to be also the smallest PV number. Moreover τ is a lower bound of the spectral radii of Coxeter transformation of wild trees. Indeed, if the tree Δ is neither of Dynkin nor of Euclidean type, then $\rho(\mathcal{C}_\Delta) \geq \tau \sim 1.17628\cdots$ (*cf.* [D]).

In our terminology, Corollary 9 in [MRS] states that the set of spectral radii of Coxeter transformations on wild stars are Salem numbers. In this way, the spectral theory of Coxeter transformations has a close connection to number theory.

Our main result is the following theorem.

THEOREM 1. *Let* $\Omega(\Delta)$ *be a bipartite orientation of a wild generalized star* Δ *with even cycles. The spectral radius of the Coxeter transformation defined by* $\Omega(\Delta)$ *is a Salem number. Denote by* T' *the set of all Salem numbers obtained this way. Furthermore, denote by* T'' *the set of all spectral radii of Coxeter transformations on wild stars. Then* $T'' \subset T' \subset T$.

PROOF. Let $\Delta = \Delta_{[p_1,p_2,\dots,p_s,C_{q_1},C_{q_2},\dots,C_{q_r}]}$ be a wild generalized star with central vertex i_0. Since Δ has no cycle with odd number of edges there is, by Theorem [C], a one-to-one correspondence between the elements of the spectrum of the graph Δ and the elements of the spectrum of the Coxeter transformation corresponding to the bipartite orientation. Denote by Δ' the graph obtained from Δ by removing the vertex i_0 (and all adjacent edges). Since all the connected components of Δ' are linear, the eigenvalues λ'_i, $1 \leq i \leq n-1$, of Δ' are in the interval $[-2, 2]$ and the corresponding eigenvalues ρ'_i of the Coxeter transformation are on the unit circle.

Reordering the eigenvalues by their size and using the *interlacing theorem* of the Frobenius-Perron theory we have

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots,$$

and thus

$$\rho_1 \geq \rho'_1 \geq \rho_2 \geq \rho'_2 \geq \cdots.$$

The inequality $\lambda'_1 \leq 2$ implies $|\rho'_1| \leq 1$ and $|\rho_2| \leq 1$, which was needed to prove the first statement of the theorem.

It is well-known that $T' \neq T$. For example, the Salem number ρ_1 listed in [B1] and defined by the reciprocal polynomial $x^{10} - x^6 - x^5 - x^4 + 1$ is not in the set T'.

To complete the proof, we are going to show that there is an element $\rho \in T' \setminus T''$. Denote by ρ the largest real zero of the non-cyclotomic irreducible factor of the Coxeter polynomial $\chi = \chi_{[2,C_{32}]}$ corresponding to the wild generalized star $\Delta_{[2,C_{32}]}$. Using the Maple software and the package given in [B], we have

$$\chi(x) = (x+1)(x^2+1)(x^4-1)(x^8+1)(x^{18}-x^{17}-x^{16}-x^2-x+1)$$

and $\rho \sim 1.68491$. By [L1] the wild stars with degree > 4 of the centre have spectral radii > 2 if they are not $\Delta_1 = \Delta_{[2,1,1,1]}$. Since $\chi_{[2,1,1,1]}(x) = (x^2+x+1)\cdot(x^4 - x^3 - x - 1)$, we have to check only the spectral radii of the wild stars with degree 3 of their centre. Since the spectral radius of a subgraph is smaller or equal than the spectral radius of a graph and since $\lim_{m\to\infty} \rho(C_{[1,m,m]}) = \lim_{m\to\infty} \rho_{[2,2,m]} < \rho$ (see [L2]) and $\rho(C_{[2,3,4]}) \sim 1.6574 < \rho < \rho_{[2,3,5]} \sim 1.6935$, we conclude that $\rho \in T'$ and $\rho \notin T''$. ∎

By Theorem [C], it turns out that each Salem polynomial $f(x)$ obtained by Theorem 1 has exactly two real zeros. Denote by $g(x)$ the non-cyclotomic irreducible monic factor (over $\mathbb{Q}$) of $f(x)$ with a real zero. The Kronecker theorem states that the zeros of a monic polynomial (with integer coefficients) with absolute value one are roots of unity. Thus the only irreducible non-cyclotomic factor of $f(x)$ is $g(x)$.

COROLLARY 1. *The Coxeter polynomial of a bipartite wild generalized star with even cycles has exactly two real zeros and one irreducible non-cyclotomic factor.*

Let us point out that the Coxeter polynomial of a tree which is not a wild star may have more than two real zeros and more than one irreducible non-cyclotomic factor.

The following statement appears in the proof of Theorem 1 of [MRS]. Here, we present an independent proof.

For $j \in \mathbb{N}$, let

$$v_j(x) = \frac{x^{j+1}-1}{x-1} = x^j + x^{j-1} + \cdots + x^2 + x + 1 \text{ and } v_0(x) = 1, v_{-1}(x) = 0.$$

THEOREM 2. *For $s \geq k \geq 0$ $(s \geq 2)$, let $p_0(t), p_1(t), \ldots, p_k(t)$ $(t \in \mathbb{N})$ be increasing sequences of positive integers (i.e., $\lim_{t\to\infty} p_i(t) = \infty$, $i = 0, 1, \ldots, k$) and $p_{k+1}, p_{k+2}, \ldots, p_s$ positive integers. Then*

$$\{\Delta_{[p_0(t),p_1(t),p_2(t),\ldots,p_k(t),p_{k+1},\ldots,p_s]} \mid t \in \mathbb{N}\}$$

is a sequence of wild stars and

(i) $\eta = \lim_{t\to\infty} \rho(C_{[p_0(t),p_1(t),p_2(t),\ldots,p_k(t),p_{k+1},\ldots,p_s]})$ *is a root of the equation*

$$1 - \sum_{i=k+1}^{s} \frac{v_{p_i-1}(x)}{v_{p_i}(x)} - \frac{k}{x} = 0, \tag{1}$$

(ii) η *is a PV number.*

PROOF. By Theorem 1 of [MRS] the only roots of equation (1) are a certain PV number and its conjugates.

We remark that the limit in (i) exists, since the spectral radius of the Coxeter transformation of a proper subgraph of a graph is not greater than that of the graph, *i.e.*, the spectral radii form a monotone increasing and bounded sequence.

To proceed with our proof, we will make use of the following:

LEMMA (*cf.* LEMMA 2.3 OF [L2]). *Let* Γ_1 *be a finite connected tree which is not of Dynkin type and* v_1 *one of its vertices of degree one. Denote by* Γ_0 *the tree obtained from* Γ_1 *by removing* v_1 *(and the respective edge). Furthermore, for* $m \geq 2$*, denote by* Γ_m *the tree obtained from* Γ_{m-1} *by adding an edge* $v_{m-1}\_\_v_m$*. Then using an orientation* Ω *of* Γ_m*, we have* $\lim_{m\to\infty} \rho(C_{\Omega(\Gamma_m)}) = x_0$*, where* x_0 *is the largest positive real root of the polynomial* $\chi_{\Gamma_1}(x) - \chi_{\Gamma_0}(x)$.

We are going to prove (i) by induction with respect to k. First let $k = 0$, *i.e.*, we have a sequence of wild stars with $s+1$ arms of lengths $p_0(t), p_1, p_2, \ldots, p_s$. By [B], the wild stars $\Delta_{[p_0,p_1,p_2,\ldots,p_s]}$ satisfy

$$\chi_{[p_0,p_1,\ldots,p_s]}(x) = \prod_{i=0}^{s} v_{p_i}(x) \left((x+1) - x \sum_{i=0}^{s} \frac{v_{p_i-1}(x)}{v_{p_i}(x)} \right). \tag{2}$$

Now, making use of the Lemma, we get that the spectral radius tends to the largest positive root of the polynomial

$$\chi_{[1,p_1,\ldots,p_s]}(x) - \chi_{[p_1,\ldots,p_s]}(x) = x^2 \prod_{i=1}^{s} v_{p_i}(x) \left(1 - \sum_{i=1}^{s} \frac{v_{p_i-1}(x)}{v_{p_i}(x)} \right);$$

thus, for $k = 0$ the statement (i) holds.

Assume that (i) is true for $k-1$, *i.e.*, $\lim_{t\to\infty} \rho(C_{[p_0(t),p_1(t),p_2(t),\ldots,p_{k-1}(t),p_k,\ldots,p_s]}) = \alpha_{p_k}$ and α_{p_k} is a root of the equation

$$g_{p_k}(x) = 1 - \sum_{i=k}^{s} \frac{v_{p_i-1}(x)}{v_{p_i}(x)} - \frac{k-1}{x}.$$

Let

$$f(x) = 1 - \sum_{i=k+1}^{s} \frac{v_{p_i-1}(x)}{v_{p_i}(x)} - \frac{k}{x}.$$

It is easy to check the identity

$$g_{p_k}(x) = f(x) + \frac{1}{xv_{p_k}(x)}. \tag{3}$$

Now set $p_k = p_k(t)$ and suppose that α is the PV root of the equation $f(x) = 0$ and α_{p_k} $(= \alpha_{p_k(t)})$ is the PV root of the equation $g_{p_k}(x) = 0$. Let $\mathcal{O}_{p_k}$ and $\mathcal{D}_{p_k}$ be the circle and the disc with radius $\frac{\alpha-1}{2^{p_k}}$ about $\alpha > 1$. By the definition of α, the polynomial $f(x)$ has no zero on $\mathcal{O}_{p_k}$. Hence

$$\min_{x\in\mathcal{O}_{p_k}} |f(x)| > 0.$$

Since $\lim_{t\to\infty} \frac{1}{xv_{p_k}(x)} = 0$ for $x > 1$ we have

$$|f(x)| > \frac{1}{|xv_{p_k}(x)|}, \quad \text{if } x \in \mathcal{O}_{p_k} \text{ and } t \text{ is large enough.}$$

Using Rouche's theorem and (3), we get that $g_{p_k}(x)$ and $f(x)$ have the same number of zeros in $\mathcal{D}_{p_k}$.

