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ON THE IMAGE AND THE FIBRES OF THE NON-NORMAL CUBIC LIFT

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ABSTRACT. We describe the image and the fibers of the non-normal cubic base change lift, constructed by Jacquet, Piatetskii-Shapiro and Shalika.

RÉSUMÉ. Nous décrivons l'image et les fibres pour le relèvement cubique non-normal, construit par Jacquet, Piatetskii-Shapiro et Shalika.

1. Introduction. Let K/F be a Galois extension of number fields. The Galois group $\mathrm{Gal}(K/F)$ acts on the space of automorphic forms on $\mathrm{GL}_2(_K)$, and hence on the collection of automorphic representations of $\mathrm{GL}_2(_K)$. Suppose K/F is a cyclic extension of prime degree. By the characterisation of the image of the base change map constructed by Langlands [L], we know that if π is a cuspidal automorphic representation of $\mathrm{GL}_2(_K)$, which is invariant with respect to the action of $\mathrm{Gal}(K/F)$, then π lies in the image of the base change map $\mathrm{BC}_{K/F}$ from automorphic representations on $\mathrm{GL}_2(_F)$ to automorphic representations on $\mathrm{GL}_2(_K)$.

Let K/F be a non-normal cubic extension of number fields. Jacquet, Piatetskii-Shapiro and Shalika [JPSH] showed the existence of the cubic base change lift $\mathrm{BC}_{K/F}$, from automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ to automorphic representations of $\mathrm{GL}_2(\mathbf{A}_K)$. This was then utilised by Tunnell, who extended the results of Langlands to establish the Artin conjecture for all octahedral representations of the Galois group. Our main purpose is to describe the image and the fibers of the lift $\mathrm{BC}_{K/F}$.

One of the problems with characterising the base change forms is the lack of a suitable Galois action on the space of automorphic forms over K, with respect to which we can define the space of invariant forms. Neither it seems to be possible to define the invariance condition in terms of the associated L-functions. We can try to get around this by considering automorphic representations instead of automorphic forms, and viewing the invariance condition locally. Let v be a prime of F splitting completely into primes w_1, w_2, w_3 of K. A necessary condition for an automorphic representation π to be in the image of $\mathrm{BC}_{K/F}$, is

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that the local components π_{w_i}, $(i = 1,2,3)$, should be mutually isomorphic as representations of $GL_2(K_{w_i}) \simeq GL_2(F_v)$. For v decomposing as w_1, w_2, with K_{w_2} a quadratic extension of K_{w_1}, we require that the local component π_{w_2} to be a quadratic local base change of π_{w_1}. With this however there is a problem with those primes of F, which remain prime in K. Here we are presented with the problem of characterising the image of the local base change for a non-normal cubic extension.

To solve this, we observe that for any Galois extension L of F, the set of primes of L of degree 1 over F is of density 1 in L. Let L denote the Galois closure of K over F. L/F is a Galois extension with Galois group isomorphic to the permutation group S_3 on 3 symbols. Denote by E the unique quadratic extension of F contained in L.

Upon base changing to L, we can use the refined forms of strong multiplicity one theorem of D. Ramakrishnan [DR], that if the local components of two irreducible, unitary, cuspidal automorphic representations on GL_2 agree on a set of primes of density at least 7/8, then the representations coincide. There is still a problem which persists with those cuspidal represntations of $GL_2(\mathbf{A}_K)$ which do not remain cuspidal upon base changing to L. One requires a refined form of multiplicity one to handle these noncuspidal automorphic representations, and this is done in [R].

2. Non-normal cubic base change. We first recall a refined form of the multiplicity one theorem for non-cuspidal automorphic representations given in [R].

PROPOSITION 1. *Suppose π_1 and π_2 are non-cuspidal automorphic representations of $GL_2(\mathbf{A}_K)$. Suppose that the local components of π_1 and π_2 coincide on a set of places of K of density strictly greater than 3/4. Then $\pi_1 \simeq \pi_2$.*

Now let K/F be a cubic non-normal extension of number fields. Let v be a place of F which splits completely in K into primes $w_1, \ldots, w_d$, where d is the degree of K over F. We have $K_{w_i} \simeq K_{w_j} \simeq F_v$ for all $1 \leq i, j \leq d$. Let π be an automorphic representation of $GL_2(_K)$. Let (I) denote the following 'invariance' hypothesis:

HYPOTHESIS I. π satisfies (I), if for any place v of F splitting completely in K into w_i, $1 \leq i \leq d$, the local components π_{w_i}, for $i = 1 \leq i \leq d$, are all isomorphic to one another, via the natural isomorphisms $GL_2(K_{w_i}) \simeq GL_2(K_{w_j})$.

Suppose K/F is a quadratic extension, and σ be the automorphism of K/F. Suppose that θ is an idele class character of K, satisfying $\theta^\sigma \neq \theta$. Then there exists a cuspidal automorphic representation $I_K^F(\theta)$ of $GL_2(\mathbf{A}_F)$ [GL], the corresponding induced automorphic representation. We have that $BC_{K/F}(I_K^F(\theta)) \simeq \pi(\theta, \theta^\sigma)$, where $\pi(\theta, \theta^\sigma)$ is the automorphic representation of $GL_2(\mathbf{A}_K)$, constructed by the method of Eisenstein series. We can now refine the description of the image of the base change lift for cyclic extensions of number fields of prime degree.

PROPOSITION 2. *Let K/F be a cyclic extension of number fields of prime degree p.*

a) Let π be a unitary, irreducible, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$ satisfying hypothesis (I). Then π lies in the image of the base change lift $BC_{K/F}$.

b) Let $\pi = \pi(\theta_1, \theta_2)$ be a non-cuspidal automorphic representation on $\mathrm{GL}_2(\mathbf{A}_K)$ satisfying hypothesis (I). Then π lies in the image of the base change lift $\mathrm{BC}_{K/F}$.

PROOF. a) follows from the strong multiplicity one theorem of [DR] and by the remarks on density made in the last section.

b) Let σ be a non-trivial element of $\mathrm{Gal}(K/F)$. By our hypothesis and Proposition 1, we have that $\{\theta_1, \theta_2\}$ is invariant under $\mathrm{Gal}(K/F)$. Hence either i) $\theta_1^\sigma = \theta_1$, $\theta_2^\sigma = \theta_2$, or ii) $\theta_1^\sigma = \theta_2$, $\theta_2^\sigma = \theta_1$. i) implies that there are exactly p^2 distinct, (non-cuspidal) automorphic representations of $\mathrm{GL}_2(_F)$ which lift to π. If K/F is not quadratic, then ii) implies i). If K/F is quadratic and only ii) holds and not i), then there is exactly one (cuspidal) automorphic representation $I_K^F(\theta_1)$, which lifts to π.

We have the following theorem characterising the image and the fibres of the non-normal cubic lift. We remark that the characterisation is quite different from that of a cyclic extension of prime degree.

THEOREM 3. *Let K/F be a cubic non-normal extension of number fields. Let L be the Galois closure of K over F and let E be the unique quadratic extenstion contained in L. Let π be an automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$. Then we have the following:*

a) If π is a irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$, satisfying hypothesis (I), then π lies in the image of the base change lift $\mathrm{BC}_{K/F}$.

b) The base change map $\mathrm{BC}_{K/F}$, furnishes a bijective correspondence between unitary, cuspidal automorphic representations of K, satisfying hypothesis (I), and which are not automorphically induced from a character on L, and unitary, cuspidal automorphic representations of F which are not automorphically induced from a character on E.

c) If π is of the form $I_L^K(\theta)$, for a grossencharacter θ of L and satisfies hypothesis (I), then θ is invariant with respect to the action of $\mathrm{Gal}(L/E)$. Let θ_1', θ_2', θ_3' be the (distinct) idele class characters on J_E satisfying $\theta_i' N_{L/E} = \theta$, where $N_{L/E}$ denotes the norm from the idele classes J_L of L to the idele classes J_E of E. The representations $I_E^F(\theta_i')$, for $i = 1, 2, 3$ are distinct, cuspidal representations on $\mathrm{GL}_2(\mathbf{A}_F)$, and are precisely the representations on $\mathrm{GL}_2(\mathbf{A}_F)$ which map to π under $\mathrm{BC}_{K/F}$.

d) If π_0 is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ *then* $\mathrm{BC}_{K/F}(\pi_0)$ *is cuspidal, unless π_0 is of the form $I_E^F(\theta_0')$, where θ_0' is a grossencharacter on E satisfying $\theta_0' = (\theta_0')^\sigma \chi_{L/E}$ or $\theta_0' = (\theta_0')^\sigma \chi_{L/E}^{-1}$.*

e) If $\pi = \pi(\theta_1, \theta_2)$ is a non-cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$ *satisfying (I), then π lies in the image of* $\mathrm{BC}_{K/F}$*. If $\theta_2 = \theta_1\chi_{L/K}$, then there are exactly two (cuspidal) automorphic representations of the form $I_E^F(\theta_0')$, where θ_0' is as in d), and which lift to π under* $\mathrm{BC}_{K/F}$*. If $\theta_2 \neq \theta_1\chi_{L/K}$, then this is a bijection.*

PROOF. We will use the results of Langlands as summarised in page 21 of [L]. By our hypothesis, $\mathrm{BC}_{L/K}(\pi)$ satsifies hypothesis (I) with respect to the extension L/E. Hence by Proposition 2, there are either exactly 3 or 9 automorphic representations on $\mathrm{GL}_2(\mathbf{A}_E)$, which map to $\mathrm{BC}_{L/K}(\pi)$ under $\mathrm{BC}_{L/E}$.

Let $\sigma \in S_3$ be the element of order 2 in $\mathrm{Gal}(L/F)$ fixing K. Since $\mathrm{BC}_{L/E}$ is equivariant with respect to the action of σ on E and on L, we see that σ acts by permutation on these odd number of automorphic representations on $\mathrm{GL}_2(\mathbf{A}_E)$. Since σ is of order 2, there is at least one automorphic representation π' on $\mathrm{GL}_2(\mathbf{A}_E)$ fixed by σ, such that $\mathrm{BC}_{L/E}(\pi') = \mathrm{BC}_{L/K}(\pi)$. π' thus lies in the image of the base change map $\mathrm{BC}_{E/F}$.