Clearly, if $t \to \infty$, then $\alpha_{p_k} \to \alpha$. Thus, by induction on k, the proof of (i) is completed.

Finally by Theorem [C] c), the spectral radii of Coxeter transformation of wild stars are > 1, and therefore $\eta > 1$. By (i) and Theorem 1 of [MRS], η must be identical with the PV root α of $f(x)$. ∎

COROLLARY 2. *If*

$$\Gamma_1(t) = \{\Delta_{[p_1(t),p_2(t),\ldots,p_s(t),m-1]} \mid t \in \mathbb{N}\} \quad \textit{and}$$
$$\Gamma_2(t) = \{\Delta_{[p_0(t),\underbrace{m,m,\ldots,m}_{s-\text{times}}]} \mid t \in \mathbb{N}\}$$

are sequences of wild stars with $\lim_{t\to\infty} p_i(t) = \infty$ *(*$i = 0, 1, 2, \ldots, s$*;* $s \geq 2$*), then*

(i)

$$\eta := \lim_{t\to\infty} \rho(C_{\Gamma_1(t)}) = \lim_{t\to\infty} \rho(C_{\Gamma_2(t)}),$$

(ii)

$$s - 1 \leq \eta < s.$$

PROOF. (i) follows from Theorem 2, since the polynomials

$$1 - \frac{v_{m-2}(x)}{v_{m-1}(x)} - \frac{s-1}{x} \quad \text{and} 1 - s\frac{v_{m-1}(x)}{v_m(x)},$$

which determine the limit of spectral radii of Γ_1 and Γ_2, have the same real zeros as the polynomial $F(x) = x^m - (s-1)x^{m-1} - \cdots - (s-1)x - (s-1)$.

We have $F(x) > 0$ if $x \geq s(\geq 2)$ and $F(s-1) < 0$, hence (ii) holds. ∎

REMARK 1. The list of trees with spectral radius of the Coxeter transformation less than the 'golden ratio' $\frac{1+\sqrt{5}}{2}$ is known (see [CDS]). These are all trees. Therefore, not each small Salem number is a value of spectral radius of Coxeter transformation of some bipartite graph. Moreover, using the results of [CDS] one may check that there are only 6 graphs, the wild generalized stars $\Delta_{[1,2,6]}$, $\Delta_{[1,2,7]}$, $\Delta_{[1,2,8]}$, $\Delta_{[1,2,9]}$, $\Delta_{[1,2,10]}$ and $\Delta_{[1,3,4]}$ whose spectral radii are less than 1.3.

REFERENCES

[B] A. Boldt, *Methods to determine Coxeter polynomials.* Linear Algebra Appl. **230**(1995), 151–164.

[B1] D. W. Boyd, *Small Salem numbers.* Duke Math. J. (2) **44**(1977), 315–328.

[C] N. A'Campo, *Sur les valeurs propres de la transformation de Coxeter.* Invent. Math. **33**(1972), 61–67.

[CDS] D. Cvetkovič, M. Doob and I. Gutman, *On graphs whose eigenvalues do not exceed* $\sqrt{2+\sqrt{5}}$. Ars Combin. **1**(1982), 225–239.

[D] V. Dlab, *Pairs of semi-simple algebras algebras.* In: Representation of Algebras, London Math. Soc. Lecture Note Ser. **168**, Cambridge University Press, 1992, 185–199.

[DR] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras.* Mem. Amer. Math. Soc. **173**(1976), 1–57.

[L1] P. Lakatos, *On spectral radius of Coxeter polynomials of trees.* Publ. Math. Debrecen **54**(1999), 181–187.

[L2] ———, *On Coxeter polynomials of wild stars.* Linear Algebra Appl. **293**(1999), 159–170.

[MRS] J. F. McKee, P. Rowlinson and C. J. Smyth, *Salem numbers and Pisot numbers from stars.* In: Number theory in progress, Vol. 1 (eds. K. Győry *et al.*), de Gruyter, Berlin, 1999, 309–319.

[S] R. Salem, *Power series with integer coefficients.* Duke Math. J. **12**(1945), 153–172.

Institute of Mathematics and Informatics
University of Debrecen
P.O. Box 12
4010 Debrecen
Hungary
email: lapi@math.klte.hu

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (3), 2001 pp. 78–83

GENERALIZED ALBANESE VARIETIES OF ROSENLICHT-LANG-SERRE

NIRANJAN RAMACHANDRAN

Presented by M. Ram Murty, FRSC

ABSTRACT. In this note, we link the generalized Albanese variety (of Rosenlicht-Lang-Serre) of a smooth variety over a field of characteristic zero with its De Rham cohomology. This is done by means of one-motives (in the sense of Deligne).

0. Introduction. Let k be a field of characteristic zero. J.-P. Serre has generalized the constructions of M. Rosenlicht and S. Lang by defining the "generalized Albanese variety" G of any variety X. This group variety G is a semiabelian variety; as such, it can be interpreted as a 1-motive (in the sense of P. Deligne [2, Section 10]), which we denote by $M_1(X)$. Its dual 1-motive is denoted $M^1(X)$. The purpose of this note is to prove:

THEOREM 1. *Let X be a smooth variety. The De Rham realization of $M^1(X)$ is naturally isomorphic to the De Rham cohomology group $H^1_{dR}(X)$ of X.*

The proof is based on results of Serre and Deligne. A. Weil was the first to observe the connection between semiabelian varieties and differential forms (of the third kind) [20].

NOTATIONS. We denote by S the scheme $\operatorname{Spec} k$; by $\bar{S}$, $\operatorname{Spec} \bar{k}$ of an algebraic closure of k. Let $\mathbb{G}$ denote the Galois group of $\bar{k}$ over k. For any scheme V over S, $\bar{V}$ denotes the base change of V to $\bar{S}$. A semiabelian variety is a commutative group variety G which is an extension of an abelian variety A by a torus T; it defines an exact sequence

$$0 \to T \to G \to A \to 0$$

where all the maps are homomorphisms of algebraic groups; one can interpret G as defining a class $[G]$ in the group $\operatorname{Ext}^1(A,T)$ of extensions of A by T (in the category of commutative algebraic groups) [19].

A 1-motive $M := [B \xrightarrow{u} G]$ consists of a $\mathbb{G}$-module B (the underlying abelian group is finitely generated and torsion-free), a semiabelian variety G and u is a homomorphism of $\mathbb{G}$-modules.

Received by the editors June 8, 2000.
Supported by NSF and the Alfred P. Sloan Doctoral Dissertation Fellowship, 1995–96.
AMS subject classification: 11Gxx, 14C25, 14C30, 14F99.

1. Generalized Albanese variety.

1.1. *Existence.* The generalized Albanese variety of X (defined by Serre) can be described by means of its universal property. Consider maps $f: X \to P$ where P is a torsor under a semiabelian variety B; the varieties B, P, and the map f are supposed to be defined over k. The "generalized Albanese variety" G of X is the universal object for such morphisms. Namely, there exists a map $u: X \to L$ where L is a torsor under G such that any map f as above factors as $\bar{f}u$ in a unique way. Here $\bar{f}: L \to P$ is supposed to be a morphism of torsors, *i.e.*, there exists a map $h: G \to B$ such that $\bar{f}$ is compatible with the actions of G and B and the map h.

In [17], Serre shows that every variety V admits a universal morphism (over $\bar{k}$) to a semiabelian variety G (this variety is defined over k) and the method of descent as in [19, pp. 102–107] assures the existence of a universal map (defined over k) to a torsor L under the universal semiabelian variety G. This semiabelian variety G is the "generalized Albanese" of V. For a smooth projective variety V, this is none other than the classical Albanese variety Alb(V).

EXAMPLE 2. Let X be a normal integral curve. There is a unique normal projective curve X' containing X as an open dense subscheme; let R be the closed complement of X in X'. The semiabelian variety G is the generalized Jacobian of Rosenlicht [16] of X' corresponding to the modulus R [18, p. 4].

The "generalized Albanese" is evidently a covariant functor.

1.2. *Explicit construction.* Since the characteristic of k is zero, any smooth variety X can be realized as the complement of a divisor E (with normal crossings) in a smooth and projective variety X' (Hironaka's resolution of singularities). When such a presentation of X is chosen, Serre [18] provides an explicit construction of the "generalized Albanese variety" G of X.

Let $w(E)$ the set of connected components of $\bar{E} \times_S \bar{S}$ or, what is the same, the set of irreducible components of $\bar{E}$. Given a set W, one defines the abelian group $\mathbb{Z}^W$ to be the group of integer valued functions on W, *i.e.*, $\mathbb{Z}^W := \text{Maps}(W, \mathbb{Z})$. For a finite set W, the groups $\mathbb{Z}(W)$ and $\mathbb{Z}^W$ are naturally dual as abelian groups.