Let $\chi_{L/E}$ be the idele class character on E corresponding to the extension L of E. We claim that $\chi_{L/E}$ is not invariant under the action of σ on E. This can be seen by looking at a prime v of F, which splits in E as w_1 and w_2, where w_1 and w_2 remain inert in L. Then the image of the norm map from $L_{w_1}^*$ into $E_{w_1}^*$ is a subgroup of index 3, whereas the kernel of the norm map from $E_{w_1}^* \times E_{w_2}^*$ to F_v^* consists of elements of the form (x, x^{-1}) as x ranges over $E_{w_1}^*$. Hence the image of the norm map from J_L to J_E does not contain the kernel of the norm map from J_E to J_F. Hence $\chi_{L/E}$ is not invariant under σ.

Suppose $\mathrm{BC}_{L/K}(\pi)$ is cuspidal. Then there is exactly one cuspidal representation π' of $\mathrm{GL}_2(\mathbf{A}_E)$ invariant under the action of σ. Let π_0 and $\pi_0 \otimes \chi_{E/F}$ be the two cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ which under $\mathrm{BC}_{L/F}$, map to $\mathrm{BC}_{L/K}(\pi)$. Since $\mathrm{BC}_{L/K}(\pi)$ is cuspidal, $BC_{K/F}$ of π_0 and $\pi_0 \otimes \chi_{E/F}$ are also cuspidal. Moreover since $\mathrm{BC}_{K/F}(\pi_0 \otimes \chi_{E/F}) \simeq \mathrm{BC}_{K/F}(\pi_0) \otimes \chi_{L/K}$, the images under $\mathrm{BC}_{K/F}$ of both π_0 and $\pi_0 \otimes \chi_{E/F}$ are different. Since only π and $\pi \otimes \chi_{L/K}$ map to $\mathrm{BC}_{L/K}(\pi)$ under $\mathrm{BC}_{L/K}$, π must be in the image of $\mathrm{BC}_{K/F}$. Thus we have proved that there exists a unique automorphic representation on $\mathrm{GL}_2(\mathbf{A}_F)$ which lifts to π and it is cuspidal.

Now we will examine the situation when π is cuspidal and $\mathrm{BC}_{L/K}(\pi)$ is not cuspidal. In this case π is the automorphic induction $I_L^K(\theta)$ of a unitary grossencharacter θ on L and $\mathrm{BC}_{L/K}(\pi) = \pi(\theta, \theta^\sigma)$, $\theta \neq \theta^\sigma$. By Proposition 2, θ comes from a grossencharacter θ' on E.

Now we observe that if θ_0' is an idele class character on E, which is invariant under σ, then $\theta_0' N_{L/E}$ is an idele class character on L, invariant under σ. Hence θ', $\theta'\chi_{L/E}$ and $\theta'\chi_{L/E}^2$ are not invariant under σ, since they map to θ under composition with the norm map $N_{L/E}$, which is not invariant by σ. Thus it is seen that the automorphic representations $I_E^F(\theta')$, $I_E^F(\theta'\chi_{L/E})$ and $I_E^F(\theta'\chi_{L/E}^2)$ are all cuspidal on $\mathrm{GL}_2(\mathbf{A}_F)$ and map to $\pi(\theta, \theta^\sigma)$ under $\mathrm{BC}_{L/F}$. Hence under $\mathrm{BC}_{K/F}$ they should map to π. This finishes the proof of a) and c).

b) and d): Let π_0 be a cuspidal automorphic representation on $\mathrm{GL}_2(\mathbf{A}_F)$. The difficulty arises when $\mathrm{BC}_{L/F}(\pi_0)$ is not cuspidal. Then π_0 is the automorphic induction $I_E^F(\theta_0)$ of a grossencharacter θ_0 on E satisfying $\theta_0 \neq \theta_0^\sigma$. We have $\mathrm{BC}_{L/F}(\pi_0) = \pi(\theta_0 \circ N_{L/E}, (\theta_0 \circ N_{L/E})^\sigma)$. Therefore $\mathrm{BC}_{K/F}(\pi_0) = I_L^K(\theta_0 N_{L/E})$. If $\theta_0 N_{L/E}$ is not left invariant by σ, then $\mathrm{BC}_{K/F}(\pi_0)$ is cuspidal. Hence the only problem is when $\theta_0 N_{L/E}$ is invariant under σ. Since only θ_0, $\theta_0\chi_{L/E}$ and $\theta_0\chi_{L/E}^2$ map to $\theta_0 N_{L/E}$ on composing with $N_{L/E}$, and by assumption θ_0 is not fixed by σ, it follows that one of $\theta_0\chi_{L/E}$ and $\theta_0\chi_{L/E}^2$ is fixed under σ. Moreover only one of them can be left fixed, as σ is a non-trivial involution acting on a set of 3 elements. Thus θ_0 should satisfy $\theta_0 = \theta_0^\sigma\chi_{L/E}$ or $\theta_0^\sigma\chi_{L/E}^{-1}$.

Conversely it can be seen that when θ_0 satisfies $\theta_0 = \theta_0^\sigma\chi_{L/E}$ or $\theta_0^\sigma\chi_{L/E}^{-1}$, then $\theta_0 N_{L/E}$ is left fixed by σ, and hence the representation $\mathrm{BC}_{K/F}(\pi_0)$ is not cuspidal.

e) By the proof of Proposition 2, let θ_1', θ_2' be the characters on E, which lift to $\theta_1 \circ N_{L/K}$, $\theta_2 \circ N_{L/K}$. It can be checked that the conditions $\theta_2' = (\theta_1')^\sigma$ and satisfies (I), is equivalent to saying that either $\theta_1 = \theta_2$ or $\theta_1 = \theta_2\chi_{L/K}$, and satisfies (I). Hence we are reduced to a situation considered above. If $\theta_1 \neq \theta_2\chi_{L/K}$, then by arguments as before, it can be seen that there is precisely one θ_1' which lifts to $\theta_1 \circ N_{L/K}$, and similarly for θ_2. Moreover the four non-cuspidal representations on $\mathrm{GL}_2(\mathbf{A}_F)$, which lift to $\pi(\theta_1', \theta_2')$ under $\mathrm{BC}_{E/F}$, lift to distinct non-cuspidal automorphic representations on $\mathrm{GL}_2(\mathbf{A}_K)$ under $\mathrm{BC}_{K/F}$.

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DENSITY OF ZEROS OF $L(s,\chi)$ NEAR $s=1$ AND THE SMALLEST CHARACTER NONRESIDUE

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ABSTRACT. We give an upper bound for the smallest character nonresidue in terms of the density of zeros of the corresponding L-function near $s=1$. We then generalize the result to Dirichlet series in the Selberg class.

RÉSUMÉ. On donne une borne supérieure pour le plus petit nonresidu d'un caractère, qui dépend de la densité de zéros près de $s=1$ de la fonction L correspondante. On généralise ce résultat aux séries de Dirichlet qui appartiennent à la classe de Selberg.

1. Introduction. Vinogradov's Hypothesis on the smallest quadratic nonresidue asserts that given $\epsilon > 0$ every sufficiently large prime number p has a positive quadratic nonresidue less than p^ϵ.

We know from the work of Ankeny [A] that Vinogradov's Hypothesis follows from the Generalized Riemann Hypothesis, in the stronger form: every large p has a positive quadratic nonresidue $\ll (\log p)^2$.

This was generalized by Lagarias, Montgomery and Odlyzko [L-M-O] who showed that if χ is a non-principal Dirichlet character (mod q) such that $L(s,\chi) \neq 0$ when $1-\delta < \sigma \leq 1$, $0 < |t| \leq \delta^2 \log q$, then there is a number n such that $\chi(n) \neq 1$, $\chi(n) \neq 0$, with $n < (A\delta \log q)^{1/\delta}$. Here A is an absolute constant. (For an exposition of this, see Theorem 1 in Chapter 9 of Montgomery [Mo].) The same type of result, in a slightly weaker form was obtained by Rodosskii [R].

In [L] Linnik states (without proof) the following conditional theorem which relates Vinogradov's Hypothesis to the density of zeros of the corresponding L-function near $s=1$:

"Let $\psi(q) \to \infty$ be a monotonic (as slowly increasing as we please) positive function. If the number of zeros of $L(s,\chi)$, χ being a real non-principal character (mod q) in the corresponding semi-circle: $|s-1| \leq \frac{\psi(q)}{\log q}$, $\operatorname{Re} s < 1$ may be estimated as $o(\psi(q))$, then Vinogradov's hypothesis is true."

In the following we give a more precise statement relating the density of the zeros to the exponent in Vinogradov's Hypothesis.

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Denote for $M > 0$ and χ a Dirichlet character (mod q):

$$\delta_\chi(M) = \frac{1}{M}\#\{\rho; |\rho - 1| \leq \frac{M}{\log q}, L(\rho,\chi) = 0\}$$

counting multiplicities but with the convention that here we do not count the possible Siegel zero of $L(s,\chi)$. Then we know (see Davenport [D]) that $\delta_\chi(M)$ satisfies the following relations:

$$\delta_\chi(M) = 0 \tag{1.1}$$

for $M \leq c_1$ and

$$\delta_\chi(M) \leq c_2 \tag{1.2}$$

uniformly for $M \leq \log q$. Here $c_1, c_2 > 0$ are absolute constants. Let n_χ be the smallest positive integer n for which $\chi(n) \neq 0$, $\chi(n) \neq 1$. We will prove the following

THEOREM 1. *There exists an absolute constant $c_3 > 0$ such that for any $\epsilon > 0$, any $M > M(\epsilon)$, any $q > q(M,\epsilon)$ and any non-principal Dirichlet character χ (mod q) we have*

$$\frac{\log n_\chi}{\log q} \leq \max\{\epsilon, c_3\delta_\chi(M)\}.$$

The above result can be generalized to the Selberg class $\mathcal{S}$, defined by Selberg in [S]. We refer to Conrey-Ghosh [C-G] and Murty [Mu] for the basic properties of $\mathcal{S}$.

We fix an element F_0 in $\mathcal{S}$ which has a pole of order $m_0 \geq 1$ at $s = 1$. Now let F be any element of $\mathcal{S}$ which is either entire or has a pole at $s = 1$ of order $m < m_0$. Then $F \neq F_0$ and we want to provide an upper bound for the smallest integer n for which $a_n(F) \neq a_n(F_0)$.

From the definition of Selberg class we know that $a_n(F) \ll n^\epsilon$, where the constant implied in this inequality is not explicit. In the following we assume that $|a_n(F)| \leq h(n)$ for a certain explicitly given function h which satisfies $h(n) \ll_\epsilon n^\epsilon$. Denote

$$\delta_F(M) = \frac{1}{M}\#\left\{\rho; |\rho - 1| \leq \frac{M}{d_F + \log Q_F}, F(\rho) = 0\right\},$$

where d_F and Q_F denote the dimension and respectively the conductor of F. We have the following

THEOREM 2. *Fix F_0 and h as above. There exists a constant $c = c(F_0, h)$ such that for any $\epsilon > 0$, any $M > M(F_0, h, \epsilon)$ and any $F \in \mathcal{S}$ with $Q_F > Q(M, F_0, h, \epsilon)$, $|a_n(F)| \leq h(n)$ for all n and $(s-1)^{m_0-1}F$ entire there is an $n \leq (Q_F e^{d_F})^{\max\{\epsilon,\delta_\chi(M)\}}$ such that $a_n(F) \neq a_n(F_0)$.*

Since $\delta_\chi(M)$ is unconditionally bounded for $M \gg 1$ by a constant $c_0 = c_0(h)$ which depends on h only, we obtain the following

COROLLARY 3. *Given F_0 and h as above there is a constant $c_1 = c_1(F_0,h)$ such that for any $F \in S$ for which $(s-1)^{m_0-1}F$ is entire and $|a_F(n)| \le h(n)$ for all n, there is an $n \le (Q_F e^{d_F})^{c_1}$ such that $a_n(F) \ne a_n(F_0)$.*

2. Proof of Theorem 1. We will need the following

LEMMA 4 (TURAN'S SECOND MAIN THEOREM). *Let $s_\nu = \sum_{n=1}^N b_n z_n^\nu$ and suppose that $1 = |z_1| \ge |z_2| \ge \cdots \ge |z_n|$. Then for any non-negative integer M there is an integer ν, $M+1 \le \nu \le M+N$ such that*

$$|s_\nu| \ge 2\left(\frac{N}{8e(M+N)}\right)^N \min_{1\le j\le N}\left|\sum_{n=1}^{j} b_n\right|.$$

Turan [T] proved this result with e^{26} in place of $8e$. The constant $8e$ was first obtained by Uchiyama [U]. For a presentation of Turan's method, see also Montgomery [Mo, Ch. 5].

Let us fix an $\epsilon_0 > 0$ and let χ be a non-principal character (mod q) such that

$$\frac{\log n_\chi}{\log q} = \lambda > \epsilon_0. \tag{2.1}$$

We differentiate k times the partial fraction formula (see [D])

$$\frac{L'}{L}(s,\chi) = B(\chi) + \sum_{L(\rho,\chi)=0}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

where the sum is extended over all the zeros of L, including the trivial zeros, to get the equality:

$$\frac{1}{k!}\sum_{n=1}^{\infty}\chi(n)\Lambda(n)(\log n)^k n^{-s} = -\sum_{\rho}\frac{1}{(s-\rho)^{k+1}} \tag{2.2}$$

which is valid for $\operatorname{Re} s > 1$. The corresponding formula for the principal character χ_0 is

$$\frac{1}{k!}\sum_{n=1}^{\infty}\chi_0(n)\Lambda(n)(\log n)^k n^{-s} = \frac{1}{(s-1)^{k+1}} - \sum_{L(\rho,\chi_0)=0}\frac{1}{(s-\rho)^{k+1}}. \tag{2.3}$$

Here we choose $s = 1 + \frac{M}{\log q}$. Then clearly the RHS of (2.3) is $\frac{1}{(s-1)^{k+1}}(1 + o_{k,M}(1))$ and the RHS of (2.2) is $-\sum_{|\rho-1|\le 1}\frac{1}{(s-p)^{k+1}} + \frac{1}{(s-1)^{k+1}}o_{k,M}(1)$ for M, k fixed, $k \ge 1$ as $q \to \infty$. From (2.1) we know that $\chi(n) = \chi_0(n)$ for $n < q^\lambda$. Therefore by subtracting (2.2) from (2.3) we derive:

$$\begin{aligned}\frac{(s-1)^{k+1}}{k!}&\sum_{n\ge q^\lambda}\frac{(\chi_0(n)-\chi(n))\Lambda(n)(\log n)^k}{n^{1+\frac{M}{\log q}}}\\ &= 1 + \sum_{|\rho-1|\le 1}\left(\frac{s-1}{s-p}\right)^{k+1} + o_{k,M}(1).\end{aligned} \tag{2.4}$$

From this we infer that there are absolute constants $c_4, c_5 > 0$ such that:

$$\left|1+\sum_{\rho}\left(\frac{s-1}{s-\rho}\right)^k\right| \le \frac{1}{e^{c_4\lambda M}}, \quad k=2,3,\dots,[c_5\lambda M]. \tag{2.5}$$

For, since $\sum_{n>n_0}\frac{\Lambda(n)}{n^{1+\epsilon}} \ll \frac{1}{\epsilon n_0^{\epsilon}}$ we have in particular $\frac{1}{\log q}\sum_{n\ge q^{\lambda}}\frac{\Lambda(n)}{n^{1+\frac{M}{\log q}}} \ll \frac{1}{Me^{\lambda M}}$. Next, we bound the sum:

$$\frac{1}{\log q}\sum_{n>q^{\lambda}}\frac{\Lambda(n)(\frac{\log n}{\log q})^k}{n^{1+\frac{M}{\log q}}}. \tag{2.6}$$

In order to do this we divide the range of summation in intervals: $[q^{\lambda}, q^{e\lambda}]$, $[q^{e\lambda}, q^{e^2\lambda}], \dots, [q^{e^{r_0}\lambda}, q], [q, q^2], \dots, [q^m, q^{m+1}]$ where $\frac{1}{e} \le e^{r_0}\lambda < 1$.

The sum over $[q, q^2]$ is $\ll \frac{2^k}{Me^M}$. This is $\ll \frac{1}{Me^{\frac{M}{2}}}$ if $k \le \frac{M}{2\log 2}$. In this case the sum over $[q, \infty]$ is also $\ll \frac{1}{Me^{\frac{M}{2}}}$. Then the sum in (2.6) is

$$\ll (1+r_0)\max_{0\le r<r_0}\frac{(\lambda e^{r+1})^k}{Me^{(\lambda e^r M)}} + \frac{1}{Me^{\frac{M}{2}}}.$$

The function $t \mapsto t^k e^{-\lambda t M}$ is decreasing in the interval $[\frac{k}{\lambda M}, \infty)$ hence for $k \le \lambda M$ the above maximum is attained for $r = 0$, and since $r_0 \ll_{\epsilon_0} 1$ the sum in (2.6) is $\ll (1+r_0)\frac{(\lambda e)^k}{Me^{\lambda M}} + \frac{1}{Me^{\frac{M}{2}}} \ll_{\epsilon_0} \frac{(\lambda e)^k}{Me^{\lambda M}}$. Then the LHS of (2.4) is $\ll_{\epsilon_0} \frac{M^k}{k!}\frac{(\lambda e)^k}{e^{\lambda M}}$. The function $y \mapsto (\frac{Me^2\lambda}{y})^y$ is increasing in the interval $(0, Me\lambda]$. If we take $k \le \frac{M\lambda}{e^2}$ then the LHS of (2.4) is $\ll_{\epsilon_0} e^{(\frac{4M\lambda}{e^2})}\frac{1}{e^{M\lambda}}$ so (2.5) holds true for $c_4 < 1 - \frac{4}{e^2}$ and $c_3 < \frac{1}{e^2}$.

Denote now the zeros of $L(s,\chi)$ by $\rho_1, \rho_2, \dots$ where $|\rho_1 - 1| \le |\rho_2 - 1| \le \cdots$ and put $z_0 = 1$, $z_j = \frac{s-1}{s-\rho_j}$. Note that $|z_j| \le 1$ for any j. From (1.1) and (1.2) it follows that $|\rho_j - 1| \ge \frac{c_6 j}{\log d}$ for any $j \ge 2$ and an absolute constant $c_6 > 0$. Hence:

$$|z_j| < \left|\frac{s-1}{\rho_j - 1}\right| \le \frac{M}{c_6 j}, \quad j \ge 2. \tag{2.7}$$

Let $N = \#\{\rho_j; |\rho_j - 1| \le \frac{M}{\log d}\}$. Thus $\delta_{\chi}(M) = \frac{N}{M}$. It is clear that for $j > N$ we have:

$$|z_j| < \frac{1}{\sqrt{2}}. \tag{2.8}$$

We consider the sum

$$\sum_{j>N}|z_j|^k. \tag{2.9}$$

Here we divide the range of summation in intervals $[N, \frac{2M}{c_6}]$, $[\frac{2M}{c_6}, \frac{4M}{c_6}], \dots$, $[\frac{2^r M}{c_6}, \frac{2^{r+1}M}{c_6}], \dots$ and use (2.8) for the first interval and (2.7) for the others, to get the upper bound $\frac{2M}{c_6}2^{-\frac{k}{2}} + \frac{M}{c_6}\sum_{r>1} 2^{-r(\frac{k}{2}-1)} \leq c_7 M e^{-\frac{k\log 2}{2}}$. This is $\leq e^{-c_8\lambda M}$ for $k > [\frac{c_5\lambda M}{2}]$ for large M and $c_8 < \frac{c_5 \log 2}{4}$. Using (2.5) we get:

$$\left|1+\sum_{j+1}^{N} z_j^k\right| \leq \frac{1}{e^{c_9\lambda M}} \tag{2.10}$$

for $k = [\frac{c_5\lambda M}{2}]+1,\ldots,[c_5\lambda M]$ and M large enough, where $c_9 = \min\{c_4, c_8\}$. We now use Lemma 4. It follows that for any positive integer s there exists $s+1 \leq k \leq s+N+1$ such that

$$\left|1+\sum_{j=1}^{N} z_j^k\right| \geq 2\left(\frac{N+1}{8e(s+N+1)}\right)^{N+1}. \tag{2.11}$$

Suppose now that $N < [\frac{c_5\lambda M}{2}]$. Then we can apply (2.11) with $s = [\frac{c_5\lambda M}{2}]$ for the range in (2.10) to find a k such that

$$\frac{1}{e^{c_9\lambda M}} \geq \left|1+\sum_{j=1}^{N} z_j^k\right| \geq 2\left(\frac{N+1}{8e(s+N+1)}\right)^{N+1} > \left(\frac{N+1}{16es}\right)^{N+1} = \frac{1}{e^{\frac{s}{t}\log(16et)}}$$

where $t = \frac{s}{N+1}$. Then $c_9\lambda M < \frac{c_5\lambda M \log(16et)}{2t}$, so $t \leq c_{10}$. Hence $N+1 \geq \frac{s}{c_{10}}$. In any case $N \geq -1 + s\min\{1, \frac{1}{c_{10}}\} \geq c_{11}\lambda M$ for large M. For $c_3 = \frac{1}{c_{11}}$ we then have $c_3\delta_\chi(M) \geq \lambda$. This completes the proof of Theorem 1.