There is a divisor class map from the $\mathbb{G}$-module $\mathbb{Z}^{w(E)}$ to the Néron-Severi group $NS(\bar{X}')$ of $\bar{X}'$

$$b: \mathbb{Z}^{w(E)} \to NS(\bar{X}') \quad r \mapsto [\mathcal{O}(r)] \tag{1}$$

which sends a divisor to the class of the invertible sheaf $\mathcal{O}(r)$; this is a $\mathbb{G}$-equivariant map.

Let I denote the kernel of the morphism b. Under the natural duality between $\mathbb{Z}^{w(E)}$ and $\mathbb{Z}(w(E))$, the group I is dual to the subgroup of divisors of $X' \times_S \bar{S}$, supported on $E \times_S \bar{S}$, which are algebraically equivalent to zero. Consequently, both these groups are of the same rank. Denote by v the natural map from I to the Picard variety $\mathrm{Pic}(\bar{X}')$ of $\bar{X}'$; this is also a map of $\mathbb{G}$-modules. This determines a 1-motive [2, Section 10.1] $M^1(X) := [I \xrightarrow{v} \mathrm{Pic}(X')]$. The dual 1-motive $M_1(X)$ [2, Section 10.2] is of the form $[0 \to H]$; the semiabelian variety H is identified in the next theorem.

THEOREM 3 (J.-P. SERRE). *The semiabelian variety H is naturally isomorphic to the generalized Albanese variety G of X [18].*

Let $A := \mathrm{Alb}(X')$ be the Albanese variety of X'. A consequence of Theorem 3 is that the kernel of the natural map $G \to A$ is the torus $T := \mathrm{Hom}(I, \mathbb{G}_m)$ [18]. Another corollary is the following: *G is isogenous (resp. isomorphic) to the product $A \times T$ if and only if the image of the map v is torsion (resp. zero).* Note that G being isogenous (resp. isomorphic) to the product $A \times_S T$ is equivalent to the class $[G] \in \mathrm{Ext}^1(A, T)$ being torsion (resp. finite).

REMARK 4. When $k = \mathbb{C}$, the "generalized Albanese variety" G of X can be generalized (!) further to the "higher Albanese manifolds" $\mathrm{Alb}^m(X)$ [7], [15], [8]; it is true that $\mathrm{Alb}^1(X) = G$ is algebraic but the higher $\mathrm{Alb}^m(X)$ $(m > 1)$ are not.

2. De Rham cohomology. We now consider the link with De Rham cohomology of X. We may assume that X is connected. We have the universal maps:

$$\begin{array}{ccc} X & \xrightarrow{u_X} & L \\ \downarrow j & & j_* \downarrow \\ X' & \xrightarrow{u_{X'}} & L', \end{array}$$

where L (resp. L') are the torsors under G (resp. A), whose existence is assured by Serre.

Denote by $\Omega^1_{X'}(\log E)$ the sheaf of one-forms on X' which are regular on X and are allowed only logarithmic poles on E. The group (vector space over k) of global sections $H^0(X'; \Omega^1_{X'}(\log E))$ consists of the classical *differential forms of the third kind* (regular on X and at most logarithmic singularities along E); there is a natural inclusion $H^0(X'; \Omega^1_{X'}(\log E)) \hookrightarrow H^0(X; \Omega^1_X)$.

Recall that the De Rham cohomology groups $H^*_{dR}(V)$ of any smooth variety V (of dimension n) are, by definition, the hypercohomology groups of the De Rham complex of V:

$$[\mathcal{O}_V \xrightarrow{d} \Omega^1_V \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_V]. \tag{2}$$

For any group variety N, let us write ω_N to denote the cotangent space at the identity element of N; it is dual to the Lie algebra *Lie* N of N. Let us define ω_L as the global one-forms on L invariant under the natural action of G; one may identify it with ω_G. Similarly, one may define $\omega_{L'}$ and identify it with ω_A.

The key to the proof of Theorem 1 is the following (*cf.* [20]).

THEOREM 5 (SERRE). *With notations as above, pullback via u_X furnishes an isomorphism of $\omega_L = \omega_G$ with $H^0(X'; \Omega^1_{X'}(\log E))$ (both regarded as subspaces of $H^0(X; \Omega^1_X)$).*

PROOF. See [18]. ∎

3. An example. An important example is given by the (smooth) modular curve $X_0(N)$ (of level N) and its smooth compactification $X_0'(N)$. The Picard variety of $X_0'(N)$ is none other than the Jacobian $J_0(N)$ of $X_0'(N)$. The group I in this context is generated by the cusps. The map v is the Abel-Jacobi map into the Jacobian $J_0(N)$. The following theorem is quite useful in the theory of modular forms [3, Section 7].

THEOREM 6 (MANIN-DRINFELD [13]). *The image of the cusps (under the Abel-Jacobi map v) in $J_0(N)$ is torsion.*

A Hodge-theoretic proof of this theorem can be found in [4]. The cusp forms of weight two and level N are the global one-forms on $X_0'(N)$ and the corresponding Eisenstein series are the differential forms of the third kind (with logarithmic poles along the cusps) on $X_0'(N)$; the previous theorem allows one to split off the Eisenstein series off the modular forms. For more details, we refer to [3, Section 7] and [1]. Generalizations of the previous theorem may be found in [12], [6], and [11]; note that the exact order of the image of v (which is torsion = finite) is of great arithmetic importance (see, for instance, Chapters 5 and 6 of [12]).

QUESTION. Does Theorem 6 admit a generalization to other Shimura varieties?

4. Proof of Theorem 1. The De Rham cohomology groups of X are naturally isomorphic to the hypercohomology groups of the *logarithmic De Rham complex* on X' [2, Section 3]:

$$\Omega^*_{X'}(\log E)\colon [\mathcal{O}_{X'} \xrightarrow{d} \Omega^1_{X'}(\log E) \xrightarrow{d} \cdots \Omega^{\dim X}_{X'}(\log E)], \tag{3}$$

$$\mathbb{H}^*(X'; \Omega^*_{X'}(\log E)) \xrightarrow{\sim} H^*_{dR}(X). \tag{4}$$

This implies that $H^1_{dR}(X)$ can be decomposed as follows [5]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X'; \Omega^1_{X'}(\log E)) & \longrightarrow & H^1_{dR}(X) & \longrightarrow & H^1(X'; \mathcal{O}_{X'}) & \longrightarrow & 0 \\
 & & j^*\uparrow & & j^*\uparrow & & \| & & \\
0 & \longrightarrow & H^0(X'; \Omega^1_{X'}) & \longrightarrow & H^1_{dR}(X') & \longrightarrow & H^1(X'; \mathcal{O}_{X'}) & \longrightarrow & 0,
\end{array}$$

where the vertical inclusion defines the weight filtration and the horizontal inclusion defines the Hodge filtration on $H^1_{dR}(X)$.

Just for the proof, we abbreviate $M^1(X')$ by $M' := [0 \to \mathrm{Pic}(\bar{X}')]$ and $M^1(X)$ by $M := [I \xrightarrow{v} \mathrm{Pic}(X')]$. Let us examine the construction of $\mathfrak{T}_{dR}(M)$ [2, 10.1.7,

p. 58]: The sheaf on S_{fppf} defined by the functor $\mathcal{E}xt^1(M, \mathbb{G}_a)$ is representable by $\mathcal{L}ie\ G'$, where $G' := \mathcal{E}xt^1(M, \mathbb{G}_m)$ (here G' is a semiabelian variety). By the Weil-Barsotti theorem [14], $\mathcal{E}xt^1(\mathrm{Pic}(X'), \mathbb{G}_m) = A$. Therefore, $\mathcal{E}xt^1(M', \mathbb{G}_a) = \mathrm{Lie}\, A$. There is an exact sequence:

$$0 \to \mathrm{Hom}(I, \mathbb{G}_a) \to \mathrm{Ext}^1(M, \mathbb{G}_a) \to \mathrm{Ext}^1(M', \mathbb{G}_a) \to 0. \tag{5}$$

The 1-motive $[0 \to G']$ is, by definition, dual to M; by Theorem 3, we may identify G' as G. Thus the exact sequence (5) may be identified with:

$$0 \to Lie\ T \to Lie\ G \to Lie\ A \to 0. \tag{6}$$

Considering the dual exact sequence, and using $(Lie\ G)^* = \omega_G$ (the dual k-vector space is indicated by $*$), we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1(M', \mathbb{G}_a)^* & \longrightarrow & \mathrm{Ext}^1(M, \mathbb{G}_a)^* & \longrightarrow & \mathrm{Hom}(I, \mathbb{G}_a) & \longrightarrow & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & \omega_A & \longrightarrow & \omega_G & \longrightarrow & \omega_T & \longrightarrow & 0. \end{array}$$

Deligne [2, Section 10.1] defines $\mathfrak{T}_{dR}(M)$ to be the push out in the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Ext}^1(M', \mathbb{G}_a)^* & \longrightarrow & \mathfrak{T}_{dR}(M') \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ext}^1(M, \mathbb{G}_a)^* & \longrightarrow & \mathfrak{T}_{dR}(M). \end{array}$$

Using the diagram on the Lie algebras induced by the previous diagram, we obtain that $\mathfrak{T}_{dR}(M)$ is the push out of the inclusions $\omega_A \hookrightarrow \omega_G$ and $\omega_A \hookrightarrow \mathfrak{T}_{dR}(M')$.