The proof of Theorem 2 follows exactly the above proof of Theorem 1. In (2.2) and (2.3) the factors $\chi(n)\Lambda(n)$ and $\chi_0(n)\Lambda(n)$ will be replaced by $\Lambda_F(n)$ and $\Lambda_{F_0}(n)$ respectively. Note that the first term in the RHS of (2.3) will now have multiplicity m_0. Such a term $\frac{1}{(s-1)^{k+1}}$ might appear in (2.2) also, but with a smaller multiplicity, which ensures that in the LHS of (2.5) we still have at least one term equal to 1. Note also that F_0 being fixed its zeros will have the same role in the proof as the zeros of $L(s,\chi_0)$ had in the proof of Theorem 1. Thus we are left in (2.5) with the zeros of F which appear again with coefficient equal to $+1$ and so Turan's second main theorem can be applied as above.

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HOMOGÉNÉISATION EN MILIEU PÉRIODIQUE AVEC COUCHE ISOLANTE ÉPAISSE

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ABSTRACT. We consider the homogenization of a heat transfer problem in a periodic medium having two connected components of comparable conductance, set to one for simplicity, which are separated by a third one, the thickness of which is of the same order as the magnitude ϵ of the basic cell, and whose conductance λ becomes infinitely small with ϵ. If $\sup \frac{\epsilon^2}{\lambda} < +\infty$, we identify, using Γ-convergence techniques, the homogenized problem.

1. Présentation. Dans ce travail, nous nous intéressons à l'homogénéisation d'un problème de transfert thermique dans un milieu périodique à deux composantes *connexes* échangeant de la chaleur par l'intermédiaire d'un troisième corps *connexe* lui aussi, séparant les deux premiers. C'est un modèle très simplifié d'un échangeur de chaleur où l'un des fluides circule à l'intérieur d'un réseau tridimensionnel de tubes conducteurs (3ème corps) interconnectés périodiquement.

Plus précisément, on se donne le cube $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^3$ de $\mathbb{R}^3$ et on suppose que $Y = Y_1 \cup S_{12} \cup Y_2 \cup S_{23} \cup Y_3$ où Y_1, Y_2, Y_3 sont trois ouverts connexes tels que $\partial Y \cap Y_i \neq \emptyset$, $\forall i = 1, 2, 3$ et S_{ij} est la surface séparant Y_i et Y_j. On suppose, *et c'est essentiel*, que Y_3 est de même ordre de grandeur que Y_1 et Y_2 et pour simplifier qu'il est d'épaisseur constante et d'ordre 1 $\big(O(1)\big)$. Y_1 et Y_2 sont occupés par deux matériaux de conductances λ_1 et λ_2 finies, de même ordre de grandeur, que l'on supposera plus simplement égales et normalisées à un. Le matériau occupant Y_3 a une conductance λ, supposée très petite. Si nous reproduisons la cellule Y par périodicité, *géométriquement et matériellement* à $\mathbb{R}^3$, nous obtenons les trois ouverts correspondants E_1, E_2, E_3. L'homothétie de rapport ϵ nous donne alors la cellule élémentaire $Y^\epsilon = \epsilon Y$, les trois ouverts élémentaires Y_i^ϵ de même ordre de grandeur $O(\epsilon)$ et les trois ouverts périodiques connexes $E_i^\epsilon = \epsilon E_i$ séparés par les surfaces S_{ij}^ϵ. Soit $\Omega \subset \mathbb{R}^3$ ouvert borné connexe, de bord $\partial\Omega$ lipschitzien. On note $\Omega_i^\epsilon = \Omega \cap E_i^\epsilon$, $\Gamma_{ij}^\epsilon = S_{ij}^\epsilon \cap \Omega$.

Pour toute distribution de la température u dans le domaine Ω, l'énergie em-

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magasinée par le matériau est, en posant $h = (\epsilon, \lambda)$:

$$F^h(u) = \int_\Omega a_h(x)|\nabla u|^2\,dx, \quad u \in H^1(\Omega),$$

avec $a_h(x)$ valant λ sur Ω_3^ϵ et 1 sur $\Omega_1^\epsilon \cup \Omega_2^\epsilon$.

En présence d'une densité de sources thermiques $f \in L^2(\Omega)$, la température d'équilibre u^h est la solution du problème de minimisation (on impose la température sur $\partial\Omega$)

$$M^h := \mathrm{Min}\{G^h(u), u \in H_0^1(\Omega)\}$$

où

$$G^h(u) = F^h(u) - 2\int_\Omega fu\,dx \quad \text{si } u \in H_0^1(\Omega)$$

et

$$F^h(u) = \int_\Omega a_h(x)|\nabla u|^2\,dx.$$

De façon équivalente, u_h est la solution variationnelle (faible) de l'équation d'Euler correspondante :

$$P^h\!: \begin{cases} -\operatorname{div}\big(a_h(x)\nabla u\big) = f & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega. \end{cases}$$

On se propose d'étudier la limite de la suite de problèmes M^h, quand $h = h(\epsilon) = \big(\epsilon, \lambda(\epsilon)\big)$ tend vers $(0,0)$. Le même problème, aux notations près, a été étudié dans [6], par la méthode de l'énergie, pour $\lambda = \epsilon^r$, $r \geq 0$. *Mais les résultats annoncés sont erronés.* Par ailleurs, on peut signaler un travail de M. Hnid [5], où est examiné un problème analogue, sous l'hypothèse $\sup \frac{r_\epsilon}{\epsilon\lambda} < +\infty$ où r_ϵ représente l'épaisseur de la couche Y_3; par conséquent le cas $r_\epsilon = O(\epsilon)$ envisagé ici est écarté à priori. Citons aussi le travail de H. Attouch et G. Buttazzo [4], où on envisage le cas $(\epsilon, r_\epsilon, \lambda) \to (0, 0, +\infty)$, avec toujours $\frac{r_\epsilon}{\epsilon} \to 0$.

Ainsi, notre travail complète ces travaux au cas où l'épaisseur de la couche intermédiaire est du même ordre de grandeur que la cellule de base. Comme dans [4] et [5], notre technique se base sur la méthode de l'épi-convergence.

2. Résultat principal et résumé de la preuve.

THÉORÈME 1. *Sous l'hypothèse* $\sup \frac{\epsilon^2}{\lambda} < +\infty$, *la suite* u_h *converge quand* $h \to (0,0)$ *dans* $L^2(\Omega)$ *faible vers* u *solution du problème :*

$$(P^0)\!: \begin{cases} -\operatorname{div}\big(j^{\mathrm{hom}}(\nabla u)\big) = f & p.p \text{ dans } \Omega \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où j^{hom} *est donnée par (3).*

On a :

$$-\operatorname{div}\big(j^{\mathrm{hom}}(\nabla u)\big) = -\frac{\partial}{\partial x_i}\big(a_{ij}^{\mathrm{hom}}.\frac{\partial u}{\partial x_j}\big), \; a_{ij}^{\mathrm{hom}} = \int_{Y_1 \cup Y_2} \{\delta_{ij} - \sum_k \delta_{ik}\frac{\partial \chi^j}{\partial y_k}\}\,dy,$$

où χ^j est solution du problème

$$\operatorname{Min}\left\{\int_{Y_1\cup Y_2} |\nabla W + Z|^2\, dx \quad W \in H^1_{\text{per}}(Y)\right\}$$

où $H^1_{\text{per}}(Y) = \{u \in H^1_{\text{loc}}(\mathbb{R}^3),\ u\ Y$ périodique$\}$ et $Z = e_j$, $j = 1,2,3$, base canonique de $\mathbb{R}^3$. ∎

REMARQUE. En l'absence de l'hypothèse $\sup \frac{\epsilon^2}{\lambda} < +\infty$, on montre (voir Section 2.2) que les fonctionnelles G^h epi-convergent vers la fonctionnelle G^{hom}. Cependant, faute de compacité de la suite u_h, nous ne pouvons plus affirmer la convergence de u_h vers la solution du problème homogénéisé.

La preuve de ce théorème comporte essentiellement deux étapes : estimation à priori de la solution et épiconvergence de la suite de fonctionnelles G^h.

2.1. *Estimations à priori.*

LEMME 1. *Supposons $\frac{\epsilon^2}{\lambda} \leq M < +\infty$. Alors la suite u_h des solutions des problèmes P^h vérifie l'estimation :*

$$\int_\Omega |u_h|^2\, dx \leq C + C\frac{\epsilon^2}{\lambda} \leq C.$$

Ce lemme résulte du lemme de majoration ci-dessous, dont la démonstration utilise un résultat crucial d'Allaire-Murat [1, lemme 3.4], et le fait que u^h minimise la fonctionnelle G^h.