One has $\mathfrak{T}_{dR}(M') \xrightarrow{\sim} H^1_{dR}(X')$ [9, II 3.11], [10, pp. 46–47] which is compatible with the isomorphism $\omega_A \xrightarrow{\sim} H^0(X'; \Omega^1_{X'})$. Combining this with Lemma 5 and the commutative diagram immediately before (5), we obtain a map $\mathfrak{T}_{dR}(M) \to H^1_{dR}(X)$ by the universal nature of $\mathfrak{T}_{dR}(M)$; this map is easily seen to be an isomorphism by the above discussion. ∎

ACKNOWLEDGEMENTS. I would like to thank S. Lichtenbaum, R. Murty, L. Washington for much encouragement. The results obtained in this paper form part of my doctoral thesis (Brown University, 1996) under the guidance of S. Lichtenbaum.

REFERENCES

1. C. Brinkmann, *Die Andersonextension und 1-Motive*. Bonner Mathematische Schriften **223**, Universität Bonn, Mathematisches Institut, Bonn, 1991.
2. P. Deligne, *Théorie de Hodge II*. Inst. Hautes Études Sci. Publ. Math. **40**(1972), 5–58; **III**. Inst. Hautes Études Sci. Publ. Math. **44**(1974), 3–77.

3. ______, *Valeurs de fonctions L et périodes d'intégrales*. Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and *L*-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Amer. Math. Soc., Providence, R.I., 1979, 313–346.
4. R. Elkik, *Le théorème de Manin-Drinfeld*. Séminaire sur les Pinceaux de Courbes Elliptiques (Paris, 1988), Astérisque **183**(1990), 59–67.
5. G. Faltings and G. Wüstholz, *Einbettungen kommutativer algebraischer Gruppen und einige ihrer Eigenschaften*. J. Reine. Angew. Math **354**(1984), 175–205.
6. E.-U. Gekeler, *Modulare Einheiten für Funktionenkörper*. J. Reine Angew. Math. **348** (1984), 94–115.
7. R. Hain, *Higher Albanese manifolds*. Hodge theory (Sant Cugat, 1985), Lecture Notes in Math. **1246**, Springer, Berlin-New York, 1987, 84–91.
8. S.-Y. Hwang-Ma, *Periods of iterated integrals of holomorphic forms on a compact Riemann surface*. Trans. Amer. Math. Soc. (2) **264**(1981), 295–300.
9. L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*. Ann. Sci. École Norm. Sup. (4) **12**(1979), 501–661.
10. ______, *Crystalline cohomology*. Proc. Symp. Pure Math. **55**(1994), 43–70.
11. M. M. Kapranov, *Cuspidal divisors on the modular varieties of elliptic modules*. Math. USSR-Izv. (3) **30**(1988), 533–547.
12. D. S. Kubert and S. Lang, *Modular Units*. Grundlehren Math. Wiss. **244**, Springer-Verlag, New York-Berlin, 1981.
13. Yu. I. Manin, *Parabolic points and zeta functions of modular curves*. Math. USSR-Izv. **6**(1972), 19–64.
14. F. Oort, *Commutative group schemes*. Lecture Notes in Math. **16**, Springer-Verlag, 1966.
15. A. N. Parshin, *On a certain generalization of the Jacobian manifold*. Amer. Math. Soc. Transl. (2) **84**(1969), 187–196.
16. M. Rosenlicht, *Generalized Jacobian varieties*. Ann. of Math. **59**(1954), 505–530.
17. J.-P. Serre, *Morphismes universels et variété d'Albanese*. Variétés de Picard, Séminaire C. Chevalley, E.N.S. Paris, 1958–59.
18. ______, *Morphismes universels et différentiels de troisième éspèce*. Variétés de Picard, Séminaire C. Chevalley, E.N.S. Paris, 1958–59.
19. ______, *Algebraic groups and class fields*. Graduate Texts in Math. **117**, Springer Verlag, New York-Berlin, 1988.
20. A. Weil, *Variétés abéliennes*. Colloque d'algèbre et théorie des nombres, CNRS, Paris, 1949, 125–128 (= Collected Papers [1950a]).

Department of Mathematics
University of Maryland
College Park, MD 20742-4015
USA
email: atma@math.umd.edu

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (3), 2001 pp. 84–90

INTEGER POINTS NEAR HYPERELLIPTIC CURVES

MARIAN VÂJÂITU AND ALEXANDRU ZAHARESCU

Presented by M. Ram Murty, FRSC

RÉSUMÉ. On se propose de décrire une manière de construire des points entiers près d'une courbe algèbrique et d'appliquer cette méthode dans le cas des courbes hyperelliptiques définies par

$$Y^2 = X^{2d+1} + P(X),$$

où $P(X) = a_0 X^d + \cdots + a_d$ est un polynôme à coefficients réels. Dans le cas où a_0 est rationel ces points entiers sont encore plus près de la courbe que prévu.

1. Introduction. Let C be an algebraic curve given by an equation $f(X,Y) = 0$ where $f(X,Y)$ is a polynomial with real coefficients, irreducible over $\mathbf{R}$. Assuming C has genus ≥ 1, by a classical result of Siegel it follows that there are only finitely many integer points on C. It is expected however that when the set $C(\mathbf{R})$ of real points of C is unbounded there are infinitely many integer points "close" to the curve C. In this paper we present a method which can be used to produce such integer points. It turns out that for a certain class of hyperelliptic curves, the method works unexpectedly well. To be precise, let $P(X)$ be a polynomial of degree d and let C be the curve given by the equation:

$$Y^2 = X^{2d+1} + P(X). \tag{1}$$

For almost all such polynomials $P(X)$ the polynomial $X^{2d+1} + P(X)$ will have distinct roots and C will be a hyperelliptic curve of genus d. Let $\theta(C) \in \mathbf{R} \cup \{-\infty\}$ be the lower bound of those $\theta \in \mathbf{R}$ for which the inequality:

$$|Y^2 - X^{2d+1} - P(X)| \leq X^\theta$$

has infinitely many integer solutions. If we let $X \in \mathbf{Z}$, $X \to \infty$ and choose $Y \in \mathbf{Z}$ such that $|Y - \sqrt{X^{2d+1} + P(X)}|$ is smallest, then clearly $|Y^2 - X^{2d+1} - P(X)| \ll X^{d+\frac{1}{2}}$ which shows that for any such C one has $\theta(C) \leq d + 1/2$. Reasoning heuristically, we would expect that when X takes the values $1, 2, \ldots, N$ some of the fractional parts $\{\sqrt{X^{2d+1} + P(X)}\}$ are as small as $1/N$, or $1/N^{1-\varepsilon}$ say. By choosing Y as before we will have integer points (X,Y) for which

Received by the editors February 8, 2001.
AMS subject classification: 11J25, 11D75, 11J54.

$|Y^2 - X^{2d+1} - P(X)| < X^{d-\frac{1}{2}+\varepsilon}$. This leads us to expect that for any C as above one has:

(2) $$\theta(C) \leq d - \frac{1}{2}.$$

A standard method to produce small fractional parts is to establish upper bounds for certain exponential sums. More precisely, the Erdős-Turán inequality relates the discrepancy of a finite set of points $U = \{u_n; 1 \leq n \leq N\} \subset [0,1]$ to upper bounds for exponential sums of the form

(3) $$S_m(U) = \sum_{n=1}^{N} e(mu_n), \quad m = 1, 2, \ldots$$

where $e(t) = e^{2\pi i t}$ (see Montgomery [6, Ch. 1]). The method has been successfully applied to produce small fractional parts of polynomials (see Schmidt [8] and Baker [1]), in other words to produce integer points close to a curve $Y = P(X)$ where $P(X)$ is a polynomial with real coefficients. In general when using exponential sums one hopes to obtain "square root cancellation" in the sums $S_m(U)$:

(4) $$|S_m(U)| \ll_\varepsilon N^{1/2}(mN)^\varepsilon$$

which would create a square root saving in the discrepancy. Thus assuming (4) in our case, *i.e.*, when $u_n = \{\sqrt{n^{2d+1} + P(n)}\}$ we would obtain $\theta(C) \leq d$ which is right in the middle between the trivial upper bound $\theta(C) \leq d + 1/2$ and the expected upper bound $\theta(C) \leq d - 1/2$. In order to break the square root barrier one needs to come up with a new method. This is done in [10] in the case of fractional parts of αn^2, where $\alpha \in \mathbf{R}$, *i.e.*, in the case of integer points close to the parabola $Y = \alpha X^2$. The curves given by (1) form another class of algebraic curves for which we can break the square root barrier.