LEMME 2 (DE MAJORATION). *Pour toute fonction u dans $H^1_0(\Omega)$, on a :*

$$\forall u \in H^1_0(\Omega),\quad \|u\|^2_{L^2(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega_1^\epsilon\cup\Omega_2^\epsilon)} + C\epsilon^2\|\nabla u\|^2_{L^2(\Omega_3^\epsilon)},$$

où C est une constante indépendante de ϵ.

2.2. *Epi-convergence de la suite G^h.* L'estimation ci-dessus suggère l'utilisation de la topologie faible de $L^2(\Omega)$. La fonctionnelle $v \mapsto \int_\Omega fv\, dx$ étant continue pour cette topologie, il suffit d'étudier l'épi-limite des fonctionnnelles F^h. On a alors le

THÉORÈME 2. *Soit $X = L^2(\Omega)$, $\tau = w - L^2(\Omega)$, $H^1_{\text{per}}(Y) = \{u \in H^1_{\text{loc}}(\mathbb{R}^3),\ u$ Y périodique$\}$ et soit*

$$(1)\quad F^\epsilon(u) = F^{\epsilon,\lambda(\epsilon)}(u)$$

$$(2)\quad = \begin{cases} \int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon} |\nabla u|^2\, dx + \lambda\int_{\Omega_3^\epsilon} |\nabla u|^2\, dx & si\ u \in H^1(\Omega) \\ +\infty & si\ u \in L^2(\Omega)\setminus H^1(\Omega). \end{cases}$$

Alors, quand $\epsilon \to 0$, $\tau - \lim_e F^h = F^{\text{hom}}$ avec :

$$F^{\text{hom}} = \begin{cases} \int_\Omega j^{\text{hom}}(\nabla u)\, dx & si\ u \in H^1(\Omega) \\ +\infty & si\ u \in L^2(\Omega)\setminus H^1(\Omega) \end{cases}$$

où :

$$j^{\hom}(Z) = \min\left\{\int_{Y_1\cup Y_2} |\nabla W + Z|^2\,dy, W \in H^1_{\text{per}}(Y)\right\}. \tag{3}$$

Soit $G^\epsilon(u) = F^\epsilon(u) - 2\int_\Omega fu$. *Pour tout* $u \in L^2(\Omega)$, *on a :*

$$\tau - \lim_e G^\epsilon(u) = F^{\hom}(u) - 2\int_\Omega fu = G^{\hom}(u).$$

Pour montrer ce résultat, on utilise l'argument de comparaison suivant :

Supposons que λ tende vers zéro en même temps que ϵ suivant une loi $\lambda = \lambda(\epsilon)$. Pour tout λ fixé et ϵ tendant vers zéro, on a :

$$F^{\epsilon,0} \leq F^{\epsilon,\lambda(\epsilon)} \leq F^{\epsilon,\lambda}.$$

Donc

$$\tau - \lim_e F^{\epsilon,0} \leq \tau - li_e F^{\epsilon,\lambda(\epsilon)} \leq \tau - ls_e F^{\epsilon,\lambda(\epsilon)} \leq \tau - \lim_e F^{\epsilon,\lambda}, \tag{4}$$

où $\tau - li_e$ (resp. $\tau - ls_e$) désigne l'épi-limite inférieure (resp. l'épi-limite supérieure) quand $\epsilon \to 0$.

On détermine donc d'abord l'épi-limite des fonctionnelles $F^{\epsilon,0}$ et $F^{\epsilon,\lambda}$, puis on montre que

$$\tau - \lim_e F^{\epsilon,\lambda} \underset{\lambda\to 0}{\longrightarrow} \tau - \lim_e F^{\epsilon,0}.$$

On en déduit alors l'epi-limite de la suite $F^{\epsilon,\lambda(\epsilon)}$, quand $\epsilon \to 0$.

3. Conclusion. Dans le cas d'un milieu périodique à trois composantes, tel que l'épaisseur de l'inclusion (troisième corps) est du même ordre que la cellule élémentaire de base, le problème limite est tout à fait "classique", à rapprocher du résultat de [2] et ne présente aucun caractère "étrange" comme annoncé dans [6]. Nos résultats sont du même type que ceux de [2] et [1].

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SUR L'ORDRE D'ANNULATION DE $L(s, F)$ EN $s = 1$

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ABSTRACT. We use Weil's explicit formula to prove a new upper bound on the analytic rank of $J_0(N)$. We stress that with this method the choice of test functions with compact support is limited. The first result assumes the generalized Riemann hypothesis.

RÉSUMÉ. Nous démontrons dans ce travail une nouvelle borne du rang analytique de $J_0(N)$, la jacobienne de la courbe modulaire $X_0(N)$, en utilisant les formules explicites de Weil. Nous faisons voir que le choix dans ces formules de fonctions test à support compact est limité. Le premier résultat dépend de l'hypothèse de Riemann généralisée.

1. Préliminaires. Soit $\mathcal{H}$ le demi-plan complexe supérieur ouvert. Considérons le groupe $\Gamma_0(N)$ défini par

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N | c \right\}.$$

Ce groupe agit sur $\mathcal{H}$ de façon évidente. La structure complexe de $\mathcal{H}$ fait de la surface quotient $\mathcal{H}/\Gamma_0(N)$ une surface de Riemann connexe à laquelle il ne manque qu'un nombre fini de points pour être une surface compacte, notée $X_0(N)$.

Une forme différentielle holomorphe de première espèce sur $X_0(N)$ s'écrit localement $f(z)dz$, où f est une fonction holomorphe de $\mathcal{H}$ qui satisfait à certaines relations de compatibilité. L'espace vectoriel (complexe) de ces f est donc de dimension finie (égale au genre g de $X_0(N)$). Ces fonctions f sont appelées formes paraboliques de poids 2 et niveau N (pour le groupe $\Gamma_0(N)$). Leur espace est noté $S_2(N)$.

Il est facile de voir que si $f \in S_2(N)$ alors $f(z+1) = f(z)$ dans $\mathcal{H}$ ce qui implique que f se développe en série de Fourier[1] $f(z) = \sum_n a_n q^n$ où $q = e^{2\pi i z}$. En fait, l'holomorphie de $f(z)dz$ implique que $a_n = 0$ dès que $n \leq 0$.

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[1] Nous n'indiquerons normalement pas la dépendance en f des coefficients a_n. Lorsque nous le ferons, ce sera en ajoutant l'information à l'indice, comme dans $a_{n,i}$ ou $a_{n,f}$.

Ce développement en série de Fourier avec l'invariance de $f(z)dz$ sous $\Gamma_0(N)$ impliquent que la fonction L de f,

$$L(s,f) = \sum_{n\geq 1} \frac{a_n}{n^s},$$

définie lorsque $\Re s > 3/2$, se prolonge analytiquement en une fonction entière et satisfait à une équation fonctionnelle.

On peut entre autre définir canoniquement dans $S_2(N)$ un produit hermitien défini positif (noté par la suite $\langle \cdot , \cdot \rangle$) et trouver une base orthogonale de formes paraboliques[2] $f_1, \ldots, f_g$ pour lesquelles l'équation fonctionnelle prend la forme suivante:

$$\Lambda(s,f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s,f) = \pm \Lambda(2-s,f).$$

De plus, lorsque N est premier, pour un choix particulier de $f_1, \ldots, f_g$, on a le développement suivant en produit eulérien, valable pour $\Re s > 3/2$:

$$\begin{aligned} L(s,f_i) &= \prod_{p|N} (1 - a_{p,i}p^{-s})^{-1} \prod_{p\nmid N} (1 - a_{p,i}p^{-s} + p^{1-2s})^{-1} \\ &= \prod_{p|N} (1 - a_{p,i}p^{-s})^{-1} \prod_{p\nmid N} (1 - \alpha_{p,i}p^{-s})^{-1}(1 - \beta_{p,i}p^{-s})^{-1}, \quad 1 \leq i \leq g. \end{aligned}$$

Pour simplifier l'exposé nous supposerons désormais N premier et qu'une base orthogonale $\mathcal{B}$ a été choisie comme ci-dessus.

Soit $J_0(N)$ la jacobienne de $X_0(N)$. Elle est de dimension g qui est de l'ordre de $N/12$. Considérons le groupe de points rationnels de cette variété abélienne, noté $J_0(N)(\mathbb{Q})$. Un théorème de Mordell-Weil permet d'affirmer que ce groupe abélien est finiment engendré, donc qu'il s'écrit comme somme directe d'une partie libre isomorphe à $\mathbb{Z}^r$ et d'une partie de torsion finie. L'entier r est appelé le rang algébrique de $J_0(N)(\mathbb{Q})$. Le sujet de cet article est l'étude de r.

$J_0(N)$ est définie sur $\mathbb{Q}$, et on peut considérer la réduite de $J_0(N)$ modulo un premier $p \in \mathbb{Z}$, qui sera une variété abélienne pour presque tout p. En comptant les points $\mathbb{F}_p$-rationnels de la réduite on peut définir la fonction L de $J_0(N)$, notée $L(s, J_0(N))$, comme une série de Dirichlet absolument convergente pour $\Re s > 3/2$.

Un théorème de Shimura prouve que (N premier)

$$L(s, J_0(N)) = \prod_{i=1}^{g} L(s,f_i) = \prod{}' L(s,f),$$

[2] Cette base est telle que $a_1 = 1$ pour toute forme f de la base, fait que nous utiliserons désormais sans rappel.

où ici comme dans la suite le $'$ signifie que les f varient dans la base $\mathcal{B}$. En particulier on a la conséquence non triviale que $L(s, J_0(N))$ se prolonge analytiquement en une fonction entière.

Nous supposerons vraie une conjecture de Birch et Swinnerton-Dyer qui affirme que r est égal à l'ordre d'annulation de $L(s, J_0(N))$ en $s = 1$, appelé pour cette raison rang analytique de $J_0(N)(\mathbb{Q})$.

De même nous admettrons l'hypothèse de Riemann (RH) pour les fonctions $L(s, f)$, selon laquelle tous les zéros de $L(s, f)$ de partie réelle comprise entre $1/2$ et $3/2$ sont situés en fait sur la droite $\Re s = 1$.

1.1. *Énoncé des résultats.*

THÉORÈME 1. *Soit* $r_f = \operatorname{ord}_{s=1} L(s, f)$. *Alors le rang de* $J_0(N)$ *est égal à* $\sum' r_f$. *Sous RH pour* $L(s, f)$ *et les fonctions L de Dirichlet* $L(s, \chi)$ *nous avons*

(1) $$\sum{}' \frac{r_f}{4\pi\langle f, f\rangle} \leq 1 + o(1)$$

et

(2) $$\sum{}' r_f \leq g + o(N) = \dim S_2(N) + o(N),$$

lorsque $N \to \infty$.

THÉORÈME 2. *Soit* $F: \mathbb{R} \to \mathbb{R}$ *une fonction continue, paire, à support contenu dans* $(-1, 1)$ *et* $\hat{F}$ *sa transformée de Fourier. Supposons que* $\hat{F}(\xi) \geq 0$ *pour tout* $\xi \in \mathbb{R}$.

Alors $F(0) \geq \hat{F}(0)$, *avec égalité si et seulement si* $F(v) = c \max(1 - |v|, 0)$, $c \in \mathbb{R}$.

2. Formules explicites et relations de quasi-orthogonalité. Les formules explicites de Riemann-Weil sont l'outil essentiel de cet article. Elles permettent de relier intimement des zéros de fonctions L au comportement des nombres premiers. En l'occurrence nous utiliserons la formulation suivante.

Soit $f \in \mathcal{B}$. L'existence du produit eulérien implique que

$$-\frac{L'}{L}(s, f) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \Lambda(n),$$

où

$$c_{p^m} = \begin{cases} a_p^m & \text{si } p \text{ divise } N, \\ \alpha_p^m + \beta_p^m & \text{sinon.} \end{cases}$$

REMARQUE. À partir du produit eulérien il est immédiat que

(3) $$c_p = a_p \quad \text{pour tout } p$$

et que

(4) $$c_{p^2} = a_p^2 - 2p \quad \text{si } p \nmid N.$$

De même nous avons la borne de Deligne $|c_n| \leq 2\sqrt{n}$ pour tout $n \geq 1$.

THÉORÈME 3 (FORMULES EXPLICITES). *Soit $F:\mathbb{R}\to\mathbb{R}$ une fonction paire telle que:*

- *il existe $\epsilon>0$ pour lequel $F(x)\exp\big((1+\epsilon)x\big)$ soit intégrable et de variation bornée,*
- *la fonction $\big(F(x)-F(0)\big)/x$ soit de variation bornée.*

Notons $\phi=\hat{F}$ la transformée de Fourier de F.

Alors

$$\sum_{L(1+i\gamma,f)=0}\phi(\gamma)=F(0)\log N-2F(0)\log(2\pi)$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+it)\phi(t)\,dt-2\sum_{n=1}^{\infty}\frac{c_n}{n}\Lambda(n)F(\log n)$$

où les γ varient dans l'ensemble des zéros avec multiplicité de $L(s,f)$ situés dans la bande critique $1/2\le\Re(1+i\gamma)\le 3/2$.

PREUVE. *Cf.* [Mes]. ∎

Les coefficients a_n des $L(s,f)$ satisfont à une relation qui est semblable aux relations d'orthogonalité des caractères. Nous la rappelons ici (*e.g.* [Iwa]).

THÉORÈME 4. *Soit $\{f_1,\dots,f_r\}$ une base orthogonale de $S_2(N)$. Alors*

$$\sum_{1\le i\le r}\frac{a_{n,i}a_{m,i}}{4\pi\langle f_i,f_i\rangle\sqrt{n}\sqrt{m}}=\delta_{mn}-2\pi\sum_{c\equiv 0(\mathrm{mod}\,N)}J_1\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{S(m,n,c)}{c}$$

où δ_{mn} est le symbole de Kronecker, J_1 est la fonction de Bessel d'ordre 1 et $S(m,n,c)$ est la somme de Kloosterman

$$S(m,n,c)=\sum_{\substack{d(\mathrm{mod}\,c)\\ d\bar{d}\equiv 1(\mathrm{mod}\,c)}}\exp\left(2\pi i\frac{md+n\bar{d}}{c}\right).$$

COROLLAIRE 1. *Dans les notations déjà introduites nous avons*

$$\sum_f{}'\frac{a_ma_n}{4\pi\langle f,f\rangle\sqrt{m}\sqrt{n}}=\delta_{mn}+O\big(N^{-3/2}\sqrt{mn}(m,n)^{\frac{1}{2}}\big).$$

3. Ingrédients de la démonstration du théorème 1.

PREUVE DE (1). Soit une fonction F à support contenu dans $(-1,1)$ dont la transformée de Fourier $\hat{F}$ est positive sur la droite réelle. On applique les formules explicites à la fonction $u\mapsto F(u/\log x)$, de transformée de Fourier $\xi\mapsto\log x\hat{F}(\xi\log x)$. Nous obtenons donc la majoration évidente, sous RH

$$r_f\phi(0)\log x\le F(0)\log N-2\sum_{n\le x}\frac{c_{n,f}}{n}\Lambda(n)F\left(\frac{\log n}{\log x}\right)+O(1),$$

d'où, en posant $\omega_f = (4\pi\langle f, f\rangle)^{-1}$,

$$\sum{}' \omega_f r_f \leq \sum{}' \omega_f \frac{F(0)}{\phi(0)} \frac{\log N}{\log x}$$

$$(5) \qquad - \frac{2}{\phi(0)\log x} \sum_{n\leq x} \Big(\sum{}' \omega_f c_{n,f} \Big) \frac{\Lambda(n)}{n} F\left(\frac{\log n}{\log x}\right) + O\left(\frac{\sum' \omega_f}{\log x}\right).$$

Le corollaire 1 implique que $\sum' \omega_f = 1 + O(N^{-3/2})$. On est donc conduit à l'analyse de la double somme

$$\sum_{p,a} \Big(\sum{}' \omega_f c_{p^a} \Big) \frac{\log p}{p^a} F\left(\frac{a \log p}{\log x}\right),$$

ce qui se fait en analysant séparément la contribution pour $a = 1$, $a = 2$, $a \geq 3$. En estimant trivialement grâce à la borne de Deligne la troisième somme on calcule qu'elle ne contribue que pour $O(1)$. Le corollaire 1, ainsi que (4), donnent pour la seconde somme une contribution de $(-\log x)/4 + O(N^{-3/2}\sqrt{x})$. Pour analyser la première somme on utilise le théorème 4 et on inverse l'ordre des sommations. Le lemme suivant est alors essentiel:

Lemme 1. *Dans les mêmes notations que dans le théorème 4 nous avons, sous RH pour les fonctions L de Dirichlet classiques,*

$$\sum_{n\leq x} \Lambda(n) S(n, m, c) \ll \varphi(c)\sqrt{x}\log^2 cx + x,$$

où la constante impliquée ne dépend pas de $m \in \mathbb{N}^*$.

Preuve. Nous avons

$$\begin{aligned}\sum_{n\leq x} \Lambda(n) S(n, m, c) &= \sum_{\substack{d\leq c\\(d,c)=1}} e^{\frac{2\pi i m d}{c}} \sum_{n\leq x} e^{\frac{2\pi i n \bar{d}}{c}} \Lambda(n)\\ &= \sum_{\substack{d\leq c\\(d,c)=1}} e^{\frac{2\pi i m d}{c}} \sum_{\substack{k\leq c\\(k,c)=1}} \psi(x; kd, c) e^{\frac{2\pi i k}{c}}\\ &= \frac{1}{\varphi(c)} \sum_{\substack{d\leq c\\(d,c)=1}} e^{\frac{2\pi i m d}{c}} \sum_{\substack{k\leq c\\(k,c)=1}} \sum_{\chi \bmod c} \overline{\chi(kd)} \psi(x, \chi) e^{\frac{2\pi i k}{c}}\end{aligned}$$

où, comme d'usage, $\psi(x, \chi) = \sum_{n\leq x} \chi(n)\Lambda(n)$,

$$= \frac{1}{\varphi(c)} \sum_{\chi \bmod c} G_1(\bar{\chi}) G_m(\bar{\chi}) \psi(x, \chi)$$

où $G_m(\chi) = \sum_{(d,c)=1} \chi(d) e^{\frac{2\pi i m d}{c}}$ désigne une variante d'une somme de Gauss. On utilise alors l'inégalité de Cauchy-Schwartz et le fait simple à démontrer par expansion de la somme que $\sum_{\chi \bmod c} |G_m(\chi)|^2 \leq \varphi(c)^2$, après avoir pris soin de majorer $|\psi(x, \chi)|$ en utilisant RH, ce qui termine la démonstration du lemme. ∎

Le lemme implique que la première somme contribue pour moins de

$$\frac{\sqrt{x}\log^2 Nx}{N} + \frac{x}{N^2}.$$

Enfin le choix de $x = N^{2-\epsilon}$ implique que

$$\sum{}' \frac{r_f}{4\pi\langle f, f\rangle} \le \frac{F(0)}{2\phi(0)} + \frac{1}{2} + o(1). \tag{6}$$

Le choix de $F(u) = \max(1 - |u|, 0)$ qui vérifie $\phi(2\xi) = \sin^2 \xi/\xi^2$ et donc $F(0) = \phi(0)$ conclut la démonstration de (1). ∎

Pour passer en moyenne naturelle, on se réfère à la méthode décrite dans [K-M]. Cela prouve le théorème 1.

4. Preuve du théorème 2. Ce théorème [3] est une conséquence d'un résultat célèbre de Krein (*e.g.* [Lev, p. 437]). Commençons par quelques rappels techniques.

DÉFINITION 1. Une fonction entière est dite de type exponentiel si

$$a = \sup\left\{b \in \mathbb{R} : \frac{|\phi(z)|}{e^{b|z|}} \ll 1\, \forall z \in \mathbb{C}\right\} < \infty.$$

a est appelé le type (ou type exponentiel) de la fonction.

Nous rappelons aussi une caractérisation de la transformée de Fourier d'une fonction à carré sommable et support compact (*e.g.* [Kat, p. 176]).

THÉORÈME 5 (PALEY-WIENER). *Une condition nécessaire et suffisante pour qu'une fonction $F \in L^2(\mathbb{R})$ soit supportée dans $(-a, a)$ est que sa transformée de Fourier soit dans $L^2(\mathbb{R})$ et de type exponentiel au plus a.*

Citons le résultat essentiel de Krein.