THEOREM 1. *Let* $P(X) \in \mathbf{R}[X]$, $\deg P = d \geq 1$ *and let* C *be the curve given by the equation (1). Then one has:*

$$\theta(C) \leq d - \frac{1}{2d+1}.$$

In case the leading coefficient of $P(X)$ is rational we have the following surprising inequality, which is sharper than (2):

THEOREM 2. *Let* $P(X) \in \mathbf{R}[X]$, $\deg P = d \geq 1$ *such that its leading coefficient is rational and let* C *be the curve given by the equation (1). Then*

(5) $$\theta(C) \leq \frac{2d^2 - 1}{2d + 1}.$$

One may wonder how Theorem 2 above interferes with some well known conjectures in this area. Let us consider the case of an elliptic curve E defined over $\mathbf{Q}$, given by the equation:

$$Y^2 = X^3 + aX + b.$$

Assume first that $a = 0$. A conjecture of Hall [4] asserts that if x, y are positive integers such that $k = x^3 - y^2$ is nonzero, then $\sqrt{x} \leq C|k|$ for some absolute constant C. In this form the conjecture might be too strong. In all the cases investigated by Gebel, Pethö and Zimmer [3] the above inequality holds true with $C = 5$, but recently Elkies [2] found the example

$$5853886516781223^3 - 447884928428402042307918^2 = 1641843$$

for which $\sqrt{x}/|x^3 - y^2| = 46.60\ldots$. Lang [5] writes Hall's conjecture in the weaker form $|k| \gg_\epsilon x^{\frac{1}{2}-\epsilon}$, which is then a special case of the well known ABC conjecture of Masser and Oesterlé [7]. All the integer points close to the curve provided by our method in the particular case $a = 0$ lie on the rational curve $Y^2 = X^3$, therefore our result does not disprove Hall's conjecture. In case $a \neq 0$ our result is related to the exceptional set in Vojta's General Conjecture [9]. Although this conjecture does not provide any nontrivial lower bound for $\theta(E)$, it describes the nature of the set of integer points unusually close to E. More precisely from Vojta's Conjecture it follows that the integer points exceptionally close to E provided by our method are forced to lie on the union of a finite set of rational curves. It turns out that this is indeed the case. For example, when E is given by an equation of the form

$$Y^2 = X^3 + X + b$$

we have the following parametrization for the integer points provided by our method which are responsible for the inequality $\theta(E) \leq \frac{1}{3}$ given by Theorem 2:

$$\begin{cases} X = t^6 + 2t^2 \\ Y = t^9 + 3t^5 + \frac{3t}{2}. \end{cases}$$

This parametrization was discovered earlier by Stark (see Lang [5]). Besides Vojta's Conjecture we also mention a conjecture of Hall-Lang-Stark (see Lang [5] and Vojta [9]) which is relevant to our discussion. The conjecture states that if E is an elliptic curve given by an equation

$$Y^2 = X^3 + aX + b$$

with a and b integers then for any integer point (X, Y) which lies on E one has:

$$|X| \ll_\epsilon \max(|a|^3, |b|^2)^{\frac{5}{3}+\epsilon}.$$

Lang originally posed the conjecture with an unknown exponent, then Stark suggested on probabilistic grounds that the exponent should be $\frac{5}{3}$. This would imply that $\theta(E) \geq \frac{3}{10}$. However, this conjecture was disproved by Elkies, who provided the following parametrization:

$$qy^2 = x^3 + ax + b,$$

where $a = 33$, $b = -18(8t-1)$, $q = 9t^2-10t+3$, $x = 324t^4-360t^3+216t^2-84t+15$ and $y = 36(54t^5 - 60t^4 + 45t^3 - 21t^2 + 6t - 1)$. Here one rescales to say $(a, b, x) = (4a, 8b, 2x)$, and then solve the Pell equation $2q = z^2$, which has infinitely many solutions. Thus for instance if we let E be the elliptic curve given by the equation

$$y^2 = x^3 + 132x$$

then we find that $\theta(E) \leq \frac{1}{4}$. Returning to the case of a general d, we mention that for $d \geq 2$ the same phenomenon appears and Vojta's General Conjecture is not disproved by our result. We will see that all the integer points provided by our method which are exceptionally close to a given hyperelliptic curve C lie on the same rational curve. Moreover, we can find a parametrization for this rational curve. Therefore the method enables us to find concrete curves which are exceptional from the point of view of Vojta's Conjecture. For example let $d = 2$ and $P(X)$ monic. So C is given by an equation of the form:

$$Y^2 = X^5 + X^2 + aX + b.$$

Then we have the following parametrization which produces integer points exceptionally close to C:

$$\begin{cases} X = \frac{16}{9}t^{10} - \frac{8}{3}t^4 \\ Y = \frac{1024}{243}t^{25} - \frac{1280}{81}t^{19} + \frac{160}{9}t^{13} - \frac{40}{9}t^7 - \frac{5}{6}t. \end{cases} \tag{6}$$

The points (X, Y) given by (6) are responsible for the inequality (5) which in this case states that $\theta(C) \leq \frac{7}{5}$. We end the Introduction with an example in case $d = 3$. Let C be given by the equation:

$$Y^2 = X^7 + X^3 + aX^2 + bX + c.$$

Then our exceptional curve responsible for the inequality $\theta(C) \leq \frac{17}{7}$ stated in (5) has the following parametrization:

$$\begin{cases} X = \frac{64}{25}t^{14} + \frac{16}{5}t^6 \\ Y = \frac{2097152}{78125}t^{49} + \frac{1835008}{15625}t^{41} + \frac{114688}{625}t^{33} + \frac{14336}{125}t^{25} + \frac{448}{25}t^{17} - \frac{56}{25}t^9 + \frac{7}{10}t. \end{cases}$$

2. **The method.** We shall apply the method to the class of hyperelliptic curves described in the introduction, although in principle it can be used to provide integer points close to more general algebraic curves C defined over $\mathbf{R}$. The method consists of the following steps:

STEP 1. C and f being given, the equation $f(X, Y) = 0$ defines implicitly Y as a function Y_f of X. One needs to find integer values for X such that the fractional part of $Y_f(X)$ is small. We take this algebraic function $Y_f(X)$ and look at its expansion at ∞. Then truncate it by considering the first k terms of this expansion, where k is to be chosen later in order to optimize the final result.

STEP 2. If ∞ is a branch point for $Y_f(X)$ with local uniformizer $X^{1/l}$ so that the above expansion is a power series in $X^{1/l}$ then write X in the form $X = u^l + v$ where u and v will later be allowed to assume integer values only. Here we force v to be smaller than a certain power of u say $|v| \le u^\gamma$, where $\gamma > 0$ is to be determined later in order to optimize the final result. Next, express each term of the form $X^{m/l}$ as a series involving u and v and then truncate each such series.

STEP 3. We are left with a finite sum of the form:

$$\sum_{m,r} c_{m,r} u^m v^r,$$

and want to find u and v such that the fractional part of this sum is small. Here m and r are integers and r assumes only positive values. The terms $c_{m,r}u^m v^r$ for which $c_{m,r}$ is rational and m is positive can be removed, in the sense that they can be made to be integers by restricting the values of u and v to some arithmetic progressions.

STEP 4. We now come to the heart of our method: restricting u and v to have almost the same prime factors, with suitable exponents, one may arrange that some terms of the form $u^m v^r$ with $m < 0$ and $c_{m,r}$ rational are also integers.

STEP 5. Use diophantine approximation techniques to kill some of the biggest remaining terms. Alternatively, one may use techniques based on exponential sums in this last step.

3. **Proof of Theorems 1 and 2.** We follow the above Steps 1–5. Let us start with the proof of Theorem 2. Let $P(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d$ with $0 \ne a_0 \in \mathbf{Q}$ and $a_1, \ldots, a_d \in \mathbf{R}$ and let C be the curve given by (1). As in Step 1 above we have:

$$Y_f(X) = (X^{2d+1} + a_0X^d + \cdots + a_d)^{\frac{1}{2}} = X^{d+\frac{1}{2}} + \frac{a_0}{2}X^{-\frac{1}{2}} + O(X^{-\frac{3}{2}}).$$

The theorem will be proved if we show that the inequality

$$\left\| X^{d+\frac{1}{2}} + \frac{a_0}{2}X^{-\frac{1}{2}} \right\| \ll_\varepsilon X^{-\frac{4d+3}{4d+2}+\varepsilon} \tag{7}$$

holds true for infinitely many positive integers X, where $\|.\|$ stands for the distance to the nearest integer. Following Step 2 we split up X as $X = u^2 + v$, where $u, v \in \mathbf{N}$. Here v is bounded by u^γ where $0 < \gamma < 1$ is a real number whose precise value will be given later. Since $X \approx u^2$, what we need to show is that one has

$$\left\| (u^2+v)^{d+\frac{1}{2}} + \frac{a_0}{2}(u^2+v)^{-\frac{1}{2}} \right\| \ll_\varepsilon u^{-\frac{4d+3}{2d+1}+\varepsilon}$$

for infinitely many pairs $(u, v) \in \mathbf{N}^2$. We have:

$$(u^2+v)^{-\frac{1}{2}} = \frac{1}{u} - \frac{v}{2u^3} + O\left(\frac{1}{u^3}\right)$$

and

$$(u^2+v)^{d+\frac{1}{2}} = u^{2d+1}\Big(1+\sum_{n\geq 1} c_i v^i u^{-2i}\Big)$$

where for any i, c_i is a nonzero rational number. Let now $\varepsilon > 0$ be small and

$$v \ll_\varepsilon u^{1-\frac{1}{2d+1}+\varepsilon}. \tag{8}$$

The exponent of u in the RHS of (8) will be our γ. We see that:

$$Y_f(n) = u^{2d+1} + \sum_{i=1}^{2d+1} c_i v^i u^{2d-2i+1} + \frac{a_0}{2u} + O_\varepsilon(u^{-\frac{4d+3}{2d+1}+(2d+2)\varepsilon}).$$

As for Step 3, we take v to be a multiple of the common denominator A of the numbers c_i, $1 \leq i \leq 2d$. Write $v = Av_1$ with $v_1 \in \mathbf{N}$. Then the numbers $c_i v^i u^{2d-2i+1}$ with $1 \leq i \leq d$ are integers and the fractional part $\{Y_f(X)\}$ reduces to:

$$\{Y_f(X)\} = \sum_{i=d+1}^{2d} c_i v^i u^{2d-2i+1} + c_{2d+1} v^{2d+1} u^{-2d-1} + \frac{a_0}{2u} + O_\varepsilon(u^{-\frac{4d+3}{2d+1}+(2d+2)\varepsilon}).$$

Step 4: We will arrange our u and v such that all the numbers

$$v_1^i u^{2d-2i+1}, \quad d+1 \leq i \leq 2d \tag{9}$$

are integers. Then we remain in Step 5 with:

$$\{Y_f(X)\} = \left\{\frac{c_{2d+1}v^{2d+1}}{u^{2d+1}} + \frac{a_0}{2u} + O_\varepsilon(u^{-\frac{4d+3}{2d+1}+(2d+2)\varepsilon})\right\}. \tag{10}$$

Now the point is that since $a_0 \in \mathbf{Q}^*$ one can kill the first two terms in the RHS of (10) by arranging u and v such that

$$\frac{c_{2d+1}v^{2d+1}}{u^{2d+1}} + \frac{a_0}{2u} = 0. \tag{11}$$

This can be done by taking $u = rw^{2d+1}$, $v = rAw^{2d}$ where

$$r = -\frac{a_0}{2A^{2d+1}c_{2d+1}} \in \mathbf{Q}^*$$

and $w \in \mathbf{N}^*$. Moreover, by choosing w such that wr^{d-m+1} is an integer for any $1 \leq m \leq d$, the relations (8), (9), (11) will hold true. We now let $w \to \infty$ subject to the above conditions. Then the corresponding $X = u^2 + v$ will satisfy (7) and the theorem is proved.

In order to prove Theorem 1 we proceed as in the proof of Theorem 2 until we get to Step 5 above. Here we can not arrange to have (11) if a_0 is irrational.

In this case what we do is the following. We take the convergents $\frac{b_n}{q_n}$ to the continued fraction of a_0 and for each n we replace a_0 by $\frac{b_n}{q_n}$ in (11). Next, we choose $w = w_n \in \mathbb{N}^*$ as small as possible such that (9) and (11) hold true. It is easy to see that the wanted w_n will be a certain small multiple of q_n. This w_n produces a pair (u_n, v_n) which gives us further an integer point (X_n, Y_n). Using the inequality $|a_0 - \frac{b_n}{q_n}| < \frac{1}{q_n^2}$ in (11) we see that the RHS of (11) is $\ll \frac{1}{u_n q_n^2}$. Now $q_n \approx w_n$, $w_n^{2d+1} \approx u_n$, $u_n^2 \approx X_n$ and we find that $\{Y_f(X_n)\} \ll X_n^{-\frac{1}{2}-\frac{1}{2d+1}}$. This gives $\theta(C) \leq d - \frac{1}{2d+1}$ which completes the proof of Theorem 1.

REFERENCES

[1] R. C. Baker, *Diophantine Inequalities.* Oxford University Press, New York, 1986.

[2] N. Elkies, *Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction.* In: Proceedings of ANTS-4 (The Netherlands, 2000), Lecture Notes in Comput. Sci. **1838** (ed. W. Bosma), Springer, 2000, 33–63.

[3] J. Gebel, A. Pethö and H. G. Zimmer, *On Mordell's equation.* Compositio Math. **110**(1998), 335–367.

[4] M. Hall, *The Diophantine equation $x^3 - y^2 = k$.* In: Computers in Number Theory (eds. A. O. L. Atkin and B. J. Birch), Academic Press, 1971, 173–198.

[5] S. Lang, *Conjectured diophantine estimates on elliptic curves.* In: Arithmetic and Geometry (eds. M. Artin and J. Tate), Birkhauser, Boston, 1983, 155–172.

[6] H. L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis.* CBMS Regional Conf. Ser. in Math. **84**, Amer. Math. Soc., Providence, 1994.

[7] J. Oesterlé, *Nouvelles approaches du "théorème" de Fermat.* Sém. Bourbaki 2/88, Astérisque **161–162**(1988), exposé #694, 165–186.

[8] W. M. Schmidt, *Small fractional parts of polynomials.* Regional Conference Series in Math. **32**, Amer. Math. Soc., Providence, 1977.

[9] P. Vojta, *Diophantine approximations and value distribution theory.* Lecture Notes in Math. **1239**, Springer-Verlag, Berlin-Heidelberg, 1987.

[10] A. Zaharescu, *Small values of $n^2\alpha$* (mod 1). Invent. Math. **121**(1995), 379–388.

Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764
RO-70700 Bucharest
Romania
email: mvajaitu@stoilow.imar.ro

Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764
RO-70700 Bucharest
Romania
and
Department of Mathematics
University of Illinois at Urbana-Champaign
Altgeld Hall
1409 W. Green Street
Urbana, IL 61801
USA
email: zaharesc@math.uiuc.edu

C. R. Math. Rep. Acad. Sci. Canada Vol. **23** (3), 2001 pp. 91–96

AN INVARIANT FOR ACTIONS OF $\mathbb{R}$ ON UHF C^*-ALGEBRAS

ANDREW J. DEAN

Presented by G. A. Elliott, FRSC

ABSTRACT. An invariant is introduced which classifies, up to equivariant isomorphism, the C^*-dynamical systems (A, α) of the following form: A is a UHF algebra, and α is an action of $\mathbb{R}$ on A that leaves invariant a sequence of full matrix subalgebras of A each of which contains the unit of A and whose union is dense in A.

1. Introduction. The classification theory of inductive limit type actions of a compact group is already well underway, *cf.* [1], [3], [4], [5], and [6]. The purpose of this note is to (hopefully) initiate the parallel development of a theory of inductive limit type actions of $\mathbb{R}$. In the compact case, the proofs have followed the pattern of classification theorems for C^*-algebras—namely, they are intertwining arguments based on K-theory. We shall also use an intertwining argument, but with an invariant that combines K-theoretic and spectral data. The invariant is defined and the intertwining argument presented in Section 2 below. The only steps which require some work are showing that the invariant is continuous with respect to the inductive limit process, and the uniqueness theorem.

2. Product-type Actions on a UHF Algebra. We begin by introducing some terminology. Let A be a UHF algebra and α an action of the reals on A. We say that α is *of product type* if there exists a sequence $(A_n)_{n=1}^{\infty}$ of full matrix subalgebras of A such that $1 \in A_1$, $A_n \subseteq A_{n+1}$ for all n, $\alpha_t(A_n) = A_n$ for all n and t, and $A = \overline{\bigcup_{n=1}^{\infty} A_n}$. In all of our C^*-dynamical systems the group will be the reals, so we simply write (A, α) for such a system, where α is an action of $\mathbb{R}$ on A. We shall write $(A, \alpha) \cong (B, \beta)$ for the statement that there is an equivariant isomorphism from A to B. We shall write A^α for the fixed-point subalgebra of A under the action α.

The following two definitions give us our invariant.

DEFINITION 2.1. Let A be a full matrix algebra and let α be an action of the reals on A. Then there exists a unique self-adjoint element $h \in A$ such that the following three conditions hold: the derivation δ_{ih} defined by $\delta_{ih}(x) = i[h, x]$ is the infinitesimal generator of α, h is positive, and the smallest eigenvalue of h is zero. Let $(n+1)^2$ be the dimension of A. Then the eigenvalues of h counted with

Received by the editors January 22, 2001.
AMS subject classification: Primary: 46L57; secondary: 46L35.

multiplicity may be written $0, a_1, a_2, \ldots, a_n$ where $0 \leq a_1 \leq \cdots \leq a_n$. Denote by $s(A, \alpha)$ the n-tuple $(a_1, \ldots, a_n) \in \mathbb{R}^n$. This n-tuple is clearly an isomorphism invariant of the dynamical system (A, α), and characterises it up to equivariant isomorphism.

DEFINITION 2.2. Let A be either a UHF algebra or a full matrix algebra, and let α be an action of the reals on A. Define a set $S(A, \alpha) \subseteq \coprod_{n=1}^{\infty} \mathbb{R}^n$ by

$$S(A,\alpha) = \left\{ s(B, \alpha|_B) \;\middle|\; \begin{array}{l} B \text{ is a full matrix algebra,} \\ 1 \in B \subseteq A, \text{ and } \alpha_t(B) = B \end{array} \right\}.$$

This definition gives a functor from the category of UHF and full matrix C^*-dynamical systems with unital injective equivariant $*$-homomorphisms to the category of subsets of $\coprod_{n=1}^{\infty} \mathbb{R}^n$ with inclusions. Note that in the case of a full matrix algebra A, $S(A, \alpha)$ is completely determined by $s(A, \alpha)$.