THÉORÈME 6 (KREIN). *Soit ϕ une fonction entière satisfaisant aux trois propriétés suivantes:*

- $\phi(t) \ge 0$, $\forall t \in \mathbb{R}$,
- *ϕ est entière de type exponentiel a,*
- *ϕ est bornée dans $\mathbb{R}$.*

Alors il existe une fonction entière, ψ, de type exponentiel $a/2$ telle que $\phi(t) = |\psi(t)|^2$, $\forall t \in \mathbb{R}$.

[3] Philippe Michel m'a communiqué successivement que Fouvry est au courant de ce résultat, mais je n'ai pas réussi à trouver sa démonstration.

PREUVE DU THÉORÈME 2. Soit F comme dans l'énoncé du théorème. On remarque que $\hat{F}$ satisfait toujours aux seconde (avec $a = 1$) et troisième hypothèses du théorème de Krein grâce au théorème de Paley-Wiener et au lemme de Riemann-Lebesgue, respectivement. La première condition étant remplie par hypothèse, cela nous permet par le théorème de Krein d'introduire la fonction ψ correspondante. Posons aussi $G = 2\pi\hat{\psi}$. Vu que ψ est à carré sommable et de type 1/2, on obtient par le théorème de Paley-Wiener que G est supportée dans $(-1/2, 1/2)$. De plus, en posant $\Gamma(x) = \overline{G(-x)}$, on a que $G * \Gamma = F$, parce que la transformée de Fourier transforme la convolution en produit.

D'où

$$\phi(0) = \int F(x)\,dx \leq \|G\|_1\,\|\Gamma\|_1 = \left|\int_{-1/2}^{1/2} G(x)\,dx\right|^2 \leq \int_{-1/2}^{1/2} |G(x)|^2\,dx = F(0)$$

démontrant ainsi le théorème, parce qu'il y a égalité si et seulement si G est constant. ∎

REMARQUE. Le théorème 2 implique que le choix de la fonction triangle $\max(1 - |u|, 0)$ dans (6) est le meilleur possible parmi les fonctions à support compact.

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FC CENTRES OF UNITS IN ALGEBRAS AND ORDERS

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RÉSUMÉ. I. N. Herstein a montré que la classe de conjugaison d'un élément non-central du groupe multiplicatif d'un anneau de division est infinie. Nous prouvons des résultats similairs pour les unités des algèbres et des ordres, et donnons des applications dans les anneaux de groupes.

1. Introduction. For a given group G, we denote by $\Delta(G)$ the *FC-centre* of G, that is:

$$\Delta(G) = \{g \in G \mid [G : C_G(g)] < \infty\}.$$

Also, for a ring R we shall denote by $\mathcal{U}(R)$ the group of units of R; *i.e.*, the set of invertible elements of R. I. N. Herstein proved in [2] that if D is a division ring then $\Delta\big(\mathcal{U}(R)\big)$ coincides with $\mathcal{Z}\big(\mathcal{U}(R)\big)$, the centre of $\mathcal{U}(R)$.

Theorem 2.2 below shows that a result similar to that of Herstein holds for finite dimensional algebras over infinite fields and this fact is extended to algebraic algebras in Corollary 2.3. In Section 3 we consider orders in finite dimensional algebras and give a partial extension of a result of Sehgal and Zassenhaus [3, Theorem 1]. In the final section, we give some applications to group rings. The proofs of these results can be found in [1] and will be published elsewhere.

2. Algebras. We approach the problem via considering the group of units $\mathcal{U}A$ of an algebra A as an algebraic group. We first note the following fact for later use.

PROPOSITION 2.1. *Let G be a connected algebraic group over an infinite field. Then every FC-element of G is central.*

Our results, in the case of algebras, are as follows.

THEOREM 2.2. *Let A be an algebra over an infinite field K.*

(i) If A is finite dimensional, then $\mathcal{U}A$ is a connected linear algebraic group and, consequently,

$$\Delta(\mathcal{U}A) = \mathcal{Z}(\mathcal{U}A).$$

Moreover, A is generated by its units, as a vector space over K and, therefore, $\mathcal{U}A$ is FC if and only if A is commutative.

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(ii) *Every torsion unit of $\Delta(\mathcal{U}A)$ commutes with each algebraic unit of A and, consequently, $\Delta(\mathcal{U}A)$ is solvable of length at most 2.*
(iii) *Every element of $\Delta(\mathcal{U}A)$ commutes with each nilpotent element of A.*

COROLLARY 2.3. *If $\mathcal{U}A$ and $\Delta(\mathcal{U}A)$ are generated by algebraic units then*

$$\Delta(\mathcal{U}A) = \mathcal{Z}(\mathcal{U}A).$$

In particular, this happens if A is an algebraic algebra. In this case A is generated by units as a vector space, and $\mathcal{U}A$ is FC if and only if A is commutative.

THEOREM 2.4. *Let A be an algebra generated by its units, as a linear space over an infinite field K such that $\mathcal{U}A$ is FC. Then, every idempotent and every nilpotent element are central in A.*

Moreover, if A is generated by its torsion units, as a linear space over K, then $\mathcal{U}A$ is FC if and only if A is commutative.

3. **Orders.** Let D be a domain and let K be its field of fraction. Let A be a K-algebra and let Λ be a D-order in A; *i.e.*, a subring such that $A = K\Lambda$.

PROPOSITION 3.1. *Let D be an infinite domain and let K be its field of fractions. Let A be a K-algebra and Λ a D-order in A. If $x \in \Delta(\mathcal{U}\Lambda)$ and $y \in A$ is nilpotent then $xy = yx$.*

As a consequence of Proposition 3.1 we obtain the following theorem.

THEOREM 3.2. *Let D be an infinite domain, K its field of fractions, A a finite dimensional $K-$algebra, Λ a $D-$order in A, $\mathcal{J} = \mathcal{J}(A)$ the Jacobson Radical of A and $\bar{A} = A/\mathcal{J}$. Assume that $\mathrm{Hom}_A(P_i, P_j) = 0$ for every pair of non-isomorphic principal modules P_i, P_j of multiplicity 1 in A. If every minimal ideal of $\bar{A}$ which is a division ring is isomorphic to K, then*

$$\Delta(\mathcal{U}\Lambda) \subset \mathcal{Z}(A).$$

COROLLARY 3.3. *Let D and K be as above, A be a finite dimensional K-algebra and Λ a order in A. Assume that $\mathrm{Hom}_A(P_i, P_j) = 0$ for every pair of non-isomorphic principal modules P_i, P_j of multiplicity 1 in A. If K is a splitting field for A, then*

$$\Delta(\mathcal{U}\Lambda) \subset \mathcal{Z}(A).$$

COROLLARY 3.4. *Let D and K be as above, A be a semisimple finite dimensional $K-$algebra and Λ an $D-$order in A. If A has no minimal ideal which is a non-commutative division ring then*

$$\Delta(\mathcal{U}\Lambda) \subset \mathcal{Z}(A).$$

THEOREM 3.5. *Let D be an infinite domain and R a D-algebra.*

(i) If R is torsion free as a D-module then

$$\Delta(M_n(R)) = \Delta(R)I,$$

where I is the identity matrix of $M_n(R)$.

(ii) If $char(D) = 0$ and $n > 1$ then

$$\Delta(M_n(R)) = \mathcal{Z}(M_n(R)).$$

4. **Group Rings.** In this section, we apply our previous results to the case of group rings. We begin with some technical lemmas.

LEMMA 4.1. *Let K be a field and let G be a subgroup of* $\mathrm{GL}(2, K)$. *Then*

(i) if $A \in \mathrm{GL}(2, K)$ is noncentral, its centralizer in $\mathrm{GL}(2, K)$ is abelian and

(ii) either $\Delta(G) = \mathcal{Z}(G)$ or G is abelian-by-finite.

LEMMA 4.2. *Let $G = K_8 \times \langle c \rangle$, where c is an element of order p, an odd prime, and $K_8 = \langle a, b \rangle$ is the quaternion group of order 8. Then*

$$\Delta(\mathcal{U}(\mathbb{Z}G)) = \mathcal{Z}\Big((\mathcal{U}(\mathbb{Z}G))\Big).$$

PROPOSITION 4.3. *Let G be a finite group and suppose that $T\Delta(\mathcal{U}(\mathbb{Z}G))$ is non-abelian. Then G is a 2-group.*

As a consequence of these results we obtain the following theorem of Williamson [4, Theorem 1].

THEOREM 4.4. *Let G be a periodic group. If $T\Delta(\mathcal{U}(\mathbb{Z}G))$ is noncentral then there exists an element $x \in G$ and an abelian subgroup A such that $G = \langle A, x : A' = 1, x^{-1}ax = a^{-1} \forall a \in A \rangle$.*

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MORE ON THE UNIT LOOP OF AN ALTERNATIVE LOOP RING

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ABSTRACT. Let L be a loop and R a commutative, associative ring of characteristic different from 2 such that the loop ring RL is alternative. It is known that the (necessarily Moufang) loop L is solvable and "torsion of bounded exponent over the centre" in the sense that, for some integer n, ℓ^n is central for all $\ell \in L$. In this paper, we describe work which explores the extent to which these two properties are shared by the unit loop of RL in the cases that R is the ring of rational integers or a field.

RÉSUMÉ. Soit L une boucle et R un anneau commutatif et associative de caracteristique $\neq 2$ tel que l'anneau de boucle RL est alternative. Il est bien connu que la boucle L (necessairement Moufang) est resoluble et "torsion d'exposant borné", c'est-à-dire, il existe un entier n tel que ℓ^n soit centrale pour tout *ell* $\in L$. Cet article etudie comment la boucle d'unitées de RL a ces deux propriètés aux cas ou R soit l'anneau des entiers rationels ou un corps.