THEOREM 2.3 (THE MAIN RESULT). *Let A and B be UHF algebras and let α and β be product type actions of the reals on A and B respectively. If $S(A, \alpha) = S(B, \beta)$, then $(A, \alpha) \cong (B, \beta)$.*

PROOF. The proof follows the familiar pattern of inductive limit classification theorems. Let A, B, α, and β be as in the statement of the theorem and let $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ be sequences of subalgebras of A and B respectively such that $1 \in A_1$, $A_n \in A_{n+1}$ for all n, each A_n is a full matrix algebra, $\alpha_t(A_n) = A_n$ for all n and t, $A = \overline{\bigcup_{n=1}^{\infty} A_n}$, and the B_n's have similar properties with respect to B and β. Denote by α_n and β_n the restrictions of α and β to A_n and B_n respectively. Then what we have is a pair of diagrams of dynamical systems as follows:

$$\begin{array}{ccccccccc} 1 \in (A_1, \alpha_1) & \longrightarrow & (A_1, \alpha_2) & \longrightarrow & (A_3, \alpha_3) & \longrightarrow & \cdots & \longrightarrow & (A, \alpha) \\ 1 \in (B_1, \beta_1) & \longrightarrow & (B_2, \beta_2) & \longrightarrow & (B_3, \beta_3) & \longrightarrow & \cdots & \longrightarrow & (B, \beta). \end{array}$$

By Lemma 2.4 below, we have that $S(A, \alpha) = \bigcup_{n=1}^{\infty} S(A_n, \alpha_n)$ and similarly $S(B, \beta) = \bigcup_{n=1}^{\infty} S(B_n, \beta_n)$. Since $S(A, \alpha) = S(B, \beta)$, our observation above tells us that for some n, $s(A_1, \alpha_1) \in S(B_n, \beta_n)$, and therefore $S(A_1, \alpha_1) \subseteq S(B_n, \beta_n)$. Arguing the same way with (B_n, β_n) in place of (A_1, α_1) we find an $n_2 > n$ such that $S(B_n, \beta_n) \subseteq S(A_{n_2}, \alpha_{n_2})$. After passing to subsequences and relabelling we get a diagram,

$$\begin{array}{ccccccccc} S(A_1, \alpha_1) & \longrightarrow & S(A_2, \alpha_2) & \longrightarrow & S(A_3, \alpha_3) & \longrightarrow & \cdots & \longrightarrow & S(A, \alpha) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & & & \downarrow \\ S(B_1, \beta_1) & \longrightarrow & S(B_2, \beta_2) & \longrightarrow & S(B_3, \beta_3) & \longrightarrow & \cdots & \longrightarrow & S(B, \beta), \end{array}$$

where all of the arrows are inclusions of sets. Using Lemma 2.5 below, we may lift these inclusions to unital injective equivariant $*$-homomorphisms between the subalgebras. This gives us a diagram,

$$\begin{array}{ccccccccc} (A_1,\alpha_1) & \longrightarrow & (A_2,\alpha_2) & \longrightarrow & (A_3,\alpha_3) & \longrightarrow & \cdots & \longrightarrow & (A,\alpha) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & & & \downarrow \\ (B_1,\beta_1) & \longrightarrow & (B_2,\beta_2) & \longrightarrow & (B_3,\beta_3) & \longrightarrow & \cdots & \longrightarrow & (B,\beta), \end{array}$$

in which the triangles do not necessarily commute. Finally, we use Lemma 2.6 below to adjust the vertical maps, proceeding left to right through the diagram, to arrive at a diagram

$$\begin{array}{ccccccccc} (A_1,\alpha_1) & \longrightarrow & (A_2,\alpha_2) & \longrightarrow & (A_3,\alpha_3) & \longrightarrow & \cdots & \longrightarrow & (A,\alpha) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & & & \uparrow\downarrow \\ (B_1,\beta_1) & \longrightarrow & (B_2,\beta_2) & \longrightarrow & (B_3,\beta_3) & \longrightarrow & \cdots & \longrightarrow & (B,\beta) \end{array}$$

in which all of the triangles commute. As is well known, *cf.* [2], such a commutative diagram gives rise to a pair of inverse $*$-isomorphisms between A and B. In the present case, since the maps in the diagram are all equivariant, the limit maps are equivariant isomorphisms.

LEMMA 2.4 (PULLING BACK THE INVARIANT). *Let A be a UHF algebra, α a one parameter automorphism group on A, and $(A_n)_{n=1}^{\infty}$ a sequence of full matrix subalgebras of A such that $1 \in A_1$, $A_n \subseteq A_{n+1}$ for all n, $\alpha_t(A_n) = A_n$ for all n and t, and $A = \overline{\bigcup_{n=1}^{\infty} A_n}$. Then $S(A,\alpha) = \bigcup_{n=1}^{\infty} S(A_n, \alpha|_{A_n})$.*

PROOF. Let A, A_n, and α be as in the statement of the theorem. Let B be a full matrix subalgebra of A such that $1 \in B$ and $\alpha_t(B) \subseteq B$ for all t. We shall show that for some sufficiently large n, $s(B, \alpha|_B) \in S(A_n, \alpha|_{A_n})$.

Let P_n denote the canonical trace preserving conditional expectation from A onto A_n. Then, as maps from A to A, the sequence (P_n) converges to the identity in the point-norm topology. We first show that, as a map from B to A_n, P_n is equivariant. Let $\{e_{ij}\}$ be a system of matrix units for B consisting of eigenoperators for $\alpha|_B$, and let $\{f_{lr}\}$ be a similar system for A_n. We express A as $A_n \otimes (A_n' \cap A)$, where we identify f_{lr} with $f_{lr} \otimes 1$. Then every element x of B can be written in the form $\sum_{l,r} f_{lr} \otimes x_{lr}$, and $P_n(x) = \sum_{l,r} f_{lr} \otimes \tau(x_{lr})1$, where τ denotes the trace. Suppose x is an eigenoperator with eigenvalue a, *i.e.*, $\alpha_t(x) = e^{2\pi i a t}x$, and consider $(f_{ll} \otimes 1)x(f_{rr} \otimes 1) = f_{lr} \otimes x_{lr}$. Since $(f_{ll} \otimes 1)$ and $(f_{rr} \otimes 1)$ are fixed by α, either $f_{lr} \otimes x_{lr}$ is 0, or it is an eigenoperator with the same eigenvalue as x. Assume that it is not zero. Then since $f_{lr} \otimes 1$ is an eigenoperator, and $f_{lr} \otimes x_{lr}$ is an eigenoperator, we conclude that $1 \otimes x_{lr}$ is an eigenoperator. If its eigenvalue is not zero, then its trace is zero and $\alpha_t\big(P_n(f_{lr} \otimes x_{lr})\big) = 0 = P_n\big(\alpha_t(f_{lr} \otimes x_{lr})\big)$. If its eigenvalue is zero, *i.e.*, if it is fixed, then $P_n\big(\alpha_t(f_{lr} \otimes x_{lr})\big) = P_n\Big(\big(\alpha_t(f_{lr})\big) \otimes x_{lr}\Big) = \big(\alpha_t(f_{lr})\big) \otimes \tau(x_{lr})1 = \alpha_t\big(P_n(f_{lr} \otimes x_{lr})\big)$. Adding up the pieces we see that $\alpha_t\big(P_n(x)\big) = P_n\big(\alpha_t(x)\big)$ for any eigenoperator x. Since every element of B is a sum of eigenoperators, we get our desired conclusion.

(Note that for the kind of action we are considering the sums of eigenoperators are clearly dense, so we have in fact shown that P_n is equivariant on all of A.)

We now set about finding a unital invariant subalgebra of some A_n with the same invariant as B. Choose n large enough to have $\|P_n(e_{1j}) - e_{1j}\|$ small for $j = 1, \ldots, k$, where $B \cong M_k$, and "small" just means small enough for each of the successive approximations below to work out. From above we know that $P_n(e_{1j})$ is an eigenoperator with the same eigenvalue as e_{1j} and that $|P_n(e_{1j})|$ is therefore in the fixed point subalgebra of α. Define a function q by

$$q: \mathbb{R} \to \mathbb{R} \quad : \quad q(t) = \begin{cases} 1 & \text{if } t > 1/2 \\ 0 & \text{if } t \leq 1/2. \end{cases}$$

Then since $|P_n(e_{1j})|$ is norm close to the projection e_{jj}, we have that q is continuous on the spectrum of $|P_n(e_{1j})|$. Furthermore, if we let q_j denote $q(|P_n(e_{1j})|)$, we see that $\big\||P_n(e_{1j})| - q_j\big\|$ is small and that q_j is a projection in A_n that is Murray-von Neumann equivalent to e_{jj} in the fixed point subalgebra of α. Let $v_{1j}|P_n(e_{1j})|$ be the polar decomposition of $P_n(e_{1j})$ in A_n, and consider the partial isometry $w_j := v_{1j}q_j$. For each j, w_j is an eigenoperator with the same eigenvalue as e_{1j}, $w_j^*w_j$ is norm close to e_{jj}, and $w_jw_j^*$ is norm close to e_{11}. Our only problem is that they, the w's, do not, *a priori*, have the same range projection and orthogonal support projections. (If they did, they would generate a subalgebra of A_n meeting all our needs.) We now proceed to fix this without changing the eigenvalues.

Let $z_{11} = w_1w_1^*$. Then since z_{11} and $w_2w_2^*$ are two projections in A_n^α that are norm close to each other, they are Murray-von Neumann equivalent in A_n^α. Let R_2 be a partial isometry in A_n^α such that $R_2(w_2w_2^*)R_2^* = z_{11}$. The cutdown $(1-z_{11})(w_2^*w_2)(1-z_{11})$ is norm close to being a projection in $(1-z_{11})A_n^\alpha(1-z_{11})$, so it is norm close to a projection, call it p_1, in $(1-z_{11})A_n^\alpha(1-z_{11})$. Furthermore, $w_2^*w_2$ is norm close to this projection p_1, so they are Murray-von Neumann equivalent in A_n^α. Let S_2 be a partial isometry in A_n^α such that $S_2S_2^* = w_2^*w_2$ and $S_2^*S_2 = p_1$. Let $z_{12} = R_2w_2S_2$. Then $z_{12}z_{12}^* = z_{11}$, $z_{12}^*z_{12} = p_1$, which is orthogonal to z_{11}, and z_{12} is an eigenoperator with the same eigenvalue as w_2 and e_{12}. Proceeding in this way we may produce $z_{11}, z_{12}, \ldots, z_{1k}$ such that each z_{1j} is a partial isometry in A_n, z_{1j} is an eigenoperator with the same eigenvalue as e_{1j}, $z_{1j}z_{1j}^* = z_{11}$, and $\sum_{j=1}^k z_{1j}^*z_{1j} = 1$ (it is a projection norm close to $\sum_{j=1}^k e_{jj} = 1$). Thus the z's form the first row of a system of matrix units for a full matrix subalgebra of A_n, call it C, having the same unit as A. It is clear that $s(C, \alpha|_C) = s(B, \alpha|_B)$, so we are done.

LEMMA 2.5 (EXISTENCE LEMMA). *Let (A, α) and (B, β) be two full matrix C^*-dynamical systems such that $S(B, \beta) \subseteq S(A, \alpha)$. Then there exists a unital equivariant $*$-homomorphism from B into A.*

PROOF. This is immediate from the definition of S.

LEMMA 2.6 (UNIQUENESS LEMMA). *Let A, B, and C be full matrix algebras, α, β, and γ actions of the reals on them, and let $\theta: A \to C$, $\varphi: A \to B$, and $\psi: B \to C$ be unital injective equivariant $*$-homomorphisms. Then there exists a unital injective equivariant $*$-homomorphism $\psi': B \to C$ such that $\psi' \circ \varphi = \theta$.*

To prove this we shall make use of the following lemma.

LEMMA 2.7 (UNIQUE FACTORIZATION LEMMA). *Let A_1, A_2, B, and C be full matrix algebras with α_1, α_2, β, and γ actions of the reals on them. If $(A_1, \alpha_1) \cong (A_2, \alpha_2)$ and $(A_1 \otimes B, \alpha_1 \otimes \beta) \cong (A_2 \otimes C, \alpha_2 \otimes \gamma)$ then $(B, \beta) \cong (C, \gamma)$.*

PROOF. Let A_1, A_2, B, C, α_1, α_2, β, and γ be as in the statement of the Lemma 2.7. Let h_1 be the unique positive element of A_1 with smallest eigenvalue zero such that δ_{ih_1} is the infinitesimal generator of α_1, and let h_2, H_1, and H_2 be the analogous elements for α_2, β, and γ, respectively. Then $h_1 \otimes 1 + 1 \otimes H_1$ is a Hamiltonian for $\alpha_1 \otimes \beta$ and $h_2 \otimes 1 + 1 \otimes H_2$ is one for $\alpha \otimes \beta$. Let $a_1, \ldots, a_k$ be the eigenvalues of h_1 listed with multiplicities in increasing order. Since $(A_1, \alpha_1) \cong (A_2, \alpha_2)$, the list for h_2 is the same. Let $b_1, \ldots, b_l$ and $c_1, \ldots, c_l$ be the lists for H_1 and H_2 respectively. Then the eigenvalues of $h_1 \otimes 1 + 1 \otimes H_1$, listed with multiplicity but not necessarily in order, are $a_j + b_m$, $1 \leq j \leq k$, $1 \leq m \leq l$, and those for $h_2 \otimes 1 + 1 \otimes H_2$ are $a_j + c_m$, $1 \leq j \leq k$, $1 \leq m \leq 1$. Since $a_1 = 0 = b_1$, we see that $h_1 \otimes 1 + 1 \otimes H_1$ is the unique positive Hamiltonian for $\alpha_1 \otimes \beta$ with smallest eigenvalue zero, and similarly $h_2 \otimes 1 + 1 \otimes H_2$ is the analogous element for $\alpha_2 \otimes \gamma$. Since $(A_1 \otimes B, \alpha_1 \otimes \beta) \cong (A_2 \otimes C, \alpha_2 \otimes \gamma)$ the two lists above must be the same. This implies that they have the same largest element, so $a_k + b_l = a_k + c_l$ and therefore $b_l = c_l$. Striking out the elements $a_j + b_l = a_j + c_l$, $1 \leq j \leq k$ from both lists, we see that the remaining sets are still the same, so we must have $a_k + b_{l-1} = a_k + c_{l-1}$, *etc.* We conclude $b_m = c_m$, for $1 \leq m \leq l$ and therefore $(B, \beta) \cong (C, \gamma)$.

PROOF OF LEMMA 2.6. Using the notation in the statement of the lemma, let $D_1 = \theta(A)$, $D_2 = (\psi \circ \varphi)(A)$, $E_1 = D_1' \cap C$, $E_2 = D_2' \cap C$, $\alpha_1 = \gamma|_{\theta(A)}$, $\alpha_2 = \gamma|_{\psi \circ \varphi(A)}$, $\varepsilon_1 = \gamma|_{E_1}$, and $e_2 = \gamma|_{E_2}$. We have $(D_1 \otimes E_1, \alpha_1 \otimes \varepsilon_1) \cong (C, \gamma) \cong (D_2 \otimes E_2, \alpha_2 \otimes \varepsilon_2)$ and $(D_1, \alpha_1) \cong (D_2, \alpha_2)$, so by the lemma above $(E_1, \varepsilon_1) \cong (E_2, \varepsilon_2)$. Let $F = \varphi(A)$, $G = F' \cap B$, $f = \beta|_F$, and $g = \beta|_G$. Then (B, β) may be identified with $(F \otimes G, f \otimes g)$, and under this identification ψ maps $(1 \otimes G, 1 \otimes g)$ equivariantly into $(1 \otimes E_2, 1 \otimes e_2)$. By the isomorphism noted above there is a sub-C^*-dynamical system of (E_1, ε_1) isomorphic to (G, g). Call this sub-system (G', g') and let σ be an equivariant isomorphism from G to G'. Define the map ψ' as follows:

$$\psi'(a \otimes b) = \theta(\varphi^{-1}(a)) \otimes \sigma(b) \in D_1 \otimes E_1,$$

where $a \in F$ and $g \in G$. Then clearly ψ' is a unital injective equivariant $*$-homomorphism such that $\psi' \circ \varphi = \theta$.

Finally, a little theorem that shows how the answer to one obvious question can be read off from the invariant.

THEOREM 2.8. *Let $\|\cdot\|_n$ denote the box norm on $\mathbf{R}^n$, i.e., $\|(a_1,\ldots,a_n)\|_n = \sup\{|a_1|,\ldots,|a_n|\}$. Let (A,α) be a C^*-dynamical system with A a UHF algebra and α a product-type action of $\mathbb{R}$. Then α is inner if and only if*

$$\sup\{\|\mathbf{a}\|_n \mid \mathbf{a}\in S(A,\alpha)\cap\mathbb{R}^n, n\in\mathbb{N}\}<\infty.$$

PROOF. Let (A_n) be a sequence of α-invariant full matrix subalgebras of A containing the unit of A and whose union is dense. Let δ denote the infinitesimal generator of α and δ_n the infinitesimal generator of $\alpha|_{A_n}$. Then $\|s(A_n,\alpha_n)\|_k \le \|\delta_n\| \le 2\|s(A_n,a_n)\|_k$, where $k+1$ is the size of the matrix algebra A_n. If the supremum in the statement of the theorem is finite, then $(\|\delta_n\|)_{n=1}^\infty$ is a bounded sequence, so δ is bounded and by Corollary 2.5.8 in [7] α is inner. Conversely, if δ is inner, it is bounded, so that $\sup\{\|\mathbf{a}\|_n \mid \mathbf{a}\in S(A,\alpha)\cap\mathbb{R}^n, n\in\mathbb{N}\}\le\|\delta\|$.

REFERENCES

1. O. Bratteli, G. A. Elliott, D. Evans and A. Kishimoto, *On the classification of inductive limits of inner actions of a compact group.* In: Current topics in operator algebras (Nara, 1990), World Sci. Publishing, River Edge, NJ, 1991, 13–24.
2. G. A. Elliott, *On the classification of C^*-algebras of real rank zero.* J. Reine Angew. Math. **443**(1993), 179–219.
3. G. A. Elliott and H. Su, *K-theoretic classification for inductive limit Z_2 actions on AF algebras.* Canad. J. Math., to appear.
4. D. Handelman and W. Rossmann, *Product type actions of finite and compact groups.* Indiana Univ. Math. J. (4) **33**(1984), 479–509.
5. ———, *Actions of compact groups on AF C^*-algebras.* Illinois J. Math. (1) **29**(1985), 51–95.
6. A. Kishimoto, *Actions of finite groups on certain inductive limit C^*-algebras.* Internat. J. Math. (3) **1**(1990), 267–292.
7. S. Sakai, *Operator algebras in dynamical systems.* Cambridge University Press, 1991.

Department of Mathematical Sciences
Lakehead University
955 Oliver Road
Thunder Bay, Ontario
P7B 5E1
email: adean@mercury.lakeheadu.ca