1. Introduction. An *alternative* ring is one which satisfies both the left and right alternative laws

$$x(xy) = x^2y \quad \text{and} \quad (yx)x = yx^2.$$

A *Moufang* loop is a loop which satisfies any of the three equivalent identities

$$\big((xy)x\big)z = x\big(y(xz)\big), \quad \big((xy)z\big)y = x\big(y(zy)\big) \quad \text{and} \quad (xy)(zx) = \big(x(yz)\big)x.$$

Given a Moufang loop L and a commutative, associative ring R, one forms the loop ring RL precisely as if L were a group. Unlike the associative situation, however, it is rarely the case that RL inherits the Moufang identity from L. When it does, and if char $R \neq 2$, L is called an *RA loop* because RL is an alternative ring. A suitable single source of information about Moufang loops, RA loops, alternative rings and alternative loop rings is the monograph [GJM96].

If L is an RA loop, the set $\mathcal{U}$(RL) of *units* (or invertible elements) in RL is a Moufang loop (containing L) and it is interesting to compare the properties

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of L and $\mathcal{U}(\mathrm{R}L)$. Since the square of any element in L is central (hence $L/\mathcal{Z}(L)$ is an elementary abelian 2-group), for instance, L is nilpotent. Necessary and sufficient conditions for the nilpotence of $\mathcal{U}(\mathrm{R}L)$, for various R, were determined a few years ago [GM95], [GM97], so we have recently turned our attention to solvability, with results in the cases that $R = \mathrm{Z}$ is the ring of rational integers or a field of arbitrary characteristic. A Moufang loop $\mathcal{U}$ is *torsion over the centre* if, for each $\mu \in \mathcal{U}$, μ^n is central for some integer $n = n(\mu)$. If n can be chosen independent of μ, then $\mathcal{U}$ is torsion *of bounded exponent* n over the centre. We have already said that an RA loop is torsion of bounded exponent 2 over its centre. It is natural then to see if the unit loop can exhibit similar behaviour.

The questions with which we are concerned in this paper have been considered for groups. Group rings whose unit groups are torsion over their centres have been investigated by Sehgal [Seh78, Section II.2], Cliff and Sehgal [CS80], Coelho [Coe82] and Bist [Bis94]. The first paper on solvability was written by Bateman [Bat71], who considered the group algebra of a finite group over a field. Solvability of the unit group of infinite groups and over other rings of coefficients has been investigated by a number of people, including Motose, Tominaga and Ninomiya [MT71], [MN72], Sehgal [Seh75], Passman [Pas77], Gonçalves [Gon86] and Bovdi [Bov92].

2. **Integral loop rings.** An element of finite order in a loop is a *torsion* element. In an RA loop, the set of all torsion elements forms a subloop called the *torsion* subloop of L. In [GM95] (see also [GJM96, Chapter XII]), the authors showed that the nilpotence of $\mathcal{U}(\mathrm{Z}L)$, the unit loop of an integral alternative loop ring, is equivalent to a number of conditions, one of which is:

* The torsion subloop T of L is an abelian group or a hamiltonian Moufang 2-loop and $x^{-1}tx = t^{\pm 1}$ for any $t \in T$ and any $x \in L$. Moreover, if T is an abelian group and $x \in L$ is an element that does not centralize T, then $x^{-1}tx = t^{-1}$ for all $t \in T$.

Recently, we have shown that solvability is equivalent to a suitable weakening of this statement [GMa].

THEOREM 1. *Let L be an RA loop with torsion subloop T. Then $\mathcal{U}(\mathrm{Z}L)$ is solvable if and only if T is either an abelian group or a hamiltonian Moufang 2-loop and every subloop of T is normal in L.*

It follows that solvability and nilpotence of $\mathcal{U}(\mathrm{Z}L)$ are equivalent for torsion RA loops. (There are examples which show that this is not always the case.) Also, and this was quite unexpected, we have shown that (2) is a condition also equivalent to $\mathcal{U}(\mathrm{Z}L)$ being torsion over its centre. Specifically, we have this result [GMb].

THEOREM 2. *Let L be an RA loop with torsion subloop T. Then the following conditions are equivalent:*

(i) $\mathcal{U}(\mathbb{Z}L)$ *is torsion of bounded exponent over the centre.*
(ii) $\mathcal{U}(\mathbb{Z}L)$ *is torsion of bounded exponent* 2 *over the centre.*
(iii) $\mathcal{U}(\mathbb{Z}L)$ *is nilpotent.*
(iv) Condition (2) holds.

We do not know when $\mathcal{U}(\mathbb{Z}L)$ is torsion of unbounded exponent over its centre.

3. Unit loops of loop algebras. Questions about loop algebras over fields require different approaches and usually have very different answers from analogous questions about integral loop rings and such was our experience with these investigations. In certain cases, for instance, we were able to establish that torsion over the centre was equivalent to torsion of bounded exponent. Our results depend on the characteristic of K and are presented in four theorems below, proofs of which can be found in [GMb]. Those of Theorems 3 and 5, which consider the case of unbounded exponent, make fundamental use of the observation that every idempotent of KL is central whenever $\mathcal{U}(KL)$ is torsion over its centre, but not torsion. At this point, we have very explicit information about the nature of units: specifically, every unit μ can be written as $\mu = \sum \mu_q q$, the q in L and the μ_q in KT, where T is the torsion subloop of L [GJM96, Lemma XII.1.1], [GM95].

THEOREM 3. *Let L be an RA loop with torsion subloop T and let K be a field of characteristic* 0. *Suppose $\mathcal{U}(KL)$ is not torsion. Then the following conditions are equivalent.*
(i) $\mathcal{U}(KL)$ *is torsion over its centre.*
(ii) T is central.
(iii) $\mathcal{U}(KL)$ *is torsion of bounded exponent* 2 *over its centre.*
(iv) $\mathcal{U}(KL)$ *is torsion of bounded exponent over its centre.*

Since the prime 2 plays a very important role in the theory of RA loops, when K has positive characteristic p, the case $p = 2$ is usually special and easy to handle.

THEOREM 4. *Let L be an RA loop and let K be a field of characteristic* 2. *Then $\mathcal{U}(KL)$ is torsion of bounded exponent* 2 *over its centre.*

THEOREM 5. *Let K be a field of positive characteristic $p \neq 2$ and let L be an RA loop with torsion subloop T. Suppose $\mathcal{U}(KL)$ is not torsion. Then $\mathcal{U}(KL)$ is torsion over its centre if and only if T is an abelian group, every idempotent of KT is central in KL and, if T is not central, then K is algebraic over its prime field.*

THEOREM 6. *Let K be a field of positive characteristic $p \neq 2$ and let L be an RA loop with torsion subloop T. Let P and $A = T_{p'}$ denote, respectively, the sets of p- and p'-elements in T. Suppose $\mathcal{U}(KL)$ is not torsion. Then the following statements are equivalent.*
(i) $\mathcal{U}(KL)$ *is torsion of bounded exponent over its centre.*

(ii) T is abelian, there exists an integer k such that x^{p^k} is central for all $x \in \Delta(L, P)$ and either

- *T is central and $\mathcal{U}(KL)$ has exponent $2p^k$ over its centre, or*
- *K is finite, $A^m = \{1\}$ for some m and $\mathcal{U}(KL)$ has exponent $2rp^k$ over its centre, where $r = |K(\zeta)| - 1$, ζ a primitive m-th root of unity.*

We return to questions of solvability of the unit loop of a loop algebra. Over a field K, when the RA loop L is torsion, we can show that $\mathcal{U}(KL)$ is solvable if and only if char $K = 2$. When L is not torsion, however, the question is considerably more complicated and not completely settled. Our results rely heavily on the ability to find *Zorn's vector matrix algebra* in loop algebras. Given a field K, Zorn's algebra $\mathfrak{Z}(K)$ is the set of 2×2 matrices of the form $\begin{bmatrix} a & \mathsf{x} \\ \mathsf{y} & b \end{bmatrix}$ where $a, b \in K$, $\mathsf{x}, \mathsf{y} \in K^3$, with obvious addition and the following multiplication:

$$\begin{bmatrix} a_1 & \mathsf{x}_1 \\ \mathsf{y}_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & \mathsf{x}_2 \\ \mathsf{y}_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + \mathsf{x}_1 \cdot \mathsf{y}_2 & a_1\mathsf{x}_2 + b_2\mathsf{x}_1 - \mathsf{y}_1 \times \mathsf{y}_2 \\ a_2\mathsf{y}_1 + b_1\mathsf{y}_2 + \mathsf{x}_1 \times \mathsf{x}_2 & b_1b_2 + \mathsf{y}_1 \cdot \mathsf{x}_2 \end{bmatrix}.$$

Here, $\cdot$ and $\times$ denote the dot and cross products respectively in K^3. Lowell Paige showed that the unit loop of any Zorn's algebra is simple modulo its centre [Pai56]. Of particular relevance here is the consequence that the unit loop of $\mathfrak{Z}(K)$ is never solvable.

Suppose K is a field of characteristic different from 2, L is an RA loop with torsion subloop T and the subloop generated by some $t \in T$ is not normal in L. We are able to construct quite explicitly a copy of $\mathfrak{Z}(K)$ inside KL. Similarly, if L contains the quaternion group of order 8 and K has characteristic 3, again we can exhibit $\mathfrak{Z}(K)$ within KL. These facts were the key in obtaining the following result.

THEOREM 7. *Let K be a field of characteristic $p \geq 0$ and suppose L is an RA loop which contains no elements of order p (in the case $p > 0$). Then $\mathcal{U}(KL)$ is solvable if and only if (i) $p = 2$ or (ii) T is an abelian group and every idempotent of KT is central in KL.*

Comparison with Theorem 4 gives the following.

COROLLARY 8. *With the hypotheses on K and L as in Theorem 7, if $\mathcal{U}(KL)$ is torsion over its centre but not torsion, then $\mathcal{U}(KL)$ is solvable.*

When L contains elements of order p, we have results only in positive characteristic. The corresponding problem for group rings is also unsettled, although A. Bovdi has made progress for nilpotent nontorsion groups [Bov92].

THEOREM 9. *Let K be a field of characteristic $p > 0$. Let L be an RA loop which is not torsion and which contains elements of order p. The $\mathcal{U}(KL)$ is solvable if and only if (i) $p = 2$ or (ii) the set P of p-elements in L is a finite central group and $\mathcal{U}(K[L/P])$ is solvable.*

